SEPARABILITY OF TWO-OSCILLATOR MIXED STATES FROM THE SUBSPACE WITH FIXED TOTAL OCCUPATION QUANTUM NUMBER

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For a system of two-oscillator, we introduce a basis labelled by the sum and the difference between the occupation quantum numbers of the oscillators. It forms a basis for the SU(2) bosonic representation. This basis could be useful in the studies of coupled oscillator systems when the total occupation quantum number of the systems is a constant of motion, for example, in the weak coupling limit. Using this basis, we are able to show that the positive partial transpose criterion is necessary and sufficient for the separability of general mixed states in the subspaces with fixed total occupation quantum number. We find that mixed states that are diagonal in this basis are the only separable states in this subspace. The result is consistent with the fact that the identification of entanglement in infinite dimensional systems can be reduced to a problem in finite dimensions. Examples of quantum states that belong to these subspaces are the so-called $N00N$-states and the SU(2) coherent states used in the studies of quantum optical systems, quantum information science and quantum nonlinear rotators.

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1. Introduction

Systems of two or more oscillators are intrinsically interesting in their collective properties such as their energy eigenvalues [1], symmetries [2, 3], and the identification of entanglement between them [4, 5]. The systems form the basis for the description of various physical systems, such as two-mode Gaussian states [4], two-mode squeezed states [6], and coupled rotators [7]. Their response as a collective system to external influence also received much attention, for example, the decoherence or generation in the entanglement

of two-mode oscillators in a thermal bath [8–11], and the coupling of the pump and signal mode to the idle mode as described by the trilinear boson model [12, 13].

A common basis that is used to represent the two-oscillator states is the tensor product of the occupation quantum number states, or the Fock space, of both oscillators. In certain settings, such as due to the nature of the interactions between a pair of coupled oscillators [14], under the rotating-wave approximation [15, 16], or when a pair of oscillators is weakly coupled to a field as in the trilinear boson model [17], the so-called virtual processes in which the quanta of both oscillators are simultaneous created or annihilated are absent. As a result, the total occupation quantum number of the two-oscillator system becomes a constant of motion of the dynamics.

In this situation, it becomes natural to introduce a basis labelled by the sum, N, and the difference, r, of the occupation quantum number of both oscillators. The space is then divided into subspaces with different N. This basis can be used to furnish the SU(2) bosonic representation [18]. It is a variant of the basis that was used to study the dynamical aspects of unstable quantum systems in the Liouville space [19, 20]. The introduction of this basis is in analogy to the introduction of the total and difference in the momentum of a system of two particles when the total momentum is conserved during the interaction, or the introduction of the center of mass and relative coordinates between two particles when the total mass of the system is much larger than the reduced mass, and hence the center-of-mass coordinate can be assumed to be stationary in comparison to the relative coordinate.

As far as we know, this basis has not been used explicitly in the literature, including the field of quantum information science. When we apply this basis to the study of the separability of two-oscillator states in the subspace of fixed N, we find that the partial positive transpose (PPT) criterion [21, 22] is necessary and sufficient for the separability of the two-oscillator states, *i.e.*, the only separable states in this subspace are the states that are diagonal in this basis. The condition for separability of general density matrices (including mixed states) in this subspace has not been demonstrated before. In view of the fact that the identification of entanglement in infinite dimensions can be reduced to a problem in finite dimensions [23], our result could be useful in this respect.

The class of states we consider are subclass of bipartite non-Gaussian states. Examples of these states are the so-called $N00N$-states [24–26], which are generalizations of the Bell's states $|1,0\rangle \pm |0,1\rangle$, and the SU(2) coherent states [7, 27–29], where both states have been realized in the laboratory. They are used in quantum optical lithography [24], quantum optical metrology [26], and in the theoretical studies of quantum information processing [30] and quantum nonlinear rotators [7].

The PPT criterion is one of the most commonly used condition in testing the separability of quantum states. It is a necessary and sufficient condition for the separability of bipartite Gaussian states [31, 32]. As for general bipartite non-Gaussian states such as those we consider, this criterion, in general, leads to an infinite series of inequalities of higher order moments [33]. The PPT criterion can also be used together with the uncertainty relations [34–36] to generate entanglement conditions. There are also other methods that give rise to separability tests of quantum states, for instance, through entanglement witness [32, 37–39], local unitary relations [40, 41], cross-norm and realignment maps and their generalizations [42–44], and general entropy functions [45].

2. The basis

Rather than working in the usual number basis of the oscillators, $|n_1, n_2\rangle \equiv |n_1\rangle \otimes |n_2\rangle$, we introduce a basis labelled by the sum and the difference between the occupation quantum numbers of both oscillators, defined respectively by

$$N \equiv n_1 + n_2 \,, \qquad r \equiv n_1 - n_2 \,. \tag{1}$$

We denote the new basis by $|u_r^N\rangle \equiv |n_1, n_2\rangle$. This basis is mathematically analogous to the total angular momentum basis of integer spin systems [46]. In fact, it provides a natural basis for the bosonic representation of SU(2) [18]. In the density matrix space, we denote the basis as

$$f_{r;s}^{(N,M)} \equiv \left|u_r^N\right\rangle \left\langle u_s^M\right| \,, \tag{2}$$

where r and s have the ranges $-N \leq r \leq N$ and $-M \leq s \leq M$. They form a complete and orthogonal set of states.

By writing the density matrix ρ as a linear combination of $f_{r;s}^{(N,M)}$,

$$\rho = \sum_{N,M=0}^{\infty} \sum_{r=-N}^{N} \sum_{s=-M}^{M} c_{r;s}^{(N,M)} f_{r;s}^{(N,M)} \,, \tag{3}$$

we arrange the real diagonal coefficients $c_{r;r}^{(N,N)}$, which represent the probability elements, and the complex coefficients, $c_{r;s}^{(N,M)}$, which represent the quantum correlation between both oscillators, in the following way

$$\rho = \left(\begin{array}{c|cc|c|c} c_{0;0}^{(0,0)} & c_{0;1}^{(0,1)} & c_{0;-1}^{(0,1)} & c_{0;2}^{(0,2)} & \cdots \\ \hline c_{1;0}^{(1,0)} & c_{1;1}^{(1,1)} & c_{1;-1}^{(1,1)} & c_{1;2}^{(1,2)} & \cdots \\ c_{-1;0}^{(1,0)} & c_{-1;1}^{(1,1)} & c_{-1;-1}^{(1,1)} & c_{-1;2}^{(1,2)} & \cdots \\ \hline c_{2;0}^{(2,0)} & c_{2;1}^{(2,1)} & c_{2;-1}^{(2,1)} & c_{2;2}^{(2,2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right). \tag{4}$$

Notice that the coefficients with different (N, M) indices lie in separate blocks, and each block represents different (N, M)-subspaces. For example, the 2×2 block in the middle of the matrix (4) refers to the $(1, 1)$-subspace. The value of N or M increases by 1 when we move to the next subspace lying below or to the right, respectively. Within each subspace, r or s decreases by 2 when we move to the next entry lying below or to the right in the sequence, respectively.

3. Partial transpose of two-oscillator states

A density matrix of two-oscillator state is separable [47] if it can be written as a convex combination of the tensor product of the density matrices of both oscillators, labelled by ρ_1 and ρ_2, respectively

$$\rho = \sum_i c_i \rho_{1,i} \otimes \rho_{2,i}\,, \tag{5}$$

where $\sum_i c_i = 1$ and $c_i > 0$. Separable states possess only classical correlation and are therefore not entangled.

A partial transpose on the second oscillator is a non-completely positive map defined by [21, 22]

$$\left[|n_1\rangle\langle m_1| \otimes |n_2\rangle\langle m_2|\right]^{\mathrm{T}_2} = |n_1\rangle\langle m_1| \otimes |m_2\rangle\langle n_2|\,. \tag{6}$$

For separable states (5), the partial transpose of ρ remains a density matrix. Hence, ρ remains positive. Consequently, the positivity of the partial transpose is a necessary condition for separable states. This is the positive partial transpose (PPT) criterion for separable states [21]. For discrete states, the criterion is also a sufficient condition for $2 \otimes 2$ or $2 \otimes 3$ systems only [22], whereas for infinite dimensional systems, it is a sufficient condition for bipartite Gaussian states [31, 32].

In terms of the $f_{r;s}^{(N,M)}$ basis, the partial transpose on the second oscillator translates into the following form

$$\left[f_{r;s}^{(N,M)}\right]^{\mathrm{T}_2} = f_{r_+,r_-}^{(N_+,N_-)}\,, \tag{7}$$

where

$$N_{\pm} = \tfrac{1}{2}(N+M) \pm \tfrac{1}{2}(r-s)\,, \tag{8}$$
$$r_{\pm} = \tfrac{1}{2}(r+s) \pm \tfrac{1}{2}(N-M)\,. \tag{9}$$

As a result, the respective coefficients of the basis in matrix (4) are transformed to their respective new positions.

We now restrict our attention to density matrices within the (N,N)-subspace only

$$\rho^{(N,N)} = \sum_{r,s=-N}^{N} c_{r;s}^{(N,N)} f_{r;s}^{(N,N)}\,. \tag{10}$$

For example, the $N00N$-states [24–26]

$$|\psi\rangle = \frac{1}{\sqrt{1+|\eta|^2}}\left(|N,0\rangle + \eta|0,N\rangle\right), \tag{11}$$

and the SU(2) coherent states [7, 27–29]

$$|\tau\rangle_N = \left(1+|\tau|^2\right)^{-\frac{N}{2}} \sum_{n=0}^{N} \sqrt{\binom{N}{n}}\tau^n |n, N-n\rangle\,, \tag{12}$$

where η and τ are complex numbers, belong to this subspace.

By denoting $\nu \equiv r-s$, we find that

$$N_{\pm} = N \pm \tfrac{1}{2}\nu\,, \tag{13}$$
$$\bar{r} \equiv r_+ = r_- = \tfrac{1}{2}(r+s)\,, \tag{14}$$

where the ranges of $N_{\pm}$ and $\bar{r}$ are given respectively by

$$0 \le N_{\pm} \le 2N\,, \tag{15}$$
$$-N \le \bar{r} \le N\,. \tag{16}$$

4. Separability of mixed states in the fixed N subspace

We now show that when at least one of the off-diagonal coefficients of any mixed states in the fixed N, or (N,N)-subspace, is non-zero, the states have negative partial transpose and are therefore inseparable or entangled by the PPT criterion.

After a partial transpose, the elements in the (N,N)-subspace of the original matrix are transformed to the other subspaces with labels ranging from 0 to $2N$, see Eq. (15). We find that each of the columns and rows of the partial transposed matrix contains no more than one non-zero element

based on the following facts, which cover all the elements in this subspace and are therefore complete. Coefficients on the diagonal have $\nu = 0$, whereas coefficients with the same non-zero ν are aligned in parallel to the diagonal.

1. Diagonal coefficients ($\nu = 0$) remain unchanged under the partial transpose.

2. Off-diagonal coefficients with the same ν ($\neq 0$) are transformed to the same off-diagonal (N_+, N_-)-subspace, where $N_\pm \neq N$.

3. Coefficients associated with different ν' ($\neq \nu$) are transformed to different subspaces with labels $N'_+ (\neq N_+)$ and $N'_- (\neq N_-)$, respectively, that do not equal to each other.

4. Within each (N_+, N_-)-subspace, the transformed coefficients are located at different $(\bar{r}, \bar{r})$ positions in the matrix, (see Eq. (14) for $\bar{r}$) so that no two coefficients occupy the same row or column in each (N_+, N_-)-subspace.

As a result, each row and column of the partial transposed matrix $[\rho^{(N,N)}]^{\mathrm{T}_2}$ contains no more than one coefficient.

The characteristic matrix $[\rho^{(N,N)}]^{\mathrm{T}_2}$ can subsequently be brought into a block diagonal form that consists of three types of blocks

$$\left| \left[\rho^{(N,N)}\right]^{\mathrm{T}_2} - \lambda I \right| = \begin{pmatrix} A & & & & \\ & B_1 & & & \\ & & B_2 & & \\ & & & \ddots & \\ & & & & C \end{pmatrix}. \tag{17}$$

There is one copy of $(N+1)$-dimensional square matrix $A = \mathrm{diag}(a-\lambda, b-\lambda, c-\lambda, \dots)$, where $a, b, c, \dots$ are the diagonal elements of ρ. There are at most $N(N+1)/2$ copies of 2-dimensional square matrix

$$B_i = \begin{pmatrix} -\lambda & \alpha_i \\ \alpha_i^* & -\lambda \end{pmatrix}, \tag{18}$$

where α_i are the non-zero off-diagonal elements of $\rho^{(N,N)}$. The rest of the infinite number of zero elements make up an infinite-dimensional square matrix $C = \mathrm{diag}(-\lambda, -\lambda, \cdots)$. Since each B_i matrix gives rise to a separate negative eigenvalue $-|\alpha_i|$ of $[\rho^{(N,N)}]^{\mathrm{T}_2}$, we conclude that, in general, density matrices in the (N, N)-subspace are inseparable whenever they have at least a non-zero off-diagonal element.

5. Examples

Let us use some simple examples to illustrate our result and its limitation. Consider a density matrix with elements in the $(0,0)$-, $(1,1)$- and $(2,2)$-subspace

$$\chi = \left(\begin{array}{c|cc|cc} d & 0 & 0 & 0 & 0 \\ \hline 0 & a & c & 0 & 0 \\ 0 & c^* & b & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e \end{array}\right), \tag{19}$$

where we omit the rest of the zero coefficients of the infinite dimensional matrix for simplicity. The positivity of χ requires that $|c|^2 \leq ab$, and that a, b, d, e are all non-negative and sum up to unity for normalization. Notice that χ is effectively a $2 \otimes 2$ bipartite system where each oscillator consists of the $|0\rangle$ and $|1\rangle$ states only.

Under a partial transpose, see Eqs. (7) to (9), we have

$$\chi^{\mathrm{T}_2} = \left(\begin{array}{c|cc|cc} d & 0 & 0 & 0 & c^* \\ \hline 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & e \end{array}\right). \tag{20}$$

The characteristic matrix can then be brought into the form (17)

$$\left|\chi^{\mathrm{T}_2} - \lambda I\right| = \left(\begin{array}{cc|cc|c} a-\lambda & 0 & 0 & 0 & 0 \\ 0 & b-\lambda & 0 & 0 & 0 \\ \hline 0 & 0 & d-\lambda & c^* & 0 \\ 0 & 0 & c & e-\lambda & 0 \\ \hline 0 & 0 & 0 & 0 & -\lambda \end{array}\right). \tag{21}$$

If the initial state lies entirely in the $(1,1)$-subspace, *i.e.*, when $d = 0 = e$, then our result shows that the state is entangled since χ^{T_2} has a negative eigenvalue $-|c|$ whenever $|c| \neq 0$. When an element from the $(0,0)$- or $(2,2)$-subspace is included, *i.e.*, either d or e is non-zero, one of the eigenvalues of χ^{T_2}, either $d/2 - \sqrt{(d/2)^2 + |c|^2}$ or $e/2 - \sqrt{(e/2)^2 + |c|^2}$, respectively, is negative when $|c| \neq 0$. The state therefore remains entangled. However, when d and e are both non-zero, $|c| \neq 0$ no longer guarantees that one of the eigenvalues is negative. In fact, χ^{T_2} becomes non-negative whenever $|c|^2 \leq de$. In this situation, even though χ's off-diagonal element c is non-zero, the state is separable according to the Peres–Horodecki criterion [21, 22], for the $2 \otimes 2$ bipartite system.

The above examples illustrate the fact that our result is applicable to the individual (N, N)-subspace only. For general density matrices that involve the linear combination of various subspaces, the entanglement property of quantum states in the (N, N)-subspace is altered. The maximally mixed state serves as another example. For bipartite states in finite dimensional space, it is known that there exists a small neighbourhood of separable states around the maximally mixed state [48, 49]. However, in the system considered here, an infinite dimensional maximally mixed state is not normalizable, whereas its finite dimensional truncated version necessarily spans over two or more subspaces with different total occupation quantum numbers. Our result is therefore not applicable to the separability of states around the neighbourhood of the maximally mixed state.

6. Conclusion

We introduce a basis for two-oscillator systems that is labelled by the sum and the difference between the occupation quantum numbers of both oscillators. It is a natural basis for systems in which the total occupation quantum number of both oscillators is a constant of motion. This representation enables us to show that the PPT criterion is necessary and sufficient for the separability of two-oscillator mixed states lying in the subspace with fixed total occupation quantum number. We find that in this subspace, the only separable states are those diagonal in this basis. Our result for finite dimensional subspaces is relevant since the entanglement of infinite dimensional systems can be reduced to a problem in finite dimensions. The result and the extension of the basis to systems of three or more oscillators could be useful in the future studies of coupled oscillator systems.

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