SUPERFLUID PICTURE FOR ROTATING SPACE-TIMES

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(Received January 2, 2014)

Dedicated to Professor Andrzej Staruszkiewicz on the occasion of his birthday

A new prescription, in the framework of condensate models for space-times, for physical stationary gravitational fields is presented. We show that the spinning cosmic string metric describes the gravitational field associated with the single vortex in a superfluid condensate model for space-time outside the vortex core. This metric differs significantly from the usual acoustic metric for the Onsager–Feynman vortex. We also consider the question of what happens when many vortices are present, and show that on large scales a Gödel-like metric emerges. In both the single and multiple vortex cases, the failure of general relativity exemplified by the presence of closed time-like curves is attributed to the breakdown of superfluid rigidity.

DOI:10.5506/APhysPolB.45.905
PACS numbers: 03.75.Nt, 04.20.Cv, 67.85.Jk

1. Introduction

The various developments of quantum field theory in curved space-time have left the false impression that general relativity and quantum mechanics are compatible. Actually though certain predictions of classical general relativity such as closed time-like curves and event horizons are in conflict with a quantum mechanical description of space-time itself. In particular,

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a quantum mechanical description of any system requires a universal time. In practice, universal time is defined by means of synchronization of atomic clocks, but such synchronization is not possible in space-times with event horizons or closed time-like curves. It has been suggested [1] that the way a global time is established in Nature is via the occurrence of off-diagonal long-range quantum coherence in the vacuum state. This leads to a very different picture of compact astrophysical objects from that predicted by general relativity [2–7].

Certainly, it was historically an unfortunate development, that an unnecessary emphasis was placed on the high energy-momentum (UV) behavior of scattering amplitudes in 'quantum Einstein gravity', which has blinded many researchers to the subtleties of the global properties of the gravitational vacuum medium. We have in mind the infrared (IR) behavior of the gravitational and the Standard Model interactions of massless elementary excitations in the physically relevant case of the finite positive vacuum energy density $\epsilon_{\rm vac} = \mu^4(\hbar c)^{-3}$. The physical gravitational vacuum state in this case is described by the de Sitter universe. It was recognized long time ago that the physical gravitational vacuum state is a highly correlated quantum state of a new kind of matter constituents of which were called gravitational atoms [4–6, 8]. The fundamental role played by quantum entanglement in the quantum state of a huge number of strongly interacting bosonic constituents of the gravitational vacuum medium in the explanation of the underlying microscopic mechanism responsible for the selection of very small values of the cosmological constant, and the emergence of gravitational fields described by metric fields $g_{\mu\nu}$ on space-times, was strongly emphasized by one of the authors [4–6, 8, 9]. In this letter, we wish to point out the salient differences between the general relativistic description of rotating space-times and the picture offered by the assumption that the vacuum state is a quantum condensate.

It has been recognized for a long time that general relativity fails to describe accurately the physical situation in the regions of extremely high tidal forces (curvature singularities) of the type of a Big Bang or the interior of a black hole. Generally, this failure of general relativity was considered inconsequential because it was supposed to occur on Planckian length scales. In this case, a rather soothing philosophy was adopted to the effect that some mysterious and still unknown quantum theory of gravitation will take care of the difficulty by 'smoothing out' the curvature singularities. It was recognized only recently that the physics of event horizons is a second example of the failure of general relativity but this time on the macroscopic length scales [2–6]. In the following, we consider a third kind of the failure of general relativity on the macroscopic length scales, associated with the occurrence of closed time-like curves (CTC). CTCs occur frequently in analytically extended space-times described by general relativity once there is rotation present in a physical system under consideration, which is quite common in Nature.

2. Quantized vortices in a superfluid and rotating space-times

The most famous example of a solution to the Einstein equations, where CTCs occur, is the Gödel rotating Universe [10] though the first example of a rotating space-time with CTCs was found by Lanczos [11]. In these cases, there is no universal time because the classical space-time manifold contains closed time-like curves. Gödel thought that this indicated that there was something wrong with the intuitive notion of time itself. However, in the following, we will show that this strange behavior can also be viewed as an example of the failure of classical general relativity on macroscopic length scales.

As shown in [2], the hydrodynamic equations for a superfluid that one derives directly from the nonlinear Schrödinger equation are not exactly the classical Euler equations, but there are quantum corrections to these equations which become important when a certain quantum coherence length becomes comparable to length scale over which the superfluid density varies. One circumstance where this happens is near the core of a quantized vortex in a rotating superfluid. Although the physical size of the vortex core in a superfluid is usually small it can also happen, for example near to the isotropic Heisenberg point in an XY quantum magnet, that the vortex core has macroscopic dimensions.

In order to generalize the condensate models of Refs. [2–6, 12] to the case of rotating space-times, we consider the nonlinear Schrödinger equation in a general stationary space-time background described by the line element

$$ds^2 = g_{00}dt^2 + 2g_{0i}dtdx^i + g_{ij}dx^i dx^j \,, \tag{2.1}$$

where $g_{\mu\nu}$ is time independent. The Lagrangian describing the condensate of nonrelativistic particles with mass M has the form

$$\begin{aligned} L \;=\; & \frac{i\hbar c_{\rm s}^2}{2} g^{00}\left[\psi^*\left(\partial_t - g^{ij}g_{oj}\partial_i\right)\psi - \psi\left(\partial_t - g^{ij}g_{oj}\partial_i\right)\psi^*\right] \\ & + \frac{\hbar^2}{2M} g^{ij}\partial_i\psi^*\partial_j\psi + \frac{\hbar}{2M} g^{0i}(\partial_t\psi^*\partial_i\psi + \partial_t\psi\partial_i\psi^*) \\ & + \left(\frac{1}{2}Mc_{\rm s}^4 g^{00} - \frac{1}{2}Mc_{\rm s}^2 + \mu\right)\psi^*\psi - U\left(|\psi|^2\right) \,, \end{aligned} \tag{2.2}$$

where $g^{\mu\nu}$ is the contravariant tensor inverse to the metric $g_{\mu\nu}$ for the background space-time, μ is the chemical potential, $U(|\psi|^2)$ is the interaction potential energy, and $c_{\rm s}$ is the velocity of sound in the condensate at the equilibrium state. The velocity of sound $c_{\rm s}$ is related to the interaction potential U by the relation $Mc_{\rm s}^2 = |\psi|^2 U^{''}(|\psi|^2)$ (and $U'(|\psi|^2) = \mu$, of course). The equation of motion for the condensate order parameter ψ is

$$i\hbar c_{\rm s}^2 g^{00}\left(\partial_t + \frac{g^{0i}}{g^{00}}\partial_i\right)\psi = \frac{\hbar^2}{2M}\frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\psi\right) + \left(U^{'} - \mu\right)\psi - \frac{\hbar}{M}g^{0i}\partial_i\partial_t\psi\,, \tag{2.3}$$

where g is the determinant of the spatial metric g_{ij}.

It will be useful to write the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu} = \mathrm{diag}(c_{\rm s}^2, -1, -1, -1)$. To first order in $h_{\mu\nu}$ the effect of the background space-time is to introduce a perturbation $-\frac{1}{2}h^{\mu\nu}T_{\mu\nu}$ in the Lagrangian, where $T_{\mu\nu}$ is the symmetrized stress-energy-momentum tensor for the condensate [12]. Writing $\psi = \sqrt{n}e^{iS}$, where $n = |\psi|^2$ is the number density of particles in the condensate, we obtain the velocity field $v_i = \frac{\hbar}{M}\partial_i S$ for the condensate flow. This representation of ψ leads to the steady state quantum hydrodynamic equations for n and v_i

$$\partial_i\left[n\left(v_i\left(1 - \frac{h_{00}}{2c_{\rm s}^2}\right) - h_{ij}v_j - h_{0i}\right)\right] = 0\,, \tag{2.4}$$

$$\frac{\hbar^2}{M\sqrt{n}}\nabla^2\sqrt{n} - \frac{\hbar^2}{M\sqrt{n}}\partial_i\left(h_{ij}\partial_j\sqrt{n}\right) + 2\left(1 - \frac{h_{00}}{2c_{\rm s}^2}\right)\left(\mu - U^{'}\right) - M\left(1 - \frac{h_{00}}{2c_{\rm s}^2}\right)\vec{v}^{\,2} + 2Mh_{0i}v_i - h_{ii}nU^{''} + Mh_{ij}v_iv_j - \frac{\hbar^2}{4M}\nabla^2 h_{ii} - \frac{\hbar^2}{2Mc_{\rm s}^2\sqrt{n}}\vec{\nabla}\cdot\left(h_{00}\vec{\nabla}\sqrt{n}\right) = 0\,. \tag{2.5}$$

Our philosophy in the following will be to find the classical metrics which produce superfluid flows with vortices when $h_{00} = 0$ and $\partial_3 g_{\mu\nu} = 0$. The metric in our action is not a dynamical field. Instead, the metric components only act as Lagrange multipliers. The role of these Lagrange multipliers is to enforce the local equilibrium in the condensate. The homogeneous vacuum state of the condensate is characterized by $|\psi| = \mathrm{const.}$, $g_{\mu\nu} = \eta_{\mu\nu}$ and $U^{'} = \mu$.

We first seek a solution of Eqs. (2.4) and (2.5) corresponding to a single vortex in the condensate. The phase S of the condensate corresponding to a single vortex has a simple form: $S = N\varphi$, where φ is the azimuthal angle defined by the formula $\varphi = \mathrm{Arctan}(\frac{x^2}{x^1})$ and N is the vortex number which is an integer. The velocity field corresponding to the vortex configuration is

$$v_i = N\frac{\hbar}{M}\partial_i\varphi\,. \tag{2.6}$$

It is convenient to use the following well known relation (here, the indices i,j take values 1 and 2)

$$\partial_i\varphi = -\epsilon_{ij}\partial_j \ln r\,, \tag{2.7}$$

which yields

$$v_i = -\frac{N\kappa}{2\pi r^2}\epsilon_{ij}x_j\,, \tag{2.8}$$

where $\kappa = \frac{h}{M}$ is the fundamental unit of quantized circulation $\oint \vec{v}\cdot d\vec{l}$ or the flux of the vorticity field $\omega_{ij} = \partial_i v_j - \partial_j v_i$. The velocity field of a vortex v_i has the form of the Aharonov–Bohm electromagnetic potential [13], while the vorticity $\omega = \frac{1}{2}\epsilon_{ij}\omega_{ij} = \epsilon_{ij}\partial_i v_j$ is an analog of the Aharonov–Bohm magnetic field produced by an infinitely thin solenoid, $\omega = \kappa\delta(x^1)\delta(x^2)$.

It turns out that because of the presence of the potentials h_{0i} and h_{ij} in the hydrodynamic equations, the superfluid density n will be nearly constant when r is greater than the coherence length $\xi = \frac{\hbar}{Mc_{\mathrm{s}}}$. Indeed, it is straightforward to show that if n is constant and the velocity has the form given in Eq. (2.8), then Eqs. (2.4) and (2.5) have a solution

$$N = 1\,, \tag{2.9}$$

$$h_{00} = 0\,, \qquad h_{0i} = v_i\,, \tag{2.10}$$

$$h_{ij} = -\frac{v_i v_j}{c_{\mathrm{s}}^2}\,. \tag{2.11}$$

These values for the potentials $h_{\mu\nu}$ are equivalent to the metric for the background space-time of the local 'spinning cosmic string' solution of the Einstein field equations [14–16] (see also [17–19]) in the region where n is constant; *i.e.* for $r \gtrsim \xi$. The line element for this solution (for $r > 0$) has the form [14–16]

$$ds^2 = (c_{\mathrm{s}}dt + Ad\varphi)^2 - dr^2 - r^2 d\varphi^2 - dz^2\,, \tag{2.12}$$

where $A = \frac{\kappa}{2\pi c_{\mathrm{s}}} = \xi$. The string-like singularity at $r = 0$ has neither mass density nor pressure, so space-time is flat for $r > 0$. However, the string rotates resulting in frame dragging. This frame dragging is represented by the appearance of a vector potential A_i [14–16] with azimuthal component $A_\varphi = A = \frac{\kappa}{2\pi c_{\mathrm{s}}}$. The frame dragging implied by the metric (2.12) is evidently closely related to the velocity field surrounding a single vortex filament in a superfluid. Indeed, De Witt pointed out some time ago [20] that the vector potential A_i associated with frame dragging can be formally identified as the vector potential for a superconductor. Kirzhnits and Yudin [21] have also studied stationary superfluid flows in the presence of gravitational fields g_{0i} produced by rotating compact, massive objects (superfluid cores of neutron stars). Balasin and Israel [22] have concluded that vortex filaments in a superfluid neutron star do produce gravimagnetic forces, contrary to the statements in the literature.

It should be noted that excitations other than collective bosonic excitations in the condensate, for example massless (massive) fermionic and bosonic excitations or impurities, will feel the gravitational field, Eq. (2.12), associated with the vortex. However, this field is not the same as the acoustic metric seen by the condensate excitations. The scattering cross-section for fermionic (bosonic) particles will be given by the Aharonov–Bohm cross-section [15, 16] as is the scattering of quasiparticle excitations of unit electric charge on Abrikosov vortices [23] in type II superconductors. In this sense, the 'spinning cosmic string' is a gravitational analog of the Abrikosov vortex [15, 16]. This is also the reason why one of the authors has called the scattering of relativistic particles by gravitational vortices the gravitational Aharonov–Bohm effect [15, 16]. The scattering cross-section for condensate excitations has been given in Ref. [24] and for the reasons just mentioned is not the same as the gravitational Aharonov–Bohm scattering cross-section [15, 16].

The space-time corresponding to the metric (2.12) does not have a universal time because closed time-like curves appear close to the axis of the gravitational vortex. What does not seem to have been noted before, though, is the fact that closed time-like curves appear in the gravitational vortex background (2.12) at exactly the radius, where a classical hydrodynamic description of the superfluid begins to fail. Indeed, the superfluid velocity (2.8) will become comparable to the velocity of sound c_{s} when the radius r is close to the quantum coherence length ξ. Therefore, superfluid rigidity and classical hydrodynamics break down as one enters the core of the vortex. Remarkably, this breakdown of a classical description of the superfluid seems to be closely related to the breakdown of causality in classical GR associated with the formation of closed time-like geodesics. The condition for the appearance of closed time-like curves in a rotating space-time is that $g_{\varphi\varphi} > 0$, which for the gravitational vortex metric (2.12) becomes the condition

$$r < r_{\mathrm{c}} = \frac{\kappa}{2\pi c_{\mathrm{s}}} = \xi \,. \tag{2.13}$$

That is, closed time-like curves appear in the gravitational vortex solution of the Einstein equations near to the axis of the string where the velocity of frame dragging exceeds the speed of light. In the superfluid picture, this corresponds to the core of the vortex where the superfluid flow velocity exceeds the speed of sound c_s. As previously discussed, this is just where a classical hydrodynamic description of the fluid flow in a quantized superfluid vortex breaks down. Indeed, the solution to the equations of quantum hydrodynamics in the presence of the potentials $h_{\mu\nu}$ given by Eqs. (2.10), (2.11) is valid only in the region where the condensate particle density n is constant. The corresponding space-time metric (2.12) is perfectly well behaved in this region ($r > \xi$). It is only after the naïve analytic continuation of the metric (2.12) to the region $r < \xi$ is attempted that the causality violating regions appear in the space-time.

This observation provokes one to ask if the appearance of closed time-like curves in solutions of the classical Einstein field equations might always be associated with a breakdown of superfluid rigidity. In particular, one might wonder if the appearance of closed time-like curves in Gödel-like universes is related to the behavior of rotating superfluids. The Gödel metric for a rotating universe can be written in the form [25]

$$ds^2 = (cdt + \Omega(r)d\varphi)^2 - dr^2 - f^2 d\varphi^2 - dz^2 \,, \tag{2.14}$$

where $\Omega(r) = \frac{4\Omega}{m^2}\sinh^2(\frac{mr}{2})$ and $f(r) = \frac{1}{m}\sinh(mr)$. In the limit of small r, $\Omega(r)$ approaches Ωr^2. The off-diagonal metric component $g_{0\varphi}$ equals the velocity potential inside a body rigidly rotating with angular velocity Ω. It can be seen that the metric component $g_{0\varphi}$ for the Gödel universe has a very different dependence on radius from that of the gravitational vortex. However, as we shall now see this very different behavior is characteristic of what happens in a rapidly rotating superfluid.

Feynman pointed out [26] that when many vortices are present, the velocity of rotation in the superfluid will approach that of a rigidly rotating body; *i.e.* $\vec{v} = \vec{\Omega} \times \vec{r}$. When the area density σ of vortices is not too high, it is reasonable to approximate the phase in Eq. (2.6) as a sum $S = \sum_a \mathrm{Arg}(w - w_a)$, $w = x^1 + ix^2$ of phases of individual vortices each with vortex number $N = 1$. Using Eq. (2.7), the velocity field in this approximation can be written in the form

$$v_i = \frac{\kappa}{2\pi}\partial_i S = -\frac{\kappa}{2\pi}\epsilon_{ij}\partial_j \sum_a \ln|\vec{x} - \vec{x_a}| \,. \tag{2.15}$$

Evaluating the vorticity $\omega = \epsilon_{ij}\partial_i v_j$ and replacing the sum in Eq. (2.15) by an integral, we obtain

$$\omega = \frac{\kappa\sigma}{2\pi}\nabla_x^2 \int d^2y \ln|\vec{x} - \vec{y}\,| \,. \tag{2.16}$$

Using the relation $\nabla_x^2 \ln|\vec{x} - \vec{y}\,| = 2\pi\delta^{(2)}(\vec{x} - \vec{y}\,)$, we obtain $\omega = \kappa\sigma$. It follows from Eqs. (2.15) and (2.16) that $v_i = -\frac{\kappa\sigma}{2}\epsilon_{ij}x_j$ which means that this velocity field is indeed that of a rigid body rotating with the angular velocity $\Omega = \frac{\kappa\sigma}{2}$.

Since the gravitational vortex solution (2.12) is spatially flat, it makes sense to construct a new solution to the Einstein equations by simply superposing the velocity fields, Eq. (2.15), corresponding to a collection of parallel gravitational vortices. Following the same line of reasoning that leads one to rigid body rotation in the case of many superfluid vortices, one would surmise, based on the identification $h_{0i} = v_i$, that in the presence of many gravitational vortices the metric of space-time would assume the form

$$ds^2 = \left(c_{\rm s}dt + \frac{1}{c_{\rm s}}\Omega r^2 d\varphi\right)^2 - dr^2 - r^2 d\varphi^2 - dz^2\,. \tag{2.17}$$

This metric is, in fact, just the Som–Raychaudhuri solution of the Einstein field equations [25, 27]. This metric can be obtained from the Gödel metric Eq. (2.14) by letting $m \to 0$. It can be seen that the velocity of frame dragging for the metric (2.17) is just the velocity inside a rigidly rotating body. The condition for the appearance of closed time-like curves, *i.e.* $g_{\varphi\varphi} > 0$, in the Som–Raychaudhuri space-time is

$$\Omega r_{\rm c} > c_{\rm s}\,. \tag{2.18}$$

That is, closed time-like curves appear when the velocity of frame dragging exceeds the speed of light. In contrast with the gravitational vortex, closed time-like curves appear in the Som–Raychaudhuri space-time at large radii. The appearance of closed time-like curves in Gödel space-times mimics the behavior of Som–Raychaudhuri space-time in that the closed time-like curves appear at large radii. In particular, for the Gödel metric (2.14) the condition for the appearance of closed time-like curves is

$$\frac{2\Omega}{m}\tanh\frac{mr_{\rm c}}{2} > c_{\rm s}\,. \tag{2.19}$$

When $m \to 0$, this condition reduces to Eq. (2.18). When $m = 2\frac{\Omega}{c}$, the radius where the velocity of frame dragging approaches the speed of light recedes to infinity, and the space-time will be free of closed time-like curves everywhere. We now wish to inquire as to the significance of the conditions (2.18) and (2.19) from the point of view of a rotating superfluid. Evidently then, a superfluid description for the metrics (2.14) and (2.17) will require an external rotating container of normal matter to create a frame dragging potential. The elementary fact that this container cannot rotate faster than the speed of light leads to the conditions (2.18) and (2.19). The occurrence of solid body-like frame dragging in the Gödel and Som–Raychaudhuri metrics may seem to be incompatible with a superfluid interpretation for space-time because $\vec{\nabla} \times \vec{v} = 2\vec{\Omega}$ for a solid body rotating with angular velocity $\vec{\Omega}$, whereas the flow velocity of a superfluid must have zero curl since it is the gradient of a phase. The resolution of this paradox is that the solid body rotation curve corresponds to a coarse-grained average of the velocities from an array of individual vortices. The superfluid as a whole will respond to the frame dragging created by the array of vortices leading to the Gödel-like metrics. In between the vortices, the flow is irrotational so $\vec{\nabla} \times \vec{v} = 0$ in the superfluid condensate.

In contrast with the case of a single vortex, the coarse-grained potentials associated with the array of vortices do not satisfy the time independent hydrodynamic Eqs. (2.4) and (2.5). Indeed, in contrast with the case of the single vortex, the term $\nabla^2 h_{ii}$ in Eq. (2.5) which comes from the quantum pressure no longer cancels the term $h_{ij} v_i v_j$ which arises as a relativistic correction to the kinetic energy density of the condensate. Although a simple superposition, Eq. (2.15), of the single vortex solution, Eqs. (2.9)–(2.11), does not satisfy the superfluid Eqs. (2.4)–(2.5), there do exist multi-vortex solutions. In particular, there exist time independent solutions representing a regular lattice of vortices, the Tkachenko lattice [28].

When an impulse of energy is applied to a very low temperature rotating superfluid condensate, then a turbulent state containing a time dependent tangle of quantum vortices can develop [29]. Such a regime is known as quantum turbulence. If space-time is indeed a condensate and the conditions for the development of quantum turbulence, *i.e.* rotation and an impulse of energy are met, then there should be characteristic observational signatures. For example, the onset of quantum turbulence in cosmological space-times would lead to a characteristic scale-free spectrum of energy density fluctuations.

The authors would like to thank their colleagues Yakir Aharonov, James Bjorken, Robert Laughlin, Emil Mottola, David Santiago and Andrzej Staruszkiewicz for valuable comments. One of us (G.C.) would like to acknowledge hospitality extended to him at the University of South Carolina where this paper has been completed in March 2004. This material is based upon work (partially) supported by the National Science Foundation under Grant No. 0140377 (P.O.M.). This work was also performed (in part) under the auspices of the U.S. Department of Energy by University of California Lawrence Livermore National Laboratory under contract No. W-7405-Eng-48 (G.C.).

Note added in proof:

The last paragraph of this paper was made very short for brevity reasons because the paper was initially formatted as a letter and thus no formulae were given to support the claim that quantum turbulence in the context of the superfluid model of gravitational fields/space-times leads '... to a characteristic scale-free spectrum of energy density fluctuations'.

In an earlier unpublished work of one of the authors [30], the correlation functions of the statistical random velocity field $v_i(\boldsymbol{x})$ in the three-dimensional conformal field theory (3D CFT) describing quantum turbulence on large scales (and the 'normal' fluid $K41$ turbulence; the celebrated Kolmogorov–Obukhov $\frac{5}{3}$ law) were computed

$$\langle v_i(\boldsymbol{x}) v_j(\boldsymbol{y}) \rangle = P_{ij}(\boldsymbol{x}, \boldsymbol{y}; \Delta_v) \, |\boldsymbol{x} - \boldsymbol{y}|^{-2\Delta_v} \, , \tag{2.20}$$

where Δ_v is the scaling dimension of the statistical (random) velocity field $v_i(\boldsymbol{x})$, and from the condition of the vanishing divergence of the velocity field (outside the vortex cores of quantized vortices) $\partial_i v_i = 0$, we compute the 3D symmetric tensor $P_{ij}(\boldsymbol{x}, \boldsymbol{y}; \Delta_v)$

$$P_{ij}(\boldsymbol{x}, \boldsymbol{y}; \Delta_v) = C[(1 - \Delta_v)\delta_{ij} + \Delta_v n_i n_j] \, , \tag{2.21}$$

where C is a constant and

$$n_i = n_i(\boldsymbol{x}, \boldsymbol{y}) = \frac{(\boldsymbol{x} - \boldsymbol{y})_i}{|\boldsymbol{x} - \boldsymbol{y}|} \, . \tag{2.22}$$

At large distances, the line integral of the statistical velocity field $\oint_{\Gamma(R)} v_i dx_i$ along the closed contour $\Gamma(R)$, which is a circle of radius R, scales like R^0 because of the Onsager–Feynman quantization condition [26]. This means that on the very large distance scale when the contour encircles the tangle of quantized vortices the scaling dimension Δ_v of the statistical velocity field $v_i(\boldsymbol{x})$ is equal to one: $\Delta_v = 1$.

The naïve scaling dimension of the kinetic energy ϵ is $\Delta_\epsilon = 2$. The composite statistical operator $\epsilon(\boldsymbol{x}) = \frac{1}{2}\rho v_i(\boldsymbol{x}) v_i(\boldsymbol{x})$, with $\rho = \text{const.}$, scales with the lowest scaling dimension $\Delta_\epsilon = 2\Delta_v$. This translates to the correlations of the composite energy density operator $\epsilon(\boldsymbol{x})$ that is equivalent to the Zel'dovich–Harrison scaling spectrum of the energy density fluctuations.

In fact, the experiments on quantum turbulence in superfluid ^{4}He reported during the COSLAB Workshop in Bilbao in July 2003 showed that at a large distance scales velocity correlations display the scaling dimension $\Delta_v = 1$. This behavior translates to the Zel'dovich–Harrison power spectrum for energy density $P_\epsilon(k) \sim k^n$, where $k = |\boldsymbol{k}|$ and $n = 1$. Recall that the power spectrum $P_\epsilon(k)$ is defined by the Fourier transform of the two-point correlation function $\langle\epsilon(\boldsymbol{x})\epsilon(\boldsymbol{y})\rangle$ by the formula

$$\langle\epsilon(\boldsymbol{k})\epsilon(\boldsymbol{k}^{'})\rangle = \delta^{(3)}(\boldsymbol{k}+\boldsymbol{k}^{'})P_\epsilon(|\boldsymbol{k}|)\,. \tag{2.23}$$

In the case of the two-point correlation function with scaling

$$\langle\epsilon(\boldsymbol{x})\epsilon(\boldsymbol{y})\rangle \sim |\boldsymbol{x}-\boldsymbol{y}|^{-2\Delta_\epsilon}\,, \tag{2.24}$$

the power spectrum $P_\epsilon(k)\sim k^n$, where the exponent $n = 2\Delta_\epsilon - 3$ is entirely given in terms of the scaling dimension Δ_ϵ. Indeed, it was shown [31] that the naïve scaling of the energy density fluctuations in the 'early universe' ($\Delta_\epsilon = 2$) corresponds to the celebrated Zel'dovich–Harrison power spectrum with the exponent $n_{\rm ZH} = 1$. There is room for 'anomalous dimensions' though [30, 31].

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