A NOTE ON THE SPECTRUM
OF A TWO-PARTICLE RASHBA HAMILTONIAN

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In a series of recent papers it was shown that, when the attractive
s-wave interaction is dominant, the spin-orbit coupled fermions form a
bound state. Attributed to a convenient momentum representation, it be-
came a common condition of agreement to express the bound state as a
function of the center-of-mass momentum Q. In this paper, we prove that
the bound state of Rashba fermions does not depend on the chosen repre-
sentation. That is, all the states characterized by nonzero Q fail to obey
the translation symmetry.

1. Introduction

The role of a bound state is in the spotlight of neutral Fermi atoms. The
motives of research, technical possibilities and important physical issues are
discussed in many papers [1–7] where — among other celebrated results —
the existence of a bound state is determined by establishing the poles of
the Fourier transform of Green's function. One can also explore a free two-
particle Green's function in deriving the so-called self-consistency condition
which is just the weak (or distributional) solution to the Schrödinger equa-
tion [6]. Although trivial in many applications, the pole itself is insufficient
to determine the bound state. According to the Aronszajn–Donoghue spec-
tral theory of rank one perturbations [8, 9] (s-wave interaction is exactly
rank one), the derivative of the free Green's function with respect to (w.r.t.)
the eigenvalue must be finite. Otherwise, the spectral points are in the sin-
gular continuous spectrum. This is exactly the case when one says [6] that
the bound state ceases to exist.

The single-particle Rashba Hamiltonian in $L_2(\mathrm{R}^2; \mathrm{C}^2) =: \mathfrak{X}$, of a particle with mass m and Rashba spin-orbit coupling α (we use the $\hbar = c = 1$ units) is invariant under Euclidean moves. If parametrized in terms of the center-of-mass $\vec{R} \in \mathrm{R}^2$ and the relative $\vec{r} \in \mathrm{R}^2$ coordinates, the same applies to the Hamiltonian in $(\mathfrak{X}, d^2\vec{R}) \otimes (\mathfrak{X}, d^2\vec{r})$ which describes the two interacting particles. For the dominant s-wave interaction, with the interaction strength γ, the Euclidean group is naturally reduced to the translation group. The latter is represented as a tensor product of two two-dimensional groups whose irreducible representations act on $\vec{R}$ and $\vec{r}$, respectively. Due to the interaction, it is clear that the two-particle Hamiltonian commutes with the generator $\vec{P} \otimes I$, where $i\vec{P}$ is the gradient in $\vec{R}$ and I is the identity operator in $(\mathfrak{X}, d^2\vec{r})$, whereas it does not commute with the generator in $\vec{r}$. The commutation relation indicates that the eigenspace of the Hamiltonian is a dense subset in the domain of the closure of the generator in the Sobolev norm (apply *e.g.* [10, Corollary A.7.3.6]). Subsequently, the generalized eigenvectors of the Hamiltonian are labeled by its spectral points λ along with the spectral points $\vec{\Lambda} \in \mathrm{R}^2$ of the closure of $\vec{P}$.

In this paper, we shall prove that only the translation invariant eigenvectors, that is those with $\vec{\Lambda} = \vec{0}$, are the eigenvectors of the two-particle Rashba Hamiltonian with point-interaction. For other two-particle interactions, this is not necessarily the case though (see discussion in Sec. 6).

2. Translation symmetry

Consider an arbitrary $f = f(\vec{R}, \vec{r})$ in the Schwartz space $\mathcal{D}(\mathrm{R}^4; \mathrm{C}^4)$ of smooth functions with compact support. Then f has a Fourier transform $\chi = \chi(\vec{Q}, \vec{k})$, where $\vec{Q} \in \mathrm{R}^2$ denotes the center-of-mass momentum and $\vec{k} \in \mathrm{R}^2$ is the relative momentum.

Since $\mathcal{D}(\mathrm{R}^4; \mathrm{C}^4)$ is a dense subspace of the Hilbert space $L_2(\mathrm{R}^4; \mathrm{C}^4)$, denoted $\mathfrak{X}_o$, and $\mathfrak{X} \otimes \mathfrak{X}$ is isomorphic to $\mathfrak{X}_o$, one can restate the irreducible representation $\exp(i\vec{P} \cdot \vec{a}) \otimes I$ (all $\vec{a} \in \mathrm{R}^2$) of the translation group by simply writing $\exp(i\vec{P} \cdot \vec{a})$. Suppose that $\vec{\Lambda} \in \mathrm{R}^2$ and that f solves $(\vec{P} - \vec{\Lambda})f = \vec{0}$. Then f is of the form

$$f\left(\vec{R}, \vec{r}\right) = e^{i(\vec{\Lambda} \cdot \vec{R})} \varphi(\vec{r}), \qquad \varphi \in \mathcal{D}\left(\mathrm{R}^2; \mathrm{C}^4\right), \qquad (2.1)$$

and the Fourier transform reads

$$\chi\left(\vec{Q}, \vec{k}\right) = (2\pi)^2 \delta\left(\vec{Q} - \vec{\Lambda}\right) \hat{\varphi}\left(\vec{k}\right), \qquad (2.2)$$

where $\hat{\varphi} \in \mathcal{D}(\mathrm{R}^2; \mathrm{C}^4)$ is the Fourier transform of φ and $\delta(\cdot)$ is a two-dimensional Dirac distribution. It appears from (2.2) that the Fourier transform χ represents a singular distribution, the existence of which is predetermined by the relation $\vec{Q} = \vec{\Lambda}$. In particular, (2.1) implies that for $\vec{\Lambda} = \vec{0}$, f is

translation invariant. In general, it also follows from (2.1) that f is labeled by the spectral point $\vec{\Lambda}$, while φ is independent of $\vec{\Lambda}$.

3. Green's function

The single-particle Rashba Hamiltonian in $\mathfrak{X}$ is realized through the differential expression $-(2m)^{-1}\Delta + \alpha U$, where $U := -i(\nabla_y \sigma_x - \nabla_x \sigma_y)$. Based on the relation $U^2 = -\Delta$, the associated Green's function can be obtained in an extremely elegant way [11]. In coordinate representation, it is a linear combination of the Bessel functions of the second kind. It can be shown[1] that the single-particle Green's function is sufficient to recover the spectrum of the two-particle Rashba Hamiltonian with point-interaction. The bound state λ corresponds to the Fourier transform of the Hamiltonian with $\vec{Q} = \vec{0}$. More precisely, given $m = 1/2$, $\alpha > 0$, $\gamma < 0$, a single bound state is found to be equal to $\lambda = -\alpha^2/(2\sin^2 \omega)$, where the parameter $-\pi/2 < \omega < 0$ solves the transcendental equation $j_\gamma + \ln(\alpha/4) = \ln|\sin \omega| - \omega|\tan \omega|$, where one defines $j_\gamma := 4\pi/|\gamma| - \Psi(1)$, and $\Psi(1)$ is the digamma function. The result is in exact agreement with the associated self-consistency condition (see the integral in [6, Eq. (31)]) obtained by considering the free two-particle Green's function in momentum representation. Moreover, the above equation allows one to evaluate the characteristic radius of the interaction potential which is found to be $\exp(-\Psi(1))/2 \simeq 0.89$ (all $\alpha > 0$, all $\gamma < 0$).

Let $g_\mu^0(\vec{Q}, \vec{k})$ be the free two-particle Green's function in momentum representation projected onto the singlet basis [6]; $\mu \in \mathbb{C} \backslash [-\alpha^2 m, \infty)$. Then g_μ^0 is a bounded everywhere defined function on $\mathbb{R}^4$. The function $g_\mu(\vec{Q}, \vec{k})$ corresponding to the projected integral kernel of the two-particle Hamiltonian with point-interaction fulfills the resolvent identity

$$g_\mu\left(\vec{Q}, \vec{k}\right) = g_\mu^0\left(\vec{Q}, \vec{k}\right)(1 - \gamma \tilde{g}_\mu(Q)),$$

where

$$g_\mu^0\left(\vec{Q}, \vec{k}\right) := (\epsilon - \mu)\left((\epsilon - \mu)^2 - \alpha^2 Q^2\right)\left((\epsilon - \mu)^4\right.$$
$$\left. -4\alpha^2 m\epsilon(\epsilon - \mu)^2 + 4\alpha^4\left(\vec{Q} \cdot \vec{k}\right)^2\right)^{-1},$$

$$\epsilon(Q, k) := \frac{k^2}{m} + \frac{Q^2}{4m},$$

$$\tilde{g}_\mu(Q) := \int\limits_{k \leq C} \frac{d^2\vec{k}}{(2\pi)^2} g_\mu\left(\vec{Q}, \vec{k}\right). \tag{3.1}$$

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[1] In preparation.

The condition $k \leq C$, where C is the UV cutoff, ensures that the integral exists and the obtained natural logarithm is finite.

It appears from the resolvent identity that the (projected) Green's function

$$g_\mu\left(\vec{Q}, \vec{k}\right) = \frac{g_\mu^0\left(\vec{Q}, \vec{k}\right)}{1 + \gamma \tilde{g}_\mu^0(Q)},$$ (3.2)

where $\tilde{g}_\mu^0$ is defined similar to $\tilde{g}_\mu$ but with g_μ replaced by g_μ^0. Then the denominator of the g_μ determines the bound state solution unless $|(\partial/\partial\mu)\tilde{g}_\mu^0|$ is sufficiently large at $\mu = \lambda$ (where λ solves $1 + \gamma \tilde{g}_\lambda^0 = 0$). We use the notion "sufficiently large" rather than "infinite" because one accounts for $k \leq C$ but not for $k < \infty$ in the integral in (3.1). By (3.2), the bound state solution is a function of the center-of-mass momentum, $\lambda = \lambda(Q)$. The solution agrees with that obtained from the single-particle Green's function only if $Q = 0$. By using a more general Aronszajn–Donoghue spectral theory, it will be shown hereafter that the values of Q other than zero do not represent bound state solutions.

4. Nevanlinna function

In virtue of (3.1), equation (3.2) points to $\tilde{g}_\mu = \tilde{g}_\mu^0/(1 + \gamma \tilde{g}_\mu^0)$. In turn, the expression for $\tilde{g}_\mu$ alludes to a well-known Aronszajn–Krein formula for the symmetric rank one bounded perturbation of a self-adjoint operator in the Hilbert space. Without referring to the extension theory of symmetric operators, one can show [8, Theorem 1.1.1] that, if given a self-adjoint operator H, then the subtraction of the resolvents of the perturbed operator $H_\gamma := H + \gamma\langle\delta, \cdot\rangle\delta$ and that of H is proportional to $1/(1 + \gamma F_\mu)$, where $F_\mu := \langle\delta, (H - \mu)^{-1}\delta\rangle$ is known as the Borel transform of a measure, and it belongs to the Nevanlinna class (that is, the complex conjugate $\overline{F_\mu} = F_{\overline{\mu}}$ and the imaginary part Im $F_\mu/\text{Im}\,\mu \geq 0$, provided μ is in the resolvent set of H_γ). Here, δ is the Dirac distribution and $\langle\cdot, \cdot\rangle$ denotes the scalar product in a concrete Hilbert space.

Let us study the function F_μ, applied to our case, in a more detailed fashion. For this, let H be a free two-particle Rashba Hamiltonian in the Hilbert space $\mathfrak{X}_o$. H is parametrized in terms of the center-of-mass $\vec{R}$ and the relative $\vec{r}$ coordinates. Then the s-wave interaction can be estimated using the Dirac distribution δ at the relative coordinate $\vec{r} \in \mathrm{R}^2$ (the interaction strength is γ). As a result, the form sum H_γ describes the two-particle system with point-interaction. To see this explicitly, it suffices to note that for $f = f(\vec{R}, \vec{r})$ in the domain of H, it holds $\langle\delta, f\rangle\delta = f(\vec{R}, \vec{0})\delta = f\delta$. The Aronszajn–Krein formula reads [9, Eq. (11.13)]

$$(H_\gamma - \mu)^{-1} = (1 + \gamma F_\mu)^{-1}(H - \mu)^{-1}.$$ (4.1)

It takes little effort to verify that equation (4.1) eventually leads to the expression of the Green's function g_μ in (3.2) but with $\tilde{g}_\mu^0$ replaced by $\tilde{F}_\mu$, where $\tilde{F}_\mu$ denotes the Nevanlinna function F_μ projected onto the singlet basis. By inspection, the Fourier transform

$$((\widehat{H - \mu})^{-1}\delta)\left(\vec{Q}, \vec{k}\right) = (2\pi)^2 \delta\left(\vec{Q}\right) \hat{G}_\mu^0\left(\vec{0}, \vec{k}\right), \qquad (4.2)$$

where $\hat{G}_\mu^0$ is the Green's function of H in momentum representation. The projection of the Green's function onto the singlet basis is given by g_μ^0. By (4.2) and the fact that the norm of arbitrary $f \in \mathfrak{X}_o$ coincides with the norm of its Fourier transform, one easily deduces that the Nevanlinna function

$$\tilde{F}_\mu = \tilde{g}_\mu^0(0). \qquad (4.3)$$

Bringing together (3.2) and (4.3), one figures out that the zero-range interaction δ admits only the bound state that corresponds to $Q = 0$, while the free Green's function g_μ^0 (recall the definition in (3.1)) accepts all possible values $\vec{Q} \in \mathrm{R}^2$. The affirmation of the result for other types of spin-orbit coupling comes from the Nevanlinna function, for the reason that the Green's function was not specified when deriving (4.3).

The obtained result, (4.3), is not accidental, as it is closely related to the eigenvectors obeying the translation symmetry.

5. Eigenspace

The fact that the bound state of two interacting Rashba particles exists only for the zero-valued center-of-mass momentum can be affirmed by the commutation property of the Hamiltonian H_γ with the generator $\vec{P}$ of the translation group. The commutation relation designates the spectral points $\vec{\Lambda} \in \mathrm{R}^2$ of $\vec{P}$ to the eigenspace of H_γ. As a result, the eigenvectors of the Hamiltonian are of the form of (2.1). Since the Hamiltonian is defined as the operator in the Hilbert space $\mathfrak{X}_o$, it should be emphasized that in this case, equation (2.1) (as well as (2.2)) is understood in the generalized sense; by the density result [10], the Schwartz space can be extended to the appropriate domain of definition of H_γ.

Let Ω be the projection onto the singlet basis. In virtue of (2.1)–(2.2), vectors f in the eigenspace $\mathrm{Ker}(H_\gamma - \lambda)$ fulfill the eigenvalue equation

$$\hat{\varphi}\left(\vec{k}\right) + \gamma \hat{G}_\lambda^0\left(\vec{\Lambda}, \vec{k}\right) \int_{\mathrm{R}^2} \frac{d^2\vec{k}'}{(2\pi)^2} \Omega \hat{\varphi}\left(\vec{k}'\right) = 0. \qquad (5.1)$$

By the fact that φ is independent of $\vec{\Lambda}$, equation (5.1) yields $(\partial/\partial\vec{\Lambda})\hat{G}^0_\lambda = \vec{0}$. That is, $\hat{G}^0_\lambda(\vec{\Lambda}, \vec{k})$ is a constant w.r.t. $\vec{\Lambda} \in \mathrm{R}^2$. The only one nontrivial solution to the latter equation w.r.t. $\vec{\Lambda} \in \mathrm{R}^2$ reads $\Lambda = 0$. By (2.1), this implies that $f = \varphi$ or else, the space of translation invariant eigenvectors is reduced from $L_2(\mathrm{R}^4; \mathrm{C}^4)$ to $L_2(\mathrm{R}^2; \mathrm{C}^4)$. The cancellation of one spatial vector becomes clear recalling that $\mathfrak{X}_o$ is isomorphic to the tensor product of spaces $(\mathfrak{X}, d^2\vec{R}) \otimes (\mathfrak{X}, d^2\vec{r})$. The Hamiltonian in the Hilbert space $(\mathfrak{X}, d^2\vec{R})$ is parametrized in terms of the center-of-mass coordinate $\vec{R}$ and it does not have bound states, while the Hamiltonian in the space $(\mathfrak{X}, d^2\vec{r})$ is parametrized in terms of the relative coordinate $\vec{r}$ and it has a bound state due to the Dirac distribution at $\vec{r}$. This latter space is exactly the one where the eigenspace of the two-particle Rashba Hamiltonian is situated.

6. Conclusion and discussion

An attempt to clarify the issue regarding the bound state of spin-orbit coupled fermions was the primary intent of this note. The results of this paper clearly demonstrate that the eigenvector of the two-particle Rashba Hamiltonian with point-interaction δ is translation invariant. The associated bound state exists only for the zero-valued center-of-mass momentum Q.

The main conclusion of the present report is that, if the two-particle Hamiltonian H_γ, where γ is the strength of interaction, possesses translation symmetry, then for any two-particle interaction of rank one, the bound state exists only for the zero-valued center-of-mass momentum Q. The same applies to other types of spin-orbit coupling. On the other hand, if the Hamiltonian is not translation invariant, the conclusion does not necessarily hold. To explain this, let us give some examples.

The rank one perturbation to the free Hamiltonian H is characterized by the term $\gamma\langle\phi, \cdot\rangle\phi$, where ϕ need not be the unit vector in $\mathfrak{X}_o$; it even need not be in the Hilbert space [8, 9, 12]. In this case, the Nevanlinna function F_μ reads $\langle\phi, (H - \mu)^{-1}\phi\rangle$. For the proper potential (that is, the vector ϕ), one can possibly derive bound states with nonzero Q. For example, let ϕ be summable on rectangle $\mathrm{R}^2 \times \mathrm{R}^2$. Then an easy calculation shows that

$$((\widehat{H-\mu})^{-1}\phi)\left(\vec{Q}, \vec{k}\right) = \hat{\phi}\left(\vec{Q}, \vec{k}\right)\hat{G}^0_\mu\left(\vec{Q}, \vec{k}\right), \qquad (6.1)$$

where $\hat{\phi}$ is the Fourier transform of $\phi = \phi(\vec{R}, \vec{r})$. It is clear from (6.1) that whenever $\phi(\vec{R}, \vec{r}) = \phi(\vec{r})$, that is, when the two-particle Hamiltonian is invariant under translations, it always holds

$$((\widehat{H-\mu})^{-1}\phi)\left(\vec{Q}, \vec{k}\right) = (2\pi)^2\delta\left(\vec{Q}\right)\hat{\phi}\left(\vec{k}\right)\hat{G}^0_\mu\left(\vec{0}, \vec{k}\right) \qquad (6.2)$$

hence $Q = 0$, no matter what type of interaction between two particles is specified, *viz.* zero-range or short-range. In particular, choosing $\phi = \delta$, (6.2) formally coincides with (4.2). On the other hand, if, for example, given the vector $\phi(\vec{R}, \vec{r}) = \exp(i(\vec{a} \cdot \vec{R}))\delta(\vec{r})$ for some $\vec{a} \in \mathrm{R}^2$, one obtains from (6.1) that $\vec{Q} = \vec{a}$, and thus the bound state depends on the controllable parameter a; in this case, the two-particle Hamiltonian is not translation invariant and the (projected) Nevanlinna function $\tilde{F}_\mu = \tilde{g}_\mu^0(a)$.

REFERENCES

[1] J.P. Vyasanakere, V.B. Shenoy, *Phys. Rev.* **B83**, 094515 (2011).

[2] Z.-Q. Yu, H. Zhai, *Phys. Rev. Lett.* **107**, 195305 (2011).

[3] M. Iskin, A.L. Subaşi, *Phys. Rev.* **A84**, 043621 (2011).

[4] L. Jiang, X.-J. Liu, H. Hu, H. Pu., *Phys. Rev.* **A84**, 063618 (2011).

[5] X. Cui, *Phys. Rev.* **A85**, 022705 (2012).

[6] S. Takey, C.-H. Lin, B.M. Anderson, V. Galitski, *Phys. Rev.* **A85**, 023626 (2012).

[7] L. Dong, L. Jiang, H. Hu, H. Pu, *Phys. Rev.* **A87**, 043616 (2013).

[8] S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators*, Cambridge University Press, New York 2000.

[9] B. Simon, *Trace Ideals and Their Applications*, 2nd ed., American Mathematical Society, 2005.

[10] P.K. Bhattacharyya, *Distributions. Generalized Functions with Applications in Sobolev Spaces*, Walter de Gruyter GmbH, Berlin, Germany, 2012.

[11] J. Brüning, V. Geyler, K. Pankrashkin, *J. Phys. A: Math. Theor.* **40**, F697 (2007).

[12] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Springer Dordrecht, Heidelberg–New York–London 2012.