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COHERENT STATES AND QUANTUM NUMBERS FOR TWIST-DEFORMED OSCILLATOR MODEL

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The coherent states for twist-deformed oscillator model provided in article by M. Daszkiewicz, C.J. Walczyk [*Acta Phys. Pol. B* **40**, 293 (2009)] are constructed. Besides, it is demonstrated that the energy spectrum of considered model is labeled by two quantum numbers — by the so-called main and azimutal quantum numbers respectively.

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The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on Quantum Gravity [2, 3] and String Theory models [4, 5], indicating that space-time at Planck scale should be noncommutative, *i.e.* it should have a quantum nature. Consequently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see *e.g.* [6, 7]) as well as with field theoretical models (see *e.g.* [8, 9]), in which the quantum space-time is employed.

In accordance with the Hopf-algebraic classification of all deformations of relativistic [10] and nonrelativistic [11] symmetries, one can distinguish three basic types of space-time noncommutativity:

1. The canonical (soft) deformation

$$[\, x_\mu, x_\nu \,] = i\theta_{\mu\nu}\,, \tag{1}$$

with constant and antisymmetric tensor $\theta_{\mu\nu}$. The explicit form of corresponding Poincare Hopf algebra has been provided in [12, 13], while its nonrelativistic limit has been proposed in [14].

2. The Lie-algebraic case

$$[\, x_\mu, x_\nu \,] = i\theta_{\mu\nu}^{\rho} x_\rho \,, \tag{2}$$

with particularly chosen constant coefficients $\theta_{\mu\nu}^{\rho}$. Particular kind of such space-time modification has been obtained as representations of κ-Poincare [15, 16] and κ-Galilei [17] Hopf algebras. Besides, the Lie-algebraic twist deformations of relativistic and nonrelativistic symmetries have been provided in [18, 19] and [14], respectively.

3. The quadratic deformation

$$[\, x_\mu, x_\nu \,] = i\theta_{\mu\nu}^{\rho\tau} x_\rho x_\tau \,, \tag{3}$$

with constant coefficients $\theta_{\mu\nu}^{\rho\tau}$. Its Hopf-algebraic realization was proposed in [20, 21] and [19] in the case of relativistic symmetry, and in [22] for its nonrelativistic counterpart.

Besides, it has been demonstrated in [23], that in the case of the so-called N-enlarged Newton–Hooke Hopf algebras $\mathcal{U}_0^{(N)}(NH_\pm)$, the twist deformation provides the new space-time noncommutativity of the form[1,2]

4.

$$[\, t, x_i \,] = 0 \,, \qquad [\, x_i, x_j \,] = if_{\kappa\pm}(t)\,\theta_{ij}(x) \,, \tag{4}$$

with time-dependent functions

$$f_{\kappa+}(t) = \kappa f\left(\sinh\left(\frac{t}{\tau}\right),\ \cosh\left(\frac{t}{\tau}\right)\right) ,$$
$$f_{\kappa-}(t) = \kappa f\left(\sin\left(\frac{t}{\tau}\right),\ \cos\left(\frac{t}{\tau}\right)\right) ,$$

$\theta_{ij}(x) \sim \theta_{ij} = \text{const}$ or $\theta_{ij}(x) \sim \theta_{ij}^{k} x_k$ and τ as well as κ denoting the cosmological constant and deformation parameter respectively. It should be also noted that different relations between all mentioned above quantum spaces 1, 2, 3 and 4 have been summarized in article [23].

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[1] $x_0 = ct$.

[2] The discussed space-times have been defined as the quantum representation spaces, the so-called Hopf modules (see *e.g.* [12, 13]), for quantum N-enlarged Newton–Hooke Hopf algebras.

Let us now turn to the quantum oscillator model defined on the twist-deformed phase space [24][3]

$$[\, t, \bar{x}_i \,] = 0 \,, \quad [\, \bar{x}_1, \bar{x}_2 \,] = i f_\kappa(t) \,, \quad [\, \bar{x}_i, \bar{p}_j \,] = i\hbar\delta_{ij} \,, \quad [\, \bar{p}_i, \bar{p}_j \,] = 0 \,. \tag{5}$$

Its dynamic is given by the following Hamiltonian function with constant mass m and frequency ω

$$\bar{H}(\bar{p}, \bar{x}) = \frac{1}{2m}\left(\bar{p}_1^2 + \bar{p}_2^2\right) + \frac{1}{2}m\omega^2\left(\bar{x}_1^2 + \bar{x}_2^2\right) \,. \tag{6}$$

In order to analyze the above system, we represent the noncommutative variables $(\bar{x}_i, \bar{p}_i)$ on classical phase space (x_i, p_i) as follows (see *e.g.* [25, 26])

$$\bar{x}_1 = \hat{x}_1 - \frac{f_\kappa(t)}{2\hbar}\hat{p}_2 \,, \qquad \bar{x}_2 = \hat{x}_2 + \frac{f_\kappa(t)}{2\hbar}\hat{p}_1 \,, \tag{7}$$

where

$$[\, \hat{x}_i, \hat{x}_j \,] = 0 = [\, \hat{p}_i, \hat{p}_j \,] \,, \qquad [\, \hat{x}_i, \hat{p}_j \,] = i\hbar\delta_{ij} \,. \tag{8}$$

Then, the Hamiltonian (6) takes the form[4]

$$H_f(t) = \frac{\left(\hat{p}_1^2 + \hat{p}_2^2\right)}{2M_f(t)} + \frac{1}{2}M_f(t)\Omega_f^2(t)\left(\hat{x}_1^2 + \hat{x}_2^2\right) - \frac{f_\kappa(t)}{2\hbar}m\omega^2\hat{L} \,, \tag{9}$$

with symbol

$$\hat{L} = \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1 \tag{10}$$

denoting angular momentum of particle. Besides, the coefficients $M_f(t)$ and $\Omega_f(t)$ present in the above formula denote the time-dependent functions given by

$$M_f(t) = \frac{m}{1 + \frac{m^2\omega^2 f_\kappa^2(t)}{4\hbar^2}} \,, \qquad \Omega_f(t) = \omega\sqrt{1 + \frac{m^2\omega^2 f_\kappa^2(t)}{4\hbar^2}} \,, \tag{11}$$

respectively, such that

$$M_f(t)\Omega_f^2(t) = m\omega^2 = \text{const} \,. \tag{12}$$

Further, we introduce a set of time-dependent creation $(a_A^\dagger(t))$ and annihilation $(a_A(t))$ operators

$$\hat{a}_\pm(t) \;=\; \frac{1}{2\sqrt{\hbar}}\left[\frac{(\hat{p}_1 \pm i\hat{p}_2)}{\sqrt{M_f(t)\Omega_f(t)}} - i\sqrt{M_f(t)\Omega_f(t)}(\hat{x}_1 \pm i\hat{x}_2)\right] \,, \tag{13}$$

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[3] See type 4 of quantum space-time.

[4] It should be noted that for $f_\kappa(t) = \theta$, we get the canonically deformed oscillator model provided in [26].

satisfying the standard commutation relations

$$[\,\hat{a}_A,\hat{a}_B\,]=0\,,\quad \left[\,\hat{a}^\dagger_A,\hat{a}^\dagger_B\,\right]=0\,,\quad \left[\,\hat{a}_A,\hat{a}^\dagger_B\,\right]=\delta_{AB}\,;\quad A,B=\pm\,. \tag{14}$$

Then, one can easily check that in terms of the operators (13) the Hamiltonian function (9) looks as follows

$$\hat{H}_f(t)=\Omega_+(t)\left(\hat{N}_+(t)+\tfrac{1}{2}\right)+\Omega_-(t)\left(\hat{N}_-(t)+\tfrac{1}{2}\right)\,, \tag{15}$$

with

$$\Omega_\pm(t)=\Omega_f(t)\mp\frac{f_\kappa(t)m\omega^2}{2\hbar}\,, \tag{16}$$

and number operators in $\pm$ direction given by

$$\hat{N}_\pm(t)=\hat{a}^\dagger_\pm(t)\hat{a}_\pm(t)\,, \tag{17}$$

respectively. Moreover, we see that the energy eigenvectors can be generated in a standard way as follows

$$|n_+,n_-,t\rangle=\frac{1}{\sqrt{n_+!}}\frac{1}{\sqrt{n_-!}}\left(\hat{a}^\dagger_+(t)\right)^{n_+}\left(\hat{a}^\dagger_-(t)\right)^{n_-}|0\rangle\,, \tag{18}$$

while the corresponding (parameterized by n_+ and n_-) eigenvalues are

$$E_{n_+,n_-}(t)=\Omega_+(t)\left(n_++\tfrac{1}{2}\right)+\Omega_-(t)\left(n_-+\tfrac{1}{2}\right)\,,\quad n_+,n_-=0,1,2,\ldots \tag{19}$$

Besides, using operator representation (13), one finds

$$(\Delta\hat{x}_i)^2_{|n_+,n_-,t\rangle}(\Delta\hat{p}_i)^2_{|n_+,n_-,t\rangle}=\frac{\hbar^2}{4}(1+n_++n_-)^2\,, \tag{20}$$

where symbol $(\Delta\hat{a})_{|\varphi\rangle}$ denotes the uncertainty of observable $\hat{a}$ in quantum state $|\varphi\rangle$. The above result means that momentum-position uncertainty relations for eigenstates (18) become saturated only for $n_+=n_-=0$, *i.e.* only for vacuum vector $|0\rangle$. Apart from that, it is easy to see that the momentum operator (10) can be written as follows

$$\hat{L}=\hbar\left(\hat{a}^\dagger_-(t)\hat{a}_-(t)-\hat{a}^\dagger_+(t)\hat{a}_+(t)\right)\,, \tag{21}$$

while its action on quantum states (18) is given by

$$\hat{L}|n_+,n_-,\rangle=\hbar(n_--n_+)|n_+,n_-,t\rangle\,. \tag{22}$$

Consequently, the energy spectrum (19) can be written in terms of eigenvalues (22) as follows

$$E_{n_+,n_-}(t) = \hbar\Omega_f(t)(n_+ + n_- + 1) + \frac{f_\kappa(t)M_f(t)\Omega_f^2(t)}{2}(n_- - n_+)\,. \tag{23}$$

Let us now solve two problems. First of them concerns the construction of the so-called coherent states for considered model, *i.e.* the quantum vectors which saturate the momentum-position Heisenberg uncertainty relations. The second problem applies to the proper interpretation of quantum numbers $n = n_+ + n_-$ and $l = n_- - n_+$ labeling the energy spectrum (23).

Hence, let us consider the quantum states of the form

$$|c_+, c_-, t\rangle = \sum_{n_+,n_-} \frac{c_+^{n_+} e^{-\frac{1}{2}|c_+|^2}}{\sqrt{n_+!}} \frac{c_-^{n_-} e^{-\frac{1}{2}|c_-|^2}}{\sqrt{n_-!}} |n_+, n_-, t\rangle\,, \tag{24}$$

which play the role of the eigenfunctions for annihilation operators (13)

$$\hat{a}_\pm(t)|c_+, c_-, t\rangle = c_\pm|c_+, c_-, t\rangle\,. \tag{25}$$

By direct calculation, one may check that

$$(\Delta p_i)^2_{|c_+,c_-,t\rangle} = \frac{\hbar M_f(t)\Omega_f(t)}{2}\,, \quad (\Delta x_i)^2_{|c_+,c_-,t\rangle} = \frac{1}{2}\frac{\hbar}{M_f(t)\Omega_f(t)}\,, \quad i = 1, 2\,, \tag{26}$$

what leads to the saturated momentum-position Heisenberg uncertainty relations

$$(\Delta p_i)^2_{|c_+,c_-,t\rangle}(\Delta x_i)^2_{|c_+,c_-,t\rangle} = \frac{\hbar^2}{4}\,, \qquad i = 1, 2\,. \tag{27}$$

Consequently, we see that the vectors (24) are, in fact, nothing else than the coherent states for twist-deformed oscillator model, satisfying

$$\left\langle \hat{H}_f \right\rangle_{|c_+,c_-,t\rangle} = E_{|0,0,t\rangle}(t) + \frac{\Omega_f(t)}{\hbar}(\Delta L)^2_{|c_+,c_-,t\rangle} + \frac{M_f(t)\Omega_f^2(t)f_\kappa(t)}{2\hbar}\langle L\rangle_{|c_+,c_-,t\rangle}\,, \tag{28}$$

with

$$\langle L\rangle_{|c_+,c_-,t\rangle} = \hbar\left(|c_-|^2 - |c_+|^2\right)\,, \tag{29}$$

$$(\Delta L)^2_{|c_+,c_-,t\rangle} = \hbar^2\left(|c_-|^2 + |c_+|^2\right)\,. \tag{30}$$

In the case of second problem, one should solve the eigenvalue equation for Hamiltonian (9) written in terms of polar coordinates

$$\hat{H}_f(t)\psi(r,\varphi,t) = E(t)\psi(r,\varphi,t)\,, \tag{31}$$

where

$$\hat{H}_f(t) = -\frac{\hbar^2}{2M_f(t)}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{\hbar^2}\frac{\hat{L}^2}{r^2}\right) + \frac{M_f(t)\Omega_f^2(t)}{2}r^2 - \frac{f_\kappa(t)M_f(t)\Omega_f^2(t)}{2\hbar}\hat{L}\,, \tag{32}$$

and

$$\hat{L} = -i\hbar\frac{\partial}{\partial\varphi}\,, \qquad \left[\hat{H},\hat{L}\right] = 0\,. \tag{33}$$

To this aim, it is convenient to take the corresponding eigenfunctions in the form

$$\psi(r,\varphi,t) = \phi(\varphi)R(r,t)\,, \tag{34}$$

with its azimutal part $\phi(\varphi)$ satisfying

$$\hat{L}\phi_l(\varphi) = \hbar l\phi(\varphi)\,, \qquad \phi_l(\varphi) = \frac{1}{\sqrt{2\pi}}e^{il\varphi}\,, \qquad l = 0,\pm1,\pm2\,,\ldots \tag{35}$$

Then, the proper equation for radial function $R(r,t)$ looks as follows

$$\left(-\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{l^2}{\rho^2}+\frac{\rho^2}{4}-\mathcal{E}_l(t)\right)R_l(\rho(t)) = 0\,,$$
$$\mathcal{E}_l(t) = \frac{E(t)-\frac{f_\kappa(t)M_f(t)\Omega_f^2(t)}{2}l}{\hbar\Omega_f(t)}\,, \tag{36}$$

where $\rho(t) = r\sqrt{2M_f(t)\Omega_f(t)/\hbar}$ plays the role of dimensionless variable. Its physical solution can be written as

$$R_l^{(n)}(\rho(t)) = w_l^{(n)}(\rho(t))e^{-\rho^2(t)/4}\,, \tag{37}$$

with $w_l^{(n)}(\rho(t))$ denoting the polynomial of degree n. Then, equation (36) reduces to the following one

$$-\frac{\partial^2 w_l^{(n)}(\rho(t))}{\partial\rho^2}+\frac{\rho^2-1}{\rho}\frac{\partial w_l^{(n)}(\rho(t))}{\partial\rho} + \frac{l^2}{\rho^2}w_l^{(n)}(\rho(t)) - (\mathcal{E}_l(t)-1)w_l^{(n)}(\rho(t)) = 0\,, \tag{38}$$

for which the solution (this time) is given by[5]

$$w_l^{(n)}(\rho(t)) = a_l^{(n)}\left(1+\sum_{k=1}^{(n-|l|)/2}\left[\prod_{s=1}^{k}\frac{n+2-(2s+|l|)}{l^2-(2s+|l|)^2}\right]\rho^{2k}(t)\right)\rho^{|l|}(t) \tag{39}$$

[5] The symbol $a_l^{(n)}$ denotes the normalization factor.

only when

$$\mathcal{E}_l(t) \to \mathcal{E}_l^{(n)}(t) = n+1\,, \quad l \in \{-n, -n+2, \ldots, n-2, n\}\,, \quad n = 0, 1, 2, 3, \ldots\,, \tag{40}$$

or (equivalently)

$$E_l^{(n)}(t) = \hbar\Omega_f(t)(n+1) + \frac{f_\kappa(t) M_f(t) \Omega_f^2(t)}{2} l\,. \tag{41}$$

Consequently, after substitution $n = n_+ + n_-$ and $l = n_- - n_+$ into eigenvalues (41), we get, in fact, the energy spectrum (23) labeled by n_+ and n_- parameters. For this reason as well as due to the formulas (35), (37) and (41), the quantities n and l may be called the "main" and "azimutal" quantum numbers respectively.

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