BOHM TRAJECTORIES AND FEYNMAN PATHS IN LIGHT OF QUANTUM ENTROPY

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(Received November 12, 2013)

A new definition of quantum entropy by a gauge constraint on a classical Boltzmann manifold is proposed. Bohm potential is derived as Fisher information, in accordance with Bohm–Hiley idea of "Active Information", and the geometries underlying Bohm trajectories and Feynman paths are compared. Given a quantum system, it is shown how the modifications of such geometries are connected to the microstates that quantum entropy provides.

DOI:10.5506/APhysPolB.45.885
PACS numbers: 03.65.–w, 03.65.Ud, 03.67.–a, 11.10.–z

1. Introduction

By starting from a geometrodynamic approach in which the entropy of a quantum system is expressed by a metric in a manifold of different Boltzmann entropies, Bohm's quantum potential (QP) appears under the constraint of a minimum condition of Fisher information as a non-Euclidean deformation of the entropic space. In this picture, the difference between classical and quantum information is similar to the difference between Euclidean and non-Euclidean geometry in the parameter space determined by the quantum entropy [1, 2].

On the other hand, starting from the work of Vigier there was recognized a strong formal analogy between the QP and the most radical and powerful tool in theoretical physics, the Feynman path integrals approach [3, 4]. In spite of the fact that the two approaches seem to originate from different visions, both the paths of Bohm and the Feynman ones underline the "non-mechanical" character of a quantum object, taking into account explicitly the non-locality which derives from the global "interweaving" of dynamic stories in a "context" or "environment" via superposition and entanglement. In other words, the concept of trajectory emerges only in appropriate conditions that can be strictly defined in terms of Quantum Field Theory. This is the conceptual limit of any "geometrodynamics" [5, 6]. The link between Bohm's quantum potential and Feynman's path integrals is explored here within a new formalism based on quantum entropy as deformation of the geometry of the configuration space.

The paper is structured as follows. In Section 2, after a brief review of the most important features of Bohm's quantum potential, we will analyse a definition of quantum entropy as a constraint on different Boltzmann entropies and derive the quantum potential as extremal of Fisher information on quantum entropy manifold. In Section 3, we will outline the connection between Bohmian paths and Feynman's path integral approach, in the light of some recent research. In Section 4, we will analyse the link between the quantum information as a deformation of the geometry and Bohm–Feynman's path integrals. Finally, in Section 5 we will derive a new minimum principle which allows us to obtain quantum mechanics.

2. Quantum potential as Fisher information in entropy space

The starting point to introduce the notion of geometric quantum information is the analysis of the general features of quantum potential, the non-local "brick" of the D. Bohm theory. As is known, in his classic works of 1952, Bohm showed that if one interprets each individual physical system as composed by a corpuscle and a wave guiding it, by writing its wave function in polar form and decomposing the Schrödinger equation, the movement of the corpuscle under the guide of the wave happens in agreement with a law of motion which assumes the following form

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0\,, \tag{1}$$

where R is the amplitude and S is the phase of the wave function, $\hbar$ is Planck's reduced constant, m is the mass of the particle and V is the classical potential. This equation is equal to the classical equation of Hamilton–

Jacobi, except for the appearance of the additional term

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \tag{2}$$

having the dimension of an energy and containing the Planck constant and, therefore, appropriately defined quantum potential. If we consider a wave function $\psi = R(\vec{x}_1, \ldots \vec{x}_N, t)e^{iS(\vec{x}_1,\ldots\vec{x}_N,t)/\hbar}$, defined on the configuration space R^{3N} of a system of N particles, inside Bohm's interpretation of non-relativistic quantum mechanics the movement of this system under the action of the wave ψ happens in agreement to the quantum Hamilton–Jacobi law of motion

$$\frac{\partial S}{\partial t} + \sum_{i=1}^{N} \frac{|\nabla_i S|^2}{2m_i} + Q + V = 0\,, \tag{3}$$

where

$$Q = \sum_{i=1}^{N} -\frac{\hbar^2}{2m_i}\frac{\nabla_i^2 R}{R} \tag{4}$$

is the many-body quantum potential [7–10]; for a general review, see [11].

For our purposes here, it will be sufficient to refer to the first Bohm, still influenced by the ideas of de Broglie and influenced by the attempt to describe the quantum phenomena in a spatio-temporal arena. Later, his ideas will go more and more in the direction of a topological non-locality described with algebraic methods in a pre-space [12, 13].

A fundamental trait which emerges from the formalism of non-relativistic de Broglie–Bohm theory is the non-locality. In the expression of the quantum potential, the appearance of the amplitude of the wave function in the denominator also explains why the quantum potential can produce strong long-range effects that do not necessarily fall off with distance and so the typical properties of entangled wave functions. Thus, even though the wave function spreads out, the effects of the quantum potential need not necessarily decrease. This is just the type of behaviour required to explain the EPR effect. In virtue of the quantum potential, Bohm's interpretation of quantum phenomena has the merit to include non-locality *ab initio* rather than to come upon it as an *a posteriori* statistical "mysterious weirdness" [1].

Moreover, the expression of the quantum potential implies that the quantum potential has a contextual nature, namely brings a global information on the process and its environment by individuating an infinite set of phase paths; and it is active information in the sense that it modifies the behaviour of the particle. In a double-slit experiment, if one of the two slits is closed, the quantum potential changes, and this information arrives instantaneously

to the particle, which behaves consequently. The active information of the quantum potential is deeply different from classical one: it is, in fact, intrinsically not-Shannon computable [14].

An essential key for the relations between quantum potential, system's geometry and information is provided by the Fisher information, which plays the role of a natural tile to build a metric able to connect the system's statistical outcomes and its global geometry. In the recent paper [2], the authors showed that the quantum potential expresses how the quantum effects deform the configuration space of processes in relation to the number of the microstates of the system under consideration. The quantum potential emerges as an information channel determined by the quantum entropy space given by the manifold of different Boltzmann entropies

$$\begin{cases} S_1 = k \log W_1(\theta_1, \theta_2, \dots \theta_p) \\ S_2 = k \log W_2(\theta_1, \theta_2, \dots \theta_p) \\ \qquad \dots \\ S_n = k \log W_n(\theta_1, \theta_2, \dots \theta_p) \end{cases}, \tag{5}$$

where W are the number of the microstates for the same parameters θ as temperatures, pressures, *etc.* ... and k is Boltzmann's constant. In this picture, quantum effects are equivalent to a geometry which is described by equation

$$\frac{\partial}{\partial x^k} + \frac{\partial^2 S_j}{\partial x^k \partial x^p} \frac{\partial x^i}{\partial S_j} = \frac{\partial}{\partial x^k} + \frac{\partial \log W_j}{\partial x_h} = \frac{\partial}{\partial x^k} + B_{j,h}\,, \tag{6}$$

where

$$B_{j,h} = \frac{\partial S_j}{\partial x_h} = \frac{\partial \log W_j}{\partial x_h} \tag{7}$$

is a Weyl-like gauge potential [15–17].

The change of the geometry corresponds with a non-local deformation of the moments stated by the quantum action

$$A = \int \rho \left[\frac{\partial A}{\partial t} + \frac{1}{2m} p_i p_j + V + \frac{1}{2m} \left(\frac{\partial \log W_k}{\partial x_i} \frac{\partial \log W_k}{\partial x_j} \right) \right] dt d^n x\,. \tag{8}$$

The quantum action assumes the minimum value when

$$\delta \int \rho \left[\frac{\partial A}{\partial t} + \frac{1}{2m} p_i p_j + V \right] dt d^n x + \delta \int \frac{\rho}{2m} \frac{\partial \log W_k}{\partial x_i} \frac{\partial \log W_k}{\partial x_j} dt d^n x = 0 \tag{9}$$

and thus

$$\frac{\partial A}{\partial t}+\frac{1}{2m}p_i p_j+V+\frac{1}{2m}\left(\frac{1}{W_k^2}\frac{\partial W_k}{\partial x_i}\frac{\partial W_k}{\partial x_j}\right)=\frac{\partial S_k}{\partial t}+\frac{1}{2m}p_i p_j+V+Q\,, \quad (10)$$

where Q is the Bohm quantum potential derived from Fisher information as extremal, that is a measure of distance in the entropy space [1].

On the basis of the formalism described by equations (5)–(10), it becomes permissible the following re-reading of the mathematical formalism in non-relativistic Bohmian quantum mechanics: the distribution probability of the wave function determines the functions W_k defining the number of microstates of the physical system under consideration, a quantum entropy is fixed by these functions W_k given by equations (5), by determining a change of the geometry — with respect to the Euclidean space of classical physics — expressed by the Weyl-like gauge potential (7) and characterized by the deformation of the moments (8). Moreover, on the basis of equation (10), one can interpret Bohm's quantum potential as an information channel determined by the functions W_k given by equations (5): these functions W_k, and therefore the quantum entropy determine the action of the quantum potential (in the extreme condition of the Fisher information) on the basis of equation[1]

$$Q=\frac{1}{2mW^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}-\frac{1}{mW}\frac{\partial^2 W}{\partial x_i \partial x_j}\,. \quad (11)$$

In this way, under the constraint of the extremal condition of Fisher information, the functions W are "informational lines" of Bohm's quantum potential in the non-Euclidean space of the entropies. In other words, each of the Boltzmann entropies appearing in the superposition vector (5) can be considered as a specific information channel of a quantum potential which describes the deformation of the geometry in the presence of quantum effects.

In the recent article, Sbitnev [18] showed that the quantum potential can be interpreted as an information channel into the movement of the particles as a consequence of the fact that it determines two quantum correctors into the energy of the particle depending on the density of the ensemble of particles associated with the wave function under consideration. Here, in analogy with Sbitnev's treatment, if we substitute equation (11) into equation (1) the quantum Hamilton–Jacobi equation of motion for the corpuscle associated with the wave function $\psi(\vec{x},t)$ becomes

$$\frac{|\nabla S|^2}{2m}+\frac{1}{2mW^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}+V-\frac{1}{mW}\frac{\partial^2 W}{\partial x_i \partial x_j}=-\frac{\partial S}{\partial t} \quad (12)$$

[1] In the next equations, for simplicity, we are going to denote the generic function (defined by equations (5)) with W.

which provides a new way to read the energy conservation law in quantum mechanics. In equation (12) two quantum corrector terms appear in the energy of the system, which are owed to the functions W linked with the quantum entropy. On the basis of equation (12), we can say that, in the approach here suggested, the deformation of the geometry determined by the functions W (and thus by the quantum entropy) produces two quantum corrector terms in the energy of the system. More precisely, the energy conservation law (12) suggests that the term $\frac{1}{2mW^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}$ can be interpreted as the quantum corrector of the kinetic energy $\frac{|\nabla S|^2}{2m}$ of the particle, while the term $-\frac{1}{mW}\frac{\partial^2 W}{\partial x_i \partial x_j}$ can be interpreted as the quantum corrector of the potential energy V. These two quantum correctors $-\frac{1}{mW}\frac{\partial^2 W}{\partial x_i \partial x_j}$ and $\frac{1}{2mW^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}$ are determined just by the functions W which are linked with the quantum entropy and describe the deformation of the geometry in the presence of quantum effects.

In Sbitnev's approach, by starting from the quantum entropy

$$S_{\mathrm{Q}} = -\tfrac{1}{2}\ln\rho \tag{13}$$

introduced as the degree of order and chaos of the vacuum supporting the density of the ensemble of particles associated with the wave function, the quantum potential is an information channel into the behaviour of the particles given by relation

$$Q = -\frac{\hbar^2}{2m}\left(\nabla S_{\mathrm{Q}}\right)^2 + \frac{\hbar^2}{2m}\left(\nabla^2 S_{\mathrm{Q}}\right) . \tag{14}$$

As a consequence, by equating equations (14) and (11), one obtains

$$\frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) = -\frac{\hbar^2}{2m}(\nabla S_{\mathrm{Q}})^2 + \frac{\hbar^2}{2m}\left(\nabla^2 S_{\mathrm{Q}}\right) . \tag{15}$$

Equation (15) allows us to show that the two quantum corrector terms of the energy of the particle $-\frac{\hbar^2}{2m}(\nabla S_{\mathrm{Q}})^2$ and $\frac{\hbar^2}{2m}(\nabla^2 S_{\mathrm{Q}})$ of Sbitnev's approach to quantum entropy correspond to two terms depending on the functions W defining the number of the microstates of the physical system, namely $\frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)$ and $\left(-\frac{1}{mW}\frac{\partial^2 W}{\partial x_i \partial x_j}\right)$ respectively

$$\frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right) = -\frac{\hbar^2}{2m}(\nabla S_{\mathrm{Q}})^2 , \tag{16}$$

$$\left(-\frac{1}{mW}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) = \frac{\hbar^2}{2m}\left(\nabla^2 S_{\mathrm{Q}}\right) . \tag{17}$$

Equations (16) and (17) state that Sbitnev's entropy expresses the degree of order and chaos of the vacuum supporting the density of the ensemble of particles associated with the wave function, in the sense that it emerges from the functions W defining the number of microstates of the physical system under consideration.

On the other hand, in analogy with Sbitnev's model, the continuity equation of Bohmian mechanics may be written as

$$\frac{\partial S_{\mathrm{Q}}}{\partial t} = -(\vec{v} \cdot \nabla S_{\mathrm{Q}}) + \frac{1}{2} \nabla \cdot \vec{v}\,, \tag{18}$$

where $\vec{v} = \frac{\nabla S}{m}$ is the particle's speed and Sbitnev's entropy S_{Q} is determined by the functions W on the basis of equations (16) and (17). In equation (18), the second term at the right-hand side describes the rate of the entropy flow due to spatial divergence of the speed. Equation (18) can be interpreted as a law which describes balance of the information flows associated with Sbitnev's entropy and thus with the functions W.

According to the approach here suggested, in non-relativistic Bohmian mechanics the geometrodynamic nature of the quantum potential (namely the fact that it has a geometric nature, contains a global information on the environment in which the experiment is performed, and at the same time it is a dynamical entity, namely its information about the process and the environment is active) is just determined by the deformation of the geometry determined in turn by the functions W (and thus by the quantum entropy (5)) on the basis of equations (11) and (12), while equation (18) indicates the information flows associated with these functions. It is also interesting to observe that, in this approach, the inverse square root of the quantity

$$L_{\mathrm{quantum}} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}} \tag{19}$$

defines a typical quantum-entropic length that can be used to evaluate the strength of quantum effects and, therefore, the modification of the geometry with respect to the Euclidean geometry characteristic of classical physics. Once the quantum-entropic length becomes non-negligible, the system goes into a quantum regime. In this picture, Heisenberg's uncertainty principle derives from the fact that we are unable to perform classical measurements to distances smaller than the quantum-entropic length. In other words, the size of a measurement has to be bigger than the quantum-entropic length

$$\Delta L \geq L_{\mathrm{quantum}} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}}\,. \tag{20}$$

The quantum regime is entered when the quantum-entropic length must be taken under consideration.

In synthesis we can say that by Fisher it is possible to characterize the deformations of the geometry in the presence of quantum effects in the space of parameters and to express Bohm's quantum potential as the information channel indicating the modification of the geometry of the configuration space determined by the quantum entropy. Novello, Salim and Falciano [19] have recently proposed a geometrical approach in which the presence of quantum effects is linked with the Weyl length $L_{\mathrm{W}} = \frac{1}{\sqrt{R}}$, and thus with the curvature scalar; in analogous way, in the approach proposed by the authors and illustrated in this section (see also [2]), the quantum effects are owed to the microstates characterizing the system under consideration and thus to the quantum entropy (5). Both the Novello, Salim and Falciano's and our approach are realistic models that aim to provide a geometrical framework to quantum mechanics in a Bohmian picture, the one in the context of Weyl integrable space, the other in the context of an entropic background in the condition of Fisher information. In analogy with Novello's, Salim's and Falciano's approach that implies that the quantum effects are the manifestations of the modification of the geometry from Euclidean to a non-Euclidean Weyl integrable space, inside the entropic approach to Bohm's quantum mechanics in the condition of Fisher information, the presence of the quantum effects is linked with a change in the geometry which is determined by the quantum entropy and, therefore, the length that can be used to evaluate the strength of quantum effects is linked with the quantum entropy.

3. About de Broglie–Bohm path integrals and Feynman path integrals

In quantum mechanics the fundamental problem regarding the propagation of the wave function can be resolved by constructing the so-called quantum mechanical path integrals. As is known, Feynman's approach to quantum mechanics uses the concept of path as a mathematical tool to calculate a propagator for the wave function

$$K(x,t,x_0,t_0) = \int D[x] e^{\frac{i}{\hbar}S[x]}\,, \tag{21}$$

where S is the classical action and the integral is taken over all possible paths between the two points $(x;t)$ and $(x_0;t_0)$. One can show that equation (21) satisfies the Schrödinger equation and can be used as a substitute for it. This means that, if we have an arbitrary initial wave function, we can obtain it at any other time by (21). The Feynman path integral is conventionally understood as a sum over all (infinite) possible paths connecting

the points $(x;t)$ and $(x_0;t_0)$, each of these contributing with an amplitude found by integrating the classical Lagrangian. However, these paths are understood not be "trajectories", in fact, in the standard interpretation of quantum mechanics such a concept does not even exist. Instead, inside the interpretative universe of the de Broglie–Bohm Theory there is a spectrum of more varied positions, from post-deterministic "realism" to "softer" positions [20–23]. Among the archipelago of "ontologies", we prefer to follow the royal road of mathematics.

As regards path integrals and theories of actual particle trajectories there is, moreover, another important (and completely different) relation: since one considers a lot of possible paths of the particle, it is perhaps one of these paths that the particle actually follows? If the Feynman path integral formalism provided a probability distribution on the space of all paths, one could assume that nature chooses one of the paths at random according to this distribution. In this way, Feynman's path integral approach would have the same ontology as Bohmian mechanics, but would be based on a stochastic law of motion. As we said, these old "ontological" issues are waiting to be reformulated in terms of QFT, and a less naive "realism". The essential point is that there are already several formalisms able to establish a "bridge" between the two theories [10, 23–25]. We will limit ourselves here to illustrate a couple of examples from Bohm toward Feynman.

In their paper Abolhasani and Golshani [26] showed that the propagation of the wave function in the context of de Broglie–Bohm theory for the one-body system can be achieved by means of a Bohmian path integral which, for two points $(x;t)$ and $(x_0;t_0)$ with a finite distance on a Bohmian path, is defined by the relation

$$\psi(\vec{x},t)=\exp\left\{\frac{i}{\hbar}\int\limits_{x_0,t_0}^{x,t}\left[\frac{(\nabla S)^2}{2m}-(Q+V)\right]dt-\int\limits_{x_0,t_0}^{x,t}\left(\frac{\nabla^2 S}{2m}\right)dt\right\}\psi(\vec{x}_0,t_0)\,, \tag{22}$$

where the first exponential can be obtained by integrating the quantum Hamilton–Jacobi equation (1) on the Bohmian path, while the second exponential can be obtained by integrating the continuity equation (derived from the imaginary part of Schrödinger equation) [21]. As one may expect, here the classical action of the Feynman path integral (21) is replaced by the quantum action (which is linked with the quantum potential Q given by equation (2). The Bohmian path integral given by equation (21) can also be extended to the case of a system of N particles in the following way

$$\psi\left(\vec{x}_1,\ldots\vec{x}_N,t\right) = \exp\left\{\frac{i}{\hbar}\int\limits_{x_0,t_0}^{x,t}\sum_{i=1}^{N}\left[\frac{(\nabla_i S)^2}{2m_i}-(Q+V)\right]dt - \int\limits_{x_0,t_0}^{x,t}\sum_{i=1}^{N}\left(\frac{\nabla_i^2 S}{2m_i}\right)dt\right\}\psi\left(\vec{x}_{01},\ldots\vec{x}_{0N},t_0\right) \quad (23)$$

with the quantum potential Q given by equation (4).

Abolhasani and Golshani have then determined $\psi(\vec{x},t)$ for a free-wave packet in one dimension in terms of its Fourier components e^{ikx} (for which $\nabla^2 S=0$ and $Q=0$). In this case, the Bohmian path integral (22) becomes

$$\psi(x,t)=\int dk e^{-i\frac{\hbar k^2}{2m}(t-t_0)}\varphi(k)e^{ikx} \quad (24)$$

(which can be called Fourier–Bohm path integrals) and the Feynman path integral becomes

$$K(x,t,x_0,t_0)=\sqrt{\frac{m}{2\pi i\hbar(t-t_0)}}\exp\left\{\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right\}. \quad (25)$$

In both, Feynman and Fourier–Bohm path integrals, infinite paths have origin from each point of space at time t_0 and these paths have all possible gradients. It is important to underline that in Abolhasani's and Golshani's treatment, the infinity of Fourier–Bohm paths is of the same order of the infinity of Feynman paths. The infinite Bohmian paths which appear in the Fourier–Bohm path integral are equivalent with the infinite Feynman paths which appear in the Feynman path integral. The only difference is that, in the Bohmian path integrals each path which initiates at x_0, with the gradient $v_i=\frac{k_i\hbar}{m}$, is multiplied by one component of $\psi_0(x_0)$, *i.e.* $\varphi(k_i)e^{ik_ix}$. Instead, in Feynman path integral the same path is multiplied by all components of $\psi_0(x_0)$, *i.e.* $\sum_i\varphi(k_i)e^{ik_ix}$.

In Abolhasani and Golshani's treatment, both the Bohmian path integral (22) and the Feynman path integral (21) lead to the same result as regards the evolution of the wave function for a free wave packet. In particular, in this model, Feynman path integral (21) can be obtained directly from the Bohmian path integral (22) on the basis of an "heuristic argument". The equivalence of the Bohmian path integral (22) and the Feynman path integral (21) for the free wave packet can also be seen as a consequence of the fact that in this simple case the quantum potential vanish.

Recently, Oltean showed that the Feynman path integral (21) can be obtained from the Bohmian path integral inside a bit more general, careful and rigorous context [27]. Oltean demonstrated that the Bohmian path integral (22) derives from a combination of two opportune lemmas regarding the propagation of the wave function modulus and the propagation of the phase of the wave function respectively, and thus by integrating the quantum Lagrangian $L_{\mathrm{Q}} = \sum_{i=1}^{N} \left[\frac{(\nabla_i S)^2}{2m_i} - Q - V \right]$ along one single path *i.e.* the particle's Bohmian trajectory. More precisely, the two lemmas are the following:

— The wave function modulus is propagated according to

$$\psi(x,t) = \left[\exp\left(-\frac{1}{2} \sum_{j=1}^{N} \int_{t_0}^{t} \nabla \cdot \vec{v}_j^{\psi} dt \right) \right] \psi(x_0, t_0)\,,$$

where $\vec{v}_j^{\psi} = \frac{\nabla_j S}{m_j}$ is the velocity of the generic particle of the system.

— The phase function is propagated according to $S(x,t) = S(x_0,t_0) + \int_{t_0}^{t} L_{\mathrm{Q}} dt$ where $L_{\mathrm{Q}} = \sum_{i=1}^{N} \left[\frac{(\nabla_i S)^2}{2m_i} - Q - V \right]$ is the quantum Lagrangian.

Then, on the basis of Oltean's results, the Feynman method of summing over all paths can be constructed by starting from the de Broglie–Bohm theory. By utilizing the propagator $K(x,t,x_0,t_0)$ of the wave function (which is defined such that $\psi(\vec{x},t) = \int_{R^3} K(\vec{x},t,\vec{x}_0,t_0)\psi(\vec{x},t)d^3\vec{x}$) and the free particle wave function $\psi(\vec{x},t) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{R^3} \exp\left(\frac{i}{\hbar}\vec{p}\cdot\vec{x}\right) \widehat{\psi}(\vec{p},t) d^3\vec{p}$ (where $\widehat{\psi}(\vec{p},t)$ is the Fourier transform of the wave function), the Feynman path integral can be obtained which, for the one-body system, is given by relation

$$K(\vec{x},t,\vec{x}_0,t_0) = \lim_{n\to\infty} \int \dots \int_{R^{3(n-1)}} \left(\frac{m}{2\pi i\hbar\Delta t} \right)^{\frac{3}{2}n} \exp\left(\frac{i}{\hbar} \int_{t_0}^{t} L_{\mathrm{C}} dt \right) \prod_{k=1}^{n-1} d^3\vec{r}_k\,, \tag{26}$$

where the classical Lagrangian is given by $L_{\mathrm{C}} = \sum_{i=1}^{N} \left[\frac{(\nabla_i S)^2}{2m_i} - V \right]$.

According to the research of Abolhasani and Golshani and of Oltean, we can conclude that, despite their different conceptual starting-points, there is a significant relation between Bohmian path integral and Feynman path integral approach (in particular, in the special case of the free particle wave function, for which the quantum potential is null). In the next section, we will demonstrate the link between the concept of quantum information by Fisher entropy developed in Section 2 and Feynman's path integral approach.

4. The link between quantum entropy and Bohm–Feynman path integrals

Starting from the expression (11) of the quantum potential in terms of the functions W, it is possible to introduce interesting perspectives about the relation between the quantum information associated with the quantum entropy space (5) and Feynman's path integrals. The introduction of the quantum entropy given by equations (5) that determines the behaviour of quantum particles lead to two equations of motion, the energy conservation law (12) and the entropy balance equation (13), which introduce a new suggestive way to read the Bohmian mechanics and suggest a link with Feynman's path integrals.

By taking into account the entropic definition (11) of the quantum potential, Abolhasani and Golshani's Bohmian path integral (22) can be rewritten in the following form

$$\psi(\vec{x},t) = \exp\left\{\frac{i}{\hbar}\int_{x_0,t_0}^{x,t}\left[\frac{(\nabla S)^2}{2m} - \frac{1}{2mW^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right]dt - \int_{x_0,t_0}^{x,t}\left(\frac{\nabla^2 S}{2m} - \frac{1}{mW}\frac{\partial^2 W}{\partial x_i \partial x_j} + V\right)dt\right\}\psi(\vec{x}_0,t_0)\,. \tag{27}$$

Equation (27) indicates clearly that functions W (and therefore the quantum entropy given by equation (5)) are associated with a Bohmian path by determining appropriate corrective terms into the kinetic energy and the potential energy (and thus into the Lagrangian) of the particle under consideration. Moreover, in the picture provided by Oltean, the following re-reading of the Bohmian path integral formalism becomes permissible: the Bohmian path integral (27) derives from the integration of a quantum Lagrangian determined by the kinetic energy, the potential energy and the quantum entropy of the particle, namely the quantum entropy producing a change of the geometry expressed by a Weyl-like gauge potential introduces a modification of the Lagrangian and thus determines a particular Bohmian trajectory associated with the Bohmian path integral (27). In other words, on the basis of equation (22), one can say that Bohmian trajectories associated with Bohmian path integrals are indeed determined by the fact that the functions W (and thus the quantum entropy) produce a modification in the energy of the system under consideration. The functions W indicating the microstates of the system under consideration change whether the energetic-informational picture and hence Feynman's integrals emerge.

As regards the entropic definition (11) of the quantum potential, one can show that by means of an opportune unification of the quantum Hamilton–Jacobi equation (12) and the entropy balance equation (18), a complexified Hamilton–Jacobi equation containing complex kinetic and potential terms can be obtained. Moreover, it can show that the two quantum correction terms of kinetic energy and potential energy both depending on the functions W and thus on the quantum entropy are the fundamental terms that modify the Feynman's path integral by expanding coordinates and momenta to an imaginary sector.

In this regard, by multiplying equation (18) for $-i\hbar$ and introducing this result into equation (12), this latest equation becomes

$$\begin{aligned}&\frac{|\nabla S|^2}{2m} + i\hbar\frac{1}{m}[\nabla S\cdot\nabla S_{\mathrm{Q}}] + \frac{1}{2mW^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\\ &+V - i\hbar\frac{1}{2}\nabla\vec{v} - \frac{1}{mW}\frac{\partial^2 W}{\partial x_i\partial x_j} = -\frac{\partial S}{\partial t}\,.\end{aligned} \tag{28}$$

Here, in analogy with Sbitnev's treatment in paper [18], we write the first three terms in (28) as gradient of a complexified action squared

$$\frac{|\nabla S|^2}{2m} + i\hbar\frac{1}{m}[\nabla S\cdot\nabla S_{\mathrm{Q}}] + \frac{1}{2mW^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} = \frac{1}{2m}(\nabla J)^2 \tag{29}$$

while as regards the other three terms of (28), we use an expansion into the Taylor's series of the potential energy extended in the complex space which is to say a small broadening into the imaginary sector (as regards the complex extension, see, for example, the reference [28])

$$V\left(\vec{x}+i\vec{\varepsilon}\right) \approx V(\vec{x}) + i\hbar\left(\vec{n}\cdot\left(\frac{s}{2m}\nabla V(\vec{x})\right)\right) - \frac{\hbar^2}{2m}\left(\frac{s^2}{2m}\nabla^2 V(\vec{x})\right) + \ldots\,, \tag{30}$$

where $\vec{\varepsilon} = \frac{\hbar}{2m}s\vec{n}$ is a small vector having the dimension of length and s is the universal constant, the reverse velocity $s = 4\pi\varepsilon_0\frac{\hbar}{e^2} = 4,57\times10^{-7}$ [s/m], e is the elementary charge carried by a single electron, ε_0 is the vacuum permittivity. As regards the second term of equation (30), we have a force $-\nabla V(\vec{x})$ multiplied by a vector $l\vec{n}$ providing thus an elementary work performed by this force at shifting on a length l along $\vec{n}$. The force multiplied by the factor $l\vec{n}$ and divided into mass m is a rate of velocity's variation per unit length, *i.e.*, it represents divergence of the velocity. So, the second term of equation (30) can be rewritten in the following form

$$\frac{s}{2m}\left(\vec{n}\cdot\nabla V\left(\vec{x}\right)\right) = \frac{1}{2}\left(\nabla\cdot\vec{v}\right)\,. \tag{31}$$

As regards the third term $\frac{s^2}{2m}\nabla^2 V(\vec{x})$, it can be made the position

$$-\frac{s^2}{2m}\nabla^2 V(\vec{x}) = -\frac{1}{\hbar^2 W}\frac{\partial^2 W}{\partial x_i \partial x_j}\,. \tag{32}$$

Now, by defining the complexified momentum $\vec{p}' = \nabla J = \nabla S + i\hbar\nabla S_{\mathrm{Q}}$ and the complexified coordinates $\vec{x}' = \vec{x} + i\vec{\varepsilon}$, equation (28) can be rewritten as a complexified Hamilton–Jacobi equation

$$-\frac{\partial J}{\partial t} = \frac{1}{2m}(\nabla J)^2 + V\left(\vec{x}'\right) = H(\vec{x}', \vec{p}', t)\,, \tag{33}$$

where $H(\vec{x}', \vec{p}', t)$ on the right-hand side is the complexified Hamiltonian. The total derivative of the complex action gives the following equation

$$\frac{dJ}{dt} = -H(\vec{x}', \vec{p}', t) + \sum_{i=1}^{n} p_i' \dot{x}_i' = L\left(\vec{x}', \dot{\vec{x}}', t\right)\,. \tag{34}$$

By integrating equations (27) and (28) we obtain the solutions

$$J = -\int_{t_0}^{t} H(\vec{x}', \vec{p}', \tau)d\tau + C_1\,, \tag{35}$$

$$J = -\int_{t_0}^{t} L\left(\vec{x}', \dot{\vec{x}}', \tau\right)d\tau + C_2\,, \tag{36}$$

where C_1 and C_2 are two integration constants that satisfy the following condition

$$C_1 - C_2 = \int_{t_0}^{t}\sum_{i=1}^{n} p_i' \dot{x}_i' dt = \int_{L}\sum_{i=1}^{n} p_i' dx_i'\,. \tag{37}$$

In equation (37), L is a curve beginning at t_0 and terminating at t.

In analogy with the Sbitnev paper, the complexified state space (associated with the complexified momenta $\vec{p}' = \nabla J = \nabla S + i\hbar\nabla S_{\mathrm{Q}}$ and described by equations (33), (34), (35), (36) and (37)) can be considered as the fundamental stage which determines the features of Bohmian trajectories: Bohmian trajectories are submitted to the principle of least action that expands on the action integral (36) containing the complexified Lagrangian function. Bohmian trajectories turn out to be geodesic trajectories of an incompressible fluid loaded by the complexified Lagrangian that,

in turn, is determined by the quantum potential expressed by equation (11), in other words by the two quantum correctors linked with the functions W and thus with the quantum entropy.

In the complexified state space defined by equations (33)–(37), a solution of the Schrödinger equation can be written as

$$\psi\left(\vec{x}', \vec{p}', t\right) = \exp\left(\frac{i}{\hbar} J\right) . \tag{38}$$

By substituting the action integral (35) into equation (38), we obtain

$$\psi\left(\vec{x}', \vec{p}', t\right) = \frac{1}{Z_1} \exp\left\{-\frac{i}{\hbar} \int_{t_0}^{t} H\left(\vec{x}', \vec{p}', \tau\right) d\tau\right\} , \tag{39}$$

where $Z_1 = \exp(-\frac{i}{\hbar} C_1)$.

In this complexified state space, the Hamilton's principle $\delta J = 0$ states that the motion of an arbitrary mechanical system occurs in such a way that the definite integral (36) becomes stationary for arbitrary possible variations of the configuration of the system, provided the initial and final configurations of the system are prescribed. This principle can also be reformulated with respect to the wave function expressed in terms of the complexified action

$$\psi\left(\vec{x}', \vec{p}', t\right) = \frac{1}{Z_2} \exp\left\{-\frac{i}{\hbar} \int_{t_0}^{t} L\left(\vec{x}', \dot{\vec{x}}', \tau\right) d\tau\right\} , \tag{40}$$

where $Z_2 = \exp(-\frac{i}{\hbar} C_2)$. In this case, the principle states: this exponent becomes stationary for arbitrary possible variations of the configuration of the system, provided the initial and final configurations of the system are prescribed. Obviously, it results from stationarity of the integral (36) stated above.

Feynman's path integral approach based on the superposition principle is that all arbitrary trajectories are accepted as possible histories of the evolving quantum system. Contributions of most paths to the integral (36) will cancel each other, unless these paths are somehow "close" to the solution of $\delta J = 0$, which is the "real" path of the system. In the semiclassical region the propagator will, therefore, be dominated by those paths which are in the immediate vicinity of the classical path; the size of this vicinity follows from the estimate $\delta J \approx \hbar$. As shown in Grosche and Steiner [29], Feynman's path integral can be written mathematically in the following way inside the

complexified state space

$$K\left(\vec{x}',t;\vec{x}_0',t_0\right)=\int\int\ldots\int D\left[\vec{x}'(\tau)\right]\exp\left\{\frac{i}{\hbar}\int_{t_0}^{t}L\left(\vec{x}',\dot{\vec{x}}',\tau\right)d\tau\right\}, \quad (41)$$

where the path-integral symbol indicates the multiple integral

$$\int\int\ldots\int D\left[\vec{x}'(\tau)\right]\Leftrightarrow\left(\frac{2\pi i\hbar\delta t}{m}\right)^{-M/2}\int_{x_0'}^{x'}d\vec{x}_1'\int_{x_0'}^{x'}d\vec{x}_2'\ldots\int_{x_0'}^{x'}d\vec{x}_M'\,. \quad (42)$$

The fundamental principle of quantum mechanics, principle of superposition, underlies the path integral (41). Whereas evolution of a classical object is described by a unique trajectory satisfying the principle of least action, the path integral tests all possible virtual classical trajectories, among which there is a unique trajectory satisfying the least action principle. Other trajectories cancel each other by their interference.

Feynman's path integral represented in the product form, like in equation (21) or in equation (41), is a collection of the integrals of Fresnel type which are generally oscillatory [30, 31]. A trick suggested by Feynman was to add a negative imaginary part to Planck constant. This converts the oscillatory integrals into the Gaussian integrals and makes the path integral convergent. Another, more generally applicable trick is to assume that each element of the diagonal mass matrix has a positive imaginary part. Under this assumption, the path integral can be convergent independently of the metric of space.

On the basis of this treatment, the relation between the geometric quantum information (whose starting-point is the quantum entropy (11)) illustrated in Section 3 and Feynman's path integral approach based on equations (41) and (42) seems simple and natural inside the complexified state space characterized by the complexified momenta $\vec{p}'=\nabla J=\nabla S+i\hbar\nabla S_{\mathrm{Q}}$ and by the complexified coordinates $\vec{x}'=\vec{x}+i\vec{\varepsilon}$: the key of reading is provided by the two Bohmian quantum correctors linked with the functions W and thus with the quantum entropy. The path integral computation stems directly from decomposition of the Schrödinger equation to the modified quantum Hamilton–Jacobi equation (12) plus the entropy balance equation (18). The two Bohmian quantum correctors linked with the quantum entropy resulted from this decomposition expand the state space to the imaginary sector. In turn, imaginary terms emergent in this computations suppress the wilder contributions to the path integral. Thus, one obtains a non-trivial N-dimensional manifold embedded in the $2N$-dimensional complex state

space where its real part is the conventional coordinate state space. The two Bohmian quantum correctors in the energy of the system determined by the functions W and thus by the quantum entropy emerge as indispensable terms that modify the Feynman's path integral by expanding coordinates and momenta to the imaginary sector.

5. The principle of least action in the complexified space

On the basis of the mathematical formalism of the complexified space, a minimum principle for the complexified action may be introduced which allows us to derive non-relativistic quantum mechanics. This minimum principle is the Hamilton's principle for the complexified action $\delta J = 0$ which states that the motion of an arbitrary mechanical system occurs in such a way that the integral (36) becomes stationary for arbitrary possible variations of the configuration of the system, provided the initial and final configurations of the system are prescribed. In this section, we want to show that this minimum principle can reproduce and derive non-relativistic quantum mechanics in the sense that it is equivalent to the Schrödinger equation (and this would open the perspective to interpret the complexified space described by this principle of least action for the complexified action and associated with the quantum entropy as the fundamental key for non-relativistic quantum mechanics).

Hamilton's principle of least action is a fundamental principle of physics that is used to obtain dynamical equations for both non-relativistic and relativistic particles and fields (two classics are [32, 33]). As regards the derivation of Schrödinger equation from Hamilton's principle of least action, recently Kobe proposed an approach based on a Lagrangian density involving second-order derivatives of the wave function [34]. Kobe's approach is different from the Feynman's path integral approach that uses a classical particle Lagrangian in the path formulation to obtain the propagator. This approach gives a unified treatment for both non-relativistic quantum theory and relativistic quantum field theory, as well as showing their unity with other branches of physics. Moreover, Kobe has shown that for time-independent quantum systems, the principle of least action reduces to the energy variational principle of non-relativistic quantum mechanics.

Here, we apply Hamilton's principle of least action to non-relativistic quantum mechanics by considering the wave function in the complexified space as a generalized coordinate and constructing a Lagrangian density such that Hamilton's principle gives the Schrödinger equation. Differently from Kobe's picture, the approach of the complexified space developed in Section 4 allows us, to be coherent and compatible with Feynman's path integral approach. In this treatment, in order to use Hamilton's principle

of least action to obtain the Schrödinger equation, we need a general Lagrangian density characterizing the complexified space. Such a Lagrangian density can be constructed from the wave function (40) of the complexified space, its complex conjugate, and their partial derivatives to any order [34]. Because it involves the wave function, the Lagrangian density emphasizes the wave aspect of quantum theory.

We postulate a simple Lagrangian density L for this system in terms of the non-relativistic wave function as

$$L = \psi^* \left(i\hbar \frac{\partial}{\partial t} - H \right) \psi \tag{43}$$

which depends only on ψ^*, ψ and its partial derivatives. Any Lagrangian density that gives the correct equation of motion is a valid one, so a Lagrangian density that is complex can still be useful [35]. To obtain the dynamical equations from a Lagrangian density, we use Hamilton's principle of least action. Hamilton's principle states the action functional $J[\psi^*, \psi]$ for any Lagrangian density (43) is stationary

$$J[\psi^*, \psi] = \int dt \int d^3 r L(\psi, \psi^*, \dots) = \text{stationary}\,, \tag{44}$$

where integration is over all time and all complexified space. The action (44) is stationary when its variation with respect to ψ^* or ψ (or both) is zero. When the specific Lagrangian density (38) is substituted into equation (44) and variation is made with respect to ψ^* we obtain

$$\delta J[\psi^*, \psi] = \int dt \int d^3 r \delta\psi^* \left(i\hbar \frac{\partial}{\partial t} - H \right) \psi = 0\,. \tag{45}$$

Since the variation $\delta\psi^*$ is arbitrary except for vanishing at the boundaries, equation (45) leads immediately to the Schrödinger equation

$$H\psi = i\hbar \frac{\partial \psi}{\partial t}\,, \tag{46}$$

where the wave function $\psi(0)$ must be specified. If the variation of the action is made with respect to ψ, then integration by parts is needed and the complex conjugate of the Schrödinger equation is obtained.

6. Conclusions

In this article, a geometric approach to quantum information based on a new vision of quantum entropy has been explored and it has been analysed connection between the Bohm potential and Feynman integrals. The Quantum Information carried by Bohm–Feynman paths can be interpreted as a measure of the deformation of the geometry associated with the quantum entropy space. In this way, the idea that Born rule indicates a sort of Quantum Equilibrium [36, 37] is supported. The emergent geometries of Bohm–Feynman paths are determined by two quantum corrector terms in the energy of the system. Finally, all that confirms the contextuality of QM, its "open" nature on QFT and reproposes the possibility to put the question of the formal analogies between Bohm trajectories and Feynman integrals in terms of energetic relations of "doubling" [5, 6] between system and environment in quantum vacuum.

REFERENCES

[1] G. Resconi, I. Licata, D. Fiscaletti, *Entropy* **15**, 3602 (2013).

[2] D. Fiscaletti, I. Licata, *Int. J. Theor. Phys.* **51**, 3587 (2012).

[3] J.P. Vigier, "Real Physical Path in Quantum Mechanics. Equivalence of Einstein–de Broglie and Feynman Points of View on Quantum Particle Behaviour", Proc. 3rd Int. Symp. on Foundations of Quantum Mechanics, Tokyo 1989.

[4] H. Rafii-Tabar, *Phys. Lett.* **A138**, 353 (1989).

[5] G. Vitiello, *Braz. J. Phys.* **35**, 351 (2005).

[6] M. Blasone, P. Jizba, G. Vitiello, *Quantum Field Theory and Its Macroscopic Manifestations*, Imperial College Press, London 2011.

[7] D. Bohm, *Phys. Rev.* **85**, 166 (1952).

[8] D. Bohm, *Phys. Rev.* **85**, 180 (1952).

[9] J. Bell, *Speakable and Unspeakable in Quantum Mechanics*, Cambridge University Press, Cambridge 1988.

[10] D. Fiscaletti, *Ukr. J. Phys.* **57**, 560 (2012).

[11] P. Holland, *The Quantum Theory of Motion: An Account of the de Broglie–Bohm Causal Interpretation of Quantum Mechanics*, Cambridge University Press, Cambridge 1995.

[12] D. Bohm, B.J. Hiley, *The Undivided Universe: an Ontological Interpretation of Quantum Theory*, Routledge, London 1993.

[13] D. Bohm, *Wholeness and the Implicate Order*, Routledge, London 1980.

[14] I. Licata, "Emergence and Computation at the Edge of Classical and Quantum Systems", in: *Physics of Emergence and Organization*, I. Licata, A. Sakaji (eds.), World Scientific, Singapore 2008, pp. 1–25.

[15] R.W. Carroll, *Fluctuations, Information, Gravity and the Quantum Potential*, Springer, Berlin 2006.

[16] C. Castro, J. Mahecha, *Prog. Phys.* **2**, 38 (2006).

[17] F.T. Falciano, M. Novello, J.M. Salim, *Found. Phys.* **40**, 1885 (2010).

[18] V.I. Sbitnev, *Int. J. Bifurcat. Chaos* **19**, 2335 (2009).

[19] M. Novello, J.M. Salim, F.T. Falciano, *Int. J. Geom. Meth. Mod. Phys.* **8**, 87 (2011).

[20] S. Goldstein, R. Tumulka, N. Zanghì, "Bohmian Trajectories as the Foundation of Quantum Mechanics", in: *Quantum Trajectories*, Pratim Kumar Chattaraj (ed.), Boca: Taylor & Francis, 2010, pp. 1–15.

[21] D. Durr, S. Goldstein, N. Zanghì, *Quantum Physics Without Quantum Philosophy*, Springer, Berlin 2013.

[22] M.C. Bosca', *Theoria* **28**, 45 (2013).

[23] B.J. Hiley, R.E. Callaghan, O.J.E. Maroney, `arXiv:quant-ph/0010020`.

[24] R. Tumulka, *Eur. J. Phys.* **26**, L11 (2005).

[25] B.J. Hiley, `arXiv:1302.2323 [quant-ph]`.

[26] M. Abolhasani, M. Golshani, *Annales de la Fondation Louis de Broglie* **28**, 1 (2003).

[27] M. Oltean, *Waterloo Mathematics Review* **1**, 19 (2011).

[28] B. Poirier, *Phys. Rev.* **A77**, 022114 (2008).

[29] C. Grosche, F. Steiner, *Handbook of Feynman Path Integrals*, Springer, Berlin 1998.

[30] A. Inomata, H. Kuratsuji, C.C. Gerry, *Path Integrals and Coherent States of SU(2) and SU(1,1)*, World Scientific, Singapore 1992.

[31] M. Matsumoto, *Int. J. Mod. Phys.* **B26**, 1250143 (2012).

[32] C. Lanczos, *The Variational Principles of Mechanics*, University of Toronto Press, Toronto 1970; reprinted by Dover Publications, 1986.

[33] W. Yourgrau, S. Mandelstam, *Variational Principles in Dynamics and Quantum Theory*, Saunders, Philadelphia 1968; reprinted by Dover Publications, 2007.

[34] D.H. Kobe, `arXiv:0712.1608v1 [quant-ph]`.

[35] H.R. Brown, P. Holland, *Am. J. Phys.* **72**, 34 (2004).

[36] D. Durr, S. Goldstein, N. Zanghì, *J. Stat. Phys.* **67**, 843 (1992).

[37] S. Goldstein, W. Struyve, *J. Stat. Phys.* **128**, 1197 (2007).