THE LOCAL AND GLOBAL GEOMETRICAL ASPECTS OF THE TWIN PARADOX IN STATIC SPACETIMES: I. THREE SPHERICALLY SYMMETRIC SPACETIMES

Leszek M. Sokołowski[†], Zdzisław A. Golda[‡]

Astronomical Observatory, Jagiellonian University
Orla 171, 30-244 Kraków, Poland
and
Copernicus Center for Interdisciplinary Studies

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We investigate local and global properties of timelike geodesics in three static, spherically symmetric spacetimes. These properties are of its own mathematical relevance and provide a solution of the physical 'twin paradox' problem. The latter means that we focus our studies on the search of the longest timelike geodesics between two given points. Due to problems with solving the geodesic deviation equation, we restrict our investigations to radial and circular (if exist) geodesics. On these curves we find general Jacobi vector fields, determine by means of them sequences of conjugate points and with the aid of the comoving coordinate system and the spherical symmetry we determine the cut points. These notions identify segments of radial and circular geodesics which are locally or globally of maximal length. In de Sitter spacetime all geodesics are globally maximal. In CAdS and Bertotti–Robinson spacetimes, the radial geodesics which infinitely many times oscillate between antipodal points in the space contain infinite number of equally separated conjugate points and there are no other cut points. Yet in these two spacetimes each outgoing or ingoing radial geodesic which does not cross the centre is globally of maximal length. Circular geodesics exist only in CAdS spacetime and contain an infinite sequence of equally separated conjugate points. The geodesic curves which intersect the circular ones at these points may either belong to the two-surface $\theta = \pi/2$ or lie outside it.

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[†] lech.sokolowski@uj.edu.pl
[‡] zdzislaw.golda@uj.edu.pl

1. Introduction

We provide detailed calculations concerning the 'twin paradox' problem in three particular static spherically symmetric (SSS) spacetimes. We consider three twins following different worldlines joining common endpoints and establish which twin gets the oldest one at the reunion. As it is well known, the problem is of purely geometrical nature and in this setting is equivalent to the search in differential Lorentzian geometry of the longest timelike curve joining two given points in the spacetime. The problem actually consists of two separate problems: a local and a global one. In the local problem, one considers a bundle of nearby (infinitesimally close) timelike curves and seeks for the longest one in the bundle. Again, it is well known that there is a well defined procedure for solving the local problem in terms of the curvature tensor, which physically determines the behaviour of geodesic worldlines of nearby free test particles both in four and in a larger number of spacetime dimensions [1]. A locally maximal timelike curve is always a geodesic and is determined by solving the geodesic deviation equation. If the endpoint (the reunion point of the twins' worldlines) does not lie in a convex normal neighbourhood of the initial point, the two points are connected by two or more geodesics of the bundle. This fact is signalled by the existence of points conjugate to the initial one lying on one of these geodesics. In other terms, a segment of a geodesic γ joining points P_0 and P_1 is locally of maximal length between these points if there are no conjugate points to P_0 on γ within the segment. All necessary theorems concerning Jacobi vector fields (the deviation vectors) and conjugate points are briefly summarized in [2], where locally maximal worldlines in Schwarzschild metric were studied.

Yet the global problem is quite different: here, one seeks for the longest curve among all possible timelike ones joining the given points P_0 and P_1. This means that one compares the lengths of curves which besides the endpoints are distant from each other. It is clear that the nonlocal nature of the problem precludes the existence of any analytic tool to establish if the given curve is globally maximal: there is no differential equation (playing the role of the deviation equation) whose solutions might indicate the longest curve. The globally maximal curve is, again, a segment of a timelike geodesic and the notion of the conjugate point is replaced by the cut point indicating the end of this segment. All what is known in global Lorentzian geometry in this respect are 'existence theorems' which provide no effective algorithm for searching for maximal geodesics. On the contrary, in general, one should consider the whole set of timelike curves with the given common endpoints and compare their lengths case by case.

High symmetry, such as the spherical one, may help in this search, however, as we shall see below, the spacetimes with the same symmetry considerably differ from each other. Spherical symmetry is singled out since in these spacetimes it is quite easy to find a transformation from the coordinates in which the spacetime metric is originally given to the comoving coordinates. In the latter coordinates, it is straightforward to find out globally maximal segments for a class of timelike geodesics.

The purpose of the work is twofold: firstly, to find out complete sets of solutions for Jacobi vector fields for two classes of timelike geodesics, determine conjugate points on them (being zeros of the Jacobi fields) and, in this way, show the locally maximal segments of these curves, then establish, where it is possible, whether these segments (or their pieces) are globally maximal. Secondly, we interpret physically these geometrical properties of geodesics in the framework of the twin paradox: which twin's worldline makes him the oldest one. The mathematical apparatus applied to deal with the global maximality problems is described in our previous paper [3] and we refer the Reader to it.

We emphasize that the search for both locally and globally maximal geodesics is doubly limited. Firstly, an exact analytic expression for the geodesic is necessary and since it is a solution to the nonlinear system of equations, it is available only in a narrow class of spacetimes having sufficiently high symmetries. Secondly, the explicit form of the geodesic is used in the geodesic deviation equation, which is nonlinear in the tangent vector to the geodesic making the equation quite complicated. All timelike geodesic solutions in Schwarzschild spacetime are known and are given in terms of Weierstrass elliptic functions (see [4] and references therein). One should not expect that the equation might be effectively solved when these functions appear in it.

For this reason, we consider only static spherically symmetric spacetimes. For their metrics, one has two classes of physically distinguished and analytically simple timelike geodesics: radial and circular (if exist) ones. Staticity not only considerably simplifies all calculations, moreover, it allows for a physically meaningful notion of rest[1]. In an SSS spacetime, we introduce three twins: twin A remains at rest on a nongeodesic worldline, twin B revolves on a circular orbit (geodesic or not) around the centre of spherical symmetry and twin C moves upwards and downwards following a radial geodesic. The twins' worldlines emanate from a common initial point and we study under what conditions they will intersect in the future. We make detailed calculations in three SSS spacetimes.

[1] An unambiguous notion of rest may also be defined in some time-dependent spacetimes, *e.g.* in Robertson–Walker world.

The paper is organized as follows. In Section 2 we present the geodesic deviation equation expressed in terms of a suitably chosen vector basis on the geodesic and its first integrals generated by the Killing vector fields. Section 3 deals with the problem of maximal geodesics in de Sitter spacetime, in Section 4 the same problem is studied in anti-de Sitter space and in Section 5 — in Bertotti–Robinson spacetime. Conclusions inferred from these cases are formulated in Section 6. In a following paper we consider other spacetimes with high symmetry, first of all the Reissner–Nordström one.

2. Equations for the Jacobi vector fields

Here, we summarize for the Reader's convenience, the formalism necessary for the search of locally maximal timelike geodesics (*cf.* [3]). A Jacobi field on a given timelike geodesic γ with a unit tangent vector field $u^\alpha(s)$ is any vector field $Z^\mu(s)$ being a solution of the geodesic deviation equation on γ,

$$\frac{D^2}{ds^2}\, Z^\mu = R^\mu{}_{\alpha\beta\gamma}\, u^\alpha\, u^\beta\, Z^\gamma \tag{1}$$

which is orthogonal to the geodesic, $Z^\mu\, u_\mu = 0$. One replaces the second absolute derivative D^2/ds^2 by the ordinary one by expanding Z^μ in a basis consisting of three spacelike orthonormal vector fields $e_a{}^\mu(s)$, $a = 1, 2, 3$ on γ, which are orthogonal to γ and are parallelly transported along the geodesic, *i.e.* (the signature is $(+---)$)

$$e_a{}^\mu\, e_{b\mu} = -\delta_{ab}\,, \qquad e_a{}^\mu\, u_\mu = 0\,, \qquad \frac{D}{ds} e_a{}^\mu = 0\,. \tag{2}$$

In this basis, $Z^\mu = \sum_a Z_a e_a{}^\mu$ and the covariant vector equation (1) is reduced to three scalar second order ODEs for the scalar coefficients $Z_a(s)$,

$$\frac{d^2}{ds^2}\, Z_a = -e_a^\mu\, R_{\mu\alpha\beta\gamma}\, u^\alpha\, u^\beta \sum_{b=1}^{3} Z_b\, e_b{}^\gamma\,. \tag{3}$$

A general Jacobi field depends on 6 integration constants appearing as a result of solving (3).

Any Killing vector field K^μ of the spacetime generates a first integral of Eq. (1) of the form [5]

$$K_\mu \frac{D}{ds} Z^\mu - Z^\mu \frac{D}{ds} K_\mu = \text{const}\,. \tag{4}$$

By applying the derivative D/ds to the function on the LHS of (4), one verifies that it is constant along the given geodesic. Also the integral of

motion may be recast in terms of the scalars Z_a. To this end, one introduces a spacetime tetrad ${e_A}^\mu$, $A = 0, 1, 2, 3$, along γ consisting of the spacelike vectors ${e_a}^\mu(s)$ supplemented by ${e_0}^\mu \equiv u^\mu$. The tetrad is orthonormal,

$$ {e_A}^\mu \, e_{B\mu} = \eta_{AB} = \mathrm{diag}(1, -1, -1, -1) \tag{5} $$

and parallelly transported along γ. Expanding Z^μ and K^μ in the tetrad, $K^\mu = \sum_{A=0}^{3} K_A \, {e_A}^\mu$ with the scalars K_A defined by $K^\mu \, e_{A\mu} = \eta_{AA} K_A$ (no summation) and inserting them into (4) one gets

$$ \sum_{a=1}^{3} \left(Z_a \, \frac{dK_a}{ds} - \frac{dZ_a}{ds} \, K_a \right) = \mathrm{const} \, , \tag{6} $$

where $K_a = -K^\mu \, e_{a\mu}$. If the spacetime admits n linearly independent Killing vector fields, one gets n integrals of motion (6). In a number of cases, we find that some of these integrals are trivial, *i.e.* may be found without the use of the appropriate Killing fields and have already been employed at the very beginning of solving the relevant equations, whereas some other first integrals generated by independent Killing vectors turn out to be dependent. Nevertheless, in general, the first integrals (6) are essential in the search for the Jacobi fields.

3. De Sitter spacetime

We use the coordinates in which the spacetime is explicitly static [6–8],

$$ ds^2 = \left(1 - H^2 r^2\right) \, dt^2 - \left(1 - H^2 r^2\right)^{-1} \, dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right) \, , \tag{7} $$

$t \in (-\infty, \infty)$, $0 \leq r < 1/H$, t and r have dimension of length. This chart covers only a part of the whole manifold; the spaces $t = \mathrm{const}$ are halves of three-spheres of constant radius $1/H$ and the spacetime is spherically symmetric. The Killing vector field $\partial/\partial t$ is timelike in the domain of the chart. The surface $r = 1/H$ is a coordinate singularity and its part $t = -\infty$ is a past event horizon for an observer staying at $r = 0$, whereas the part $t = +\infty$ is a future event horizon for the observer. The fact that the spacetime expands has a considerable influence on long journeys and on the possibility of communication during these journeys [9].

The static twin A remains at $r = r_0 > 0$, $\theta = \pi/2$, $\phi = \phi_0$, whereas the twin B moves on a circular orbit $r = r_0$ in the 2-surface $\theta = \pi/2$ with angular velocity $\omega = d\phi/dt$. The general geodesic equation for the radial coordinate

$$ -\frac{\ddot{r}}{1 - H^2 r^2} - \frac{H^2 r \dot{r}^2}{\left(1 - H^2 r^2\right)^2} + H^2 r \dot{t}^2 + r \dot{\theta}^2 + r \dot{\phi}^2 \sin^2 \theta = 0 \, , \tag{8} $$

where $\dot{f} \equiv df/ds$ throughout the paper, shows that circular worldlines $r = r_0 > 0$ cannot be geodesic curves, what is in concordance with the expansion of the spacetime. We assume for B that its nongeodesic motion is $\phi = \omega t + \text{const}$ with $\omega = \text{const} > 0$, then $\dot{\phi} = \omega \dot{t}$.

All the motions take place in the 'plane' $\theta = \pi/2$, hence the universal integral of motion is

$$g_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta = \left(1 - H^2 r^2\right)\, \dot{t}^2 - \frac{\dot{r}^2}{1 - H^2 r^2} - r^2 \dot{\phi}^2 = 1\,. \tag{9}$$

For the circular B's worldline the integral (9) yields

$$\dot{t}^2 = \left[1 - r_0^2\left(H^2 + \omega^2\right)\right]^{-1} \tag{10}$$

and this relation imposes an upper limit on ω,

$$\omega < \frac{1}{r_0}\sqrt{1 - H^2 r_0^2}\,. \tag{11}$$

Assuming that (11) holds and denoting

$$\beta \equiv \left[1 - r_0^2\left(H^2 + \omega^2\right)\right]^{-1/2} \tag{12}$$

one gets for B

$$t(s) - t_0 = \beta s \qquad \text{and} \qquad \phi - \phi_0 = \omega(t - t_0)\,. \tag{13}$$

The coordinate time period T of the B's circulation follows from $\phi(t_0 + T) = \phi_0 + 2\pi$ and is $T = 2\pi/\omega$, hence the proper time measured by B after making one full circle satisfies $T = \beta s_{\mathrm{B}}(T)$ and is

$$s_{\mathrm{B}}(T) = \frac{2\pi}{\beta\omega}\,. \tag{14}$$

The length of the static A's worldline in the period T is

$$s_{\mathrm{A}}(T) = \int\limits_{t_0}^{t_0+T} \sqrt{1 - H^2 r_0^2}\, dt = \frac{2\pi}{\omega}\sqrt{1 - H^2 r_0^2}\,.$$

Comparison of the lengths

$$\frac{s_{\mathrm{A}}(T)}{s_{\mathrm{B}}(T)} = \left(\frac{1 - H^2 r_0^2}{1 - r_0^2(H^2 + \omega^2)}\right)^{1/2} > 1 \tag{15}$$

confirms in this case the conjecture in [10, 11] that the moving faster twin is younger at the reunion than the static twin. In general, however, the conjecture is false.

The radially moving twin C has $\phi = \phi_0$ and the timelike Killing vector $K^\alpha = \delta^\alpha_0$ (normalized to 1 at the coordinate singularity $r = 0$) generates for its geodesic worldline the integral of energy $K^\alpha p_\alpha = E/c$, where p^α is the C's four-momentum. If the twin C has mass m, one defines a dimensionless constant of energy, $k \equiv E/(mc^2)$ and then

$$\dot{t} = \frac{k}{1 - H^2 r^2}\,. \tag{16}$$

Inserting (16) into (8) and (9), one gets for a radial geodesic

$$\left(1 - H^2 r^2\right) \ddot{r} + H^2 r \left(\dot{r}^2 - k^2\right) = 0 \tag{17}$$

and

$$\dot{r}^2 = H^2 r^2 + k^2 - 1\,. \tag{18}$$

The geodesic C emanates from the initial point $P_0(t = t_0,\ r = r_0 > 0,\ \phi = \phi_0)$. If $\dot{r}(t_0) = 0$ then $k^2 = 1 - H^2 r_0^2$ and $\dot{r} \neq 0$ at later times implies $r > r_0$, *i.e.* the twin C will be eternally receding to infinity and will never return. We, therefore, assume that at t_0 it starts radially inwards with velocity $\dot{r}(t_0) = -u < 0$. From (18), u and k are related by

$$k^2 = u^2 - H^2 r_0^2 + 1\,. \tag{19}$$

If $k = 1$ the centre $r = 0$ may be asymptotically reached for the proper time s and coordinate time t tending to infinity since $r = r_0 \exp(-Hs)$. For $k > 1$ the twin crosses the centre and moves outwards at $\phi = \phi_0 + \pi$. We, therefore, assume $k < 1$. Under this assumption, at $t = t_m$ the twin reaches the smallest distance from the centre, $r = r_m$, where $\dot{r}(t_m) = 0$ and the spacetime expansion makes it move outwards; at $t = t_1$ it returns to $r = r_0$ and this is spacetime point $P_1(t = t_1,\ r = r_0,\ \theta = \pi/2,\ \phi = \phi_0)$. Clearly $t_1 - t_m = t_m - t_0$ and the radial journey duration is $\Delta t = t_1 - t_0 = 2(t_m - t_0)$. The lowest point r_m is determined by $0 = H^2 r_m^2 + k^2 - 1$, hence

$$r_m = \frac{1}{H}\sqrt{1 - k^2} \tag{20}$$

and one sees from (19) that indeed $r_m < r_0$. The integral of energy on C is confined to the interval $\sqrt{1 - H^2 r_0^2} < k < 1$.

The geodesic C from P_0 to P_1, *i.e.* on both the ingoing and outgoing segment, may be continuously parametrized as

$$r = r_m \cosh\eta\,, \qquad \eta \in [-\alpha, \alpha]\,, \qquad \alpha > 0\,. \tag{21}$$

By definition, $r(-\alpha) = r(\alpha) = r_0$, then $\cosh\alpha = r_0/r_m$. Inserting (21) into (18) and applying (20), one finds $d\eta/ds = H$. The length of C is counted from P_0, where $\eta = -\alpha$, then $\eta = Hs - \alpha$. The dependence $t(s)$ on the geodesic follows from (16),

$$t(s) = \frac{k}{H}\int \frac{d\eta}{1-(1-k^2)\cosh^2\eta} = \frac{1}{H}\ \mathrm{artanh}\left[\frac{1}{k}\tanh(Hs-\alpha)\right] + \mathrm{const}\,. \tag{22}$$

This function is well defined since the argument satisfies the inequality $|\frac{1}{k}\tanh\eta| < 1$. In fact, $|\tanh\eta| \le \tanh\alpha$ and one infers from (20) that

$$1 - \frac{1}{k^2}\tanh^2\alpha = 1 - \frac{r_0^2 - r_m^2}{k^2 r_0^2} = \frac{1-k^2}{k^2H^2r_0^2}\left(1 - H^2r_0^2\right) > 0\,.$$

Inverting the function (21), one gets the length of C from r_0 to r_m

$$s(r_0, r_m) = \frac{\alpha}{H} = \frac{1}{H}\ \mathrm{arcosh}\,\frac{r_0}{r_m} \tag{23}$$

and the length of the geodesic from P_0 to P_1 expressed in terms of r_0 and k is

$$s_{\mathrm{C}} = 2s(r_0, r_m) = -\frac{1}{H}\ln\left(1-k^2\right) + \frac{2}{H}\ln\left[Hr_0 + \sqrt{H^2r_0^2 + k^2 - 1}\right]\,. \tag{24}$$

Applying properties of hyperbolic functions, one arrives at

$$\begin{aligned} t(\eta) - t_0 &= \frac{1}{H}\ \mathrm{artanh}\left(\frac{1}{k}\tanh\eta\right) + \frac{1}{H}\ \mathrm{artanh}\left(\frac{1}{k}\tanh\alpha\right) \\ &= \frac{1}{H}\ \mathrm{artanh}\left(\frac{k(\tanh\eta + \tanh\alpha)}{k^2 + \tanh\eta\,\tanh\alpha}\right) \end{aligned} \tag{25}$$

and the coordinate time of flight from P_0 to P_1 is

$$\Delta t = 2(t_m - t_0) = \frac{2}{H}\ \mathrm{artanh}\left(\frac{1}{k}\tanh\alpha\right) = \frac{1}{H}\ln\frac{k+\tanh\alpha}{k-\tanh\alpha}\,, \tag{26}$$

where

$$\tanh\alpha = \frac{1}{Hr_0}\sqrt{H^2r_0^2 + k^2 - 1}\,. \tag{27}$$

The three twins depart from P_0, yet one cannot assume that they will meet together at P_1. A and C will meet at this event whereas, in general, B and C cannot. We shall discuss the latter problem below, now we compare the

proper times of A and C. The length of A's worldline between P_0 and P_1 is $s_{\rm A}(\Delta t) = (1 - H^2 r_0^2)^{1/2} \Delta t$ and

$$\frac{s_{\rm C}}{s_{\rm A}(\Delta t)} = \left(1 - H^2 r_0^2\right)^{-1/2} \left[2 \ln(H r_0 + Q) - \ln\left(1 - k^2\right)\right] \left(\ln \frac{k H r_0 + Q}{k H r_0 - Q}\right)^{-1}, \tag{28}$$

where $Q \equiv \sqrt{H^2 r_0^2 + k^2 - 1}$. A number of numerical examples confirms the expectation that always $s_{\rm C} > s_{\rm A}(\Delta t)$, *e.g.* for $H r_0 = 0.99$ and $k = 0.5$ the ratio is $s_{\rm C}/s_{\rm A} = 2.2654$.

The twins B and C will meet at P_1 if B's angular velocity is $\omega = 2\pi/\Delta t$ for Δt given in (26) and if ω is smaller than the upper limit (11) what amounts to

$$\ln \frac{k H r_0 + Q}{k H r_0 - Q} > \frac{2\pi H r_0}{\sqrt{1 - H^2 r_0^2}}. \tag{29}$$

For fixed r_0 it is a restriction of the C's energy k. In the example above, $H r_0 = 0.99$ and $k = 0.5$ the twin B cannot meet C after one revolution. The two twins will meet at P_1 if the energy k is sufficiently close to 1: if $k = 1 - \varepsilon$ with $0 < \varepsilon \ll 1$ then

$$\ln \frac{k H r_0 + Q}{k H r_0 - Q} \approx \ln \left(\frac{2 H r_0}{\varepsilon} - \frac{1 + H^2 r_0^2}{1 - H^2 r_0^2}\right)$$

and inequality (29) holds.

3.1. Jacobi fields on the timelike geodesics in de Sitter space

Timelike geodesics in de Sitter spacetime have no conjugate points. In fact, according to Proposition 4.4.2 in [12] (cited as Proposition 4 in [2]) the necessary conditions are not satisfied: whereas the condition $R_{\mu\alpha\nu\beta}\, u^\alpha u^\beta = \frac{R}{12}(g_{\mu\nu} - u_\mu u_\nu) \neq 0$ holds for any timelike geodesic, the expression $R_{\alpha\beta} u^\alpha u^\beta = R/4 = -3H^2$ is always negative. For completeness, we also investigate the existence of conjugate points on null geodesics. According to Proposition 4.4.5 in [12], the null tangent vector k^α should satisfy two necessary conditions: $R_{\alpha\beta} k^\alpha k^\beta \geq 0$ and it holds since the scalar is zero and $k^\mu k^\nu k_{[\alpha} R_{\beta]\mu\nu[\lambda} k_{\sigma]} \neq 0$ at a point and this does not hold since the tensor vanishes identically. Hence, de Sitter spacetime has no future (nor past) nonspacelike conjugate points and because the other two assumptions of Theorem 11.16 in [13] (cited as Theorem 6 in [3]) concerning the spacetime hold, one concludes that each timelike geodesic is the unique longest (*i.e.* maximal) curve connecting its endpoints (it contains no cut points). In other terms, the endpoint P_1 is in a convex normal neighbourhood of an arbitrarily chosen point P_0 (assuming $P_0 \prec\prec P_1$).

We now determine a general Jacobi field on any timelike geodesic. In de Sitter space, the geodesic deviation equation (3) for the scalar coefficients $Z_a(s)$ is the same for all geodesic curves, whether radial or not, and for any choice of the spacelike triads on them (providing they satisfy (2)), and has the form

$$\frac{d^2 Z_a}{ds^2} - H^2\, Z_a = 0\,. \tag{30}$$

The generic solution is

$$Z_a = C_a\, e^{Hs} + C'_a\, e^{-Hs}\,. \tag{31}$$

Then, the general deviation vector field on arbitrary timelike geodesic vanishing at the initial point $P_0(s=0)$ is

$$Z^\mu(s) = \sum_{a=1}^{3} C_a\, e_a{}^\mu(s) \sinh Hs \tag{32}$$

and it is clear that there are no conjugate points on it since the neighbouring geodesics exponentially diverge. The gravitation in this spacetime is repulsive.

Finally, we explicitly determine the general Jacobi field on a radial timelike geodesic. For generality, we consider the geodesic followed by the twin C, *i.e.* consisting of the ingoing segment and the outgoing one. The vector tangent to C is, from (16), (18) and (21),

$$\begin{aligned} u^\alpha &= \left[\frac{k}{1-H^2r^2}, \varepsilon\sqrt{H^2r^2+k^2-1}, 0, 0\right] \\ &= \left[\frac{k}{1-H^2r_m^2\cosh^2\eta}, \sqrt{1-k^2}\sinh\eta, 0, 0\right] \end{aligned} \tag{33}$$

with $\eta = Hs - \alpha$ and $\varepsilon = -1$ on the ingoing segment, $-\alpha \le \eta < 0$ and $\varepsilon = +1$ on the outgoing piece, $0 < \eta \le \alpha$. The basis vector triad on C satisfying (2) is chosen as

$$\begin{aligned} e_1{}^\mu &= \left[\frac{\sqrt{1-k^2}}{1-H^2r_m^2\cosh^2\eta}\,\sinh\eta, k, 0, 0\right], \\ e_2{}^\mu &= \left[0, 0, \frac{1}{r_m\cosh\eta}, 0\right], \\ e_3{}^\mu &= \left[0, 0, 0, \frac{1}{r_m\cosh\eta}\right], \end{aligned} \tag{34}$$

$-\alpha \le \eta \le +\alpha$. One inserts (34) into (32).

4. Anti-de Sitter spacetime

Actually, we consider the covering anti-de Sitter (CAdS) spacetime and use the chart covering the entire manifold and exhibiting its static nature; the radial coordinate is suitably chosen to our purposes [8, 12, 14]

$$ds^2 = \frac{r^2+a^2}{a^2}\,dt^2 - \frac{a^2}{r^2+a^2}\,dr^2 - r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\,, \tag{35}$$

where $t \in (-\infty,+\infty)$, $r \in [0,\infty)$, t, r and a have dimension of length. Hypersurfaces of simultaneity $t = \text{const}$ are the hyperbolic Lobatchevski spaces H^3. There are no horizons and $r = 0$ is a coordinate singularity.

The nongeodesic static twin A stays at $r = r_0 > 0$ and $\phi = \phi_0$ and in a coordinate time interval T its worldline has length

$$s_{\rm A}(T) = \sqrt{1+\left(\frac{r_0}{a}\right)^2}\,T\,. \tag{36}$$

Any geodesic motion takes place in the 2-surface $\theta = \pi/2$. For a timelike geodesic, one has the integral of energy E generated by the timelike Killing field $K^\alpha = \delta^\alpha_0$ (normalized to 1 at $r = 0$), $K^\alpha p_\alpha = E/c$, then $k \equiv E/(mc^2)$ and

$$\dot t \equiv \frac{dt}{ds} = \frac{a^2k}{r^2+a^2}\,. \tag{37}$$

The rotational Killing field $\partial/\partial\phi$ with components $\xi^\alpha = \delta^\alpha_3$ is normalized as in Minkowski space and gives rise to conserved angular momentum $J = -\xi^\alpha p_\alpha$. Introducing a dimensionless angular momentum h by $ah = J/(mc)$, one gets

$$\dot\phi = \frac{ah}{r^2}\,. \tag{38}$$

The geodesic equation for the radial coordinate, upon employing (37) and (38) is

$$\left(r^2+a^2\right)\ddot r - r\dot r^2 + k^2 r - \frac{h^2}{r^3}\left(r^2+a^2\right)^2 = 0 \tag{39}$$

and the universal integral of motion $g_{\alpha\beta}\,\dot x^\alpha \dot x^\beta = 1$ reads

$$\dot r^2 = k^2 - \frac{r^2+a^2}{a^2} - \frac{h^2}{r^2}\left(r^2+a^2\right)\,. \tag{40}$$

A circular geodesic at any $r = r_0 > 0$ does exist and is stable [3] and the constants of motion are, in this case, determined from (39) and (40) by a and r_0 as

$$k = \frac{r_0^2+a^2}{a^2} \qquad \text{and} \qquad h = \frac{r_0^2}{a^2}\,. \tag{41}$$

As the starting point for the three worldlines we choose $P_0(t_0=0, r=r_0>0, \theta=\pi/2,\ \phi=\phi_0)$ then the circular geodesic B has the form

$$t = s \qquad \text{and} \qquad \phi - \phi_0 = \frac{s}{a} = \frac{t}{a}\,, \tag{42}$$

what implies constant angular velocity $\omega = d\phi/dt = 1/a$. The period of one revolution is $T = 2\pi a$ and the length of geodesic B corresponding to one circle, *i.e.* between $t = 0$ and $t = T$ is $s_{\rm B} = T = 2\pi a$. Both T and $s_{\rm B}$ are the same for all circular geodesic curves independently of the radius r_0; it is a trace of the original anti-de Sitter spacetime where all timelike curves are closed with a period $2\pi a$. Clearly $s_{\rm A}(2\pi a) > s_{\rm B}$.

The third twin C moves on a radial geodesic $h = 0$ and the equations describing it are reduced to

$$\left(r^2 + a^2\right)\ddot{r} - r\dot{r}^2 + k^2 r = 0 \tag{43}$$

and

$$\dot{r}^2 = k^2 - \frac{r^2 + a^2}{a^2}\,. \tag{44}$$

Let at P_0 the twin C be initially at rest, $\dot{r}(t=0) = 0$, then its energy k is given by $k^2 = r_0^2/a^2 + 1$ and its acceleration is directed downwards, $\ddot{r}(0) = -r_0/a^2 < 0$, implying falling down. This shows that gravitation in CAdS is attractive: a body left at rest falls radially to the centre, reaches the centre $r = 0$ and flies away in the opposite direction $\phi = \phi_0 + \pi$. From (44) it follows $\dot{r}^2 = (r_0^2 - r^2)/a^2$ and this implies $r \le r_0$. At the antipodal point $r = r_0$, $\phi = \phi_0 + \pi$ (denoted below as P_3) there is again $\dot{r} = 0$ and $\ddot{r} = -r_0/a^2$ and the body falls down back and returns to the starting point at the space, $r = r_0$, $\phi = \phi_0$. We, therefore, assume that C moves as in Schwarzschild spacetime: it radially flies away with initial velocity $\dot{r}(0) = u > 0$, reaches a maximum height $r = r_{\rm M}$ at $t = t_{\rm M}$ and falls down back to r_0 at the event $P_1(t = t_1,\ r = r_0,\ \phi = \phi_0)$ where $t_1 = 2t_{\rm M}$. The quantities r_0, k and u are now related by

$$u^2 = \frac{1}{a^2}\left(a^2k^2 - r_0^2 - a^2\right) \tag{45}$$

and the highest point of the trajectory is

$$r_{\rm M}^2 = a^2\left(k^2 - 1\right)\,. \tag{46}$$

The condition $r_{\rm M} > r_0 > 0$ implies

$$k^2 > \frac{r_0^2 + a^2}{a^2}\,. \tag{47}$$

One sees from (46) that $r_{\mathrm{M}} < \infty$ for $k < \infty$ what implies that a massive particle with finite energy cannot escape to the spatial infinity $r = \infty$ ([8] (paragraph 5.2), [15]). This property of CAdS is in marked contrast to Schwarzschild spacetime, where the corresponding relationship is [2] $r_{\mathrm{M}} = 2M/(1-k^2)$ and r_{M} tends to infinity for $k \to 1$ from below.

On the segment P_0P_1 (and possibly outside it), the radial geodesic C is conveniently parametrized by an angle η,

$$r(\eta) = r_{\mathrm{M}} \cos^2 \eta \equiv \tfrac{1}{2} r_{\mathrm{M}} (\cos 2\eta + 1) \tag{48}$$

in the interval $-\alpha/2 \leq \eta \leq +\alpha/2$. Then $r_{\mathrm{M}} = r(0)$ and the endpoints P_0 and P_1 correspond to $r_0 = r(-\alpha/2) = r(+\alpha/2)$ and the boundary angle α is determined by

$$\cos\alpha = \frac{2r_0}{r_{\mathrm{M}}} - 1 \tag{49}$$

and $\cos\alpha$ is bounded from above by

$$\cos\alpha = \frac{1}{r_{\mathrm{M}}}(2r_0 - r_{\mathrm{M}}) = \frac{1}{r_{\mathrm{M}}}\left(2r_0 - a\sqrt{k^2-1}\right) < \frac{1}{r_{\mathrm{M}}}\left(2r_0 - a\sqrt{\frac{r_0^2}{a^2}}\right) = \frac{r_0}{r_{\mathrm{M}}},$$

so that $\arccos \frac{r_0}{r_{\mathrm{M}}} < \alpha < \pi$. The geodesic C consists of the outgoing segment, $\eta \in [-\alpha/2, 0)$ and the ingoing one, $\eta \in (0, +\alpha/2]$. The time component of the vector u^α tangent to C is given in (37); its radial component is from (44) $\dot{r} = \varepsilon(k^2 - r^2/a^2 - 1)^{1/2}$, where $\varepsilon = +1$ on the outgoing segment and $\varepsilon = -1$ on the ingoing one. Applying (48) in $\dot{r}$ and noticing that $\varepsilon(\sin^2\eta)^{1/2} = -\sin\eta$ both on the outgoing and ingoing segment, one finally arrives at

$$u^\alpha \equiv \dot{x}^\alpha = \left[\frac{k}{(k^2-1)\cos^4\eta + 1}, -\left(k^2-1\right)^{1/2} \sin\eta \left(1 + \cos^2\eta\right)^{1/2}, 0, 0\right] \tag{50}$$

valid along the whole geodesic line. The derivative $ds/d\eta$ along C may be found from the expression for ds^2 on C by inserting $dr/d\eta = -r_{\mathrm{M}} \sin 2\eta$ and $dt/d\eta = dt/ds \times ds/d\eta$ and applying (37); the resulting equation is solved by

$$\frac{ds}{d\eta} = \frac{2a|\cos\eta|}{(1+\cos^2\eta)^{1/2}}. \tag{51}$$

If $\sin\eta$ is a monotonic function in an interval $\eta_1 < \eta < \eta_2$ the length of the corresponding geodesic arc is

$$s(\eta_1, \eta_2) = \int_{\eta_1}^{\eta_2} ds = 2\sigma a \left[\arcsin\left(\frac{\sin\eta_2}{\sqrt{2}}\right) - \arcsin\left(\frac{\sin\eta_1}{\sqrt{2}}\right)\right], \tag{52}$$

where $\sigma = +1$ if $\cos\eta > 0$ in the interval and $\sigma = -1$ for $\cos\eta < 0$. Then the length of the geodesic C from P_0 to P_1 is

$$\begin{aligned} s_{\mathrm{C}} &= s\left(-\frac{\alpha}{2}, +\frac{\alpha}{2}\right) = 2s\left(-\frac{\alpha}{2}, 0\right) = 2s_{\mathrm{M}} = 4a \arcsin\left(\frac{\sin\frac{\alpha}{2}}{\sqrt{2}}\right) \\ &= 2a \arccos\left(\frac{r_0}{r_{\mathrm{M}}}\right), \end{aligned} \tag{53}$$

where $r_0/r_{\mathrm{M}} = r_0/(a\sqrt{k^2-1})$. To calculate the coordinate time interval corresponding to a given segment of the radial geodesic one writes $dt/d\eta = dt/ds \times ds/d\eta$ and inserts (37), (48) and (51), then

$$\frac{dt}{d\eta} = -\frac{8\varepsilon a k \sin 2\eta}{(k^2-1)(\cos 2\eta + 1)^2 + 4}\left[4 - (\cos 2\eta + 1)^2\right]^{-1/2}, \tag{54}$$

here as above $\varepsilon = +1$ on the outgoing segment ($\sin 2\eta < 0$) and $\varepsilon = -1$ on the outgoing segment ($\sin 2\eta > 0$). One then gets

$$\Delta t(\eta_1, \eta_2) = \int_{\eta_1}^{\eta_2} \frac{dt}{d\eta}\, d\eta = \varepsilon a \left[\arctan(f(\eta_2)) - \arctan(f(\eta_1))\right], \tag{55}$$

here

$$f(\eta) \equiv \frac{k(\cos 2\eta + 1)}{\sqrt{4 - (\cos 2\eta + 1)^2}}.$$

The time of flight from P_0 to P_1 is

$$\begin{aligned} t_1 &= \Delta t\left(-\frac{\alpha}{2}, +\frac{\alpha}{2}\right) = 2t_{\mathrm{M}} = \pi a - 2a \arctan\left(\frac{kr_0}{\sqrt{r_{\mathrm{M}}^2 - r_0^2}}\right) \\ &= 2a \arccos\left(\frac{kr_0}{(k^2-1)^{1/2}\left(a^2 + r_0^2\right)^{1/2}}\right). \end{aligned} \tag{56}$$

For $r_0 \to \infty$, one has $k^2 \to (r_0/a)^2 \to \infty$ and $r_0/r_{\mathrm{M}} \to 1$, hence $s_{\mathrm{C}} \to 0$ and $t_1 \to 0$. CAdS spacetime has the peculiar feature that both the length of any radial timelike geodesic (emanating from $r_0 < \infty$ and consisting of the outgoing segment and ingoing one returning to r_0) and the coordinate time of flight on it are bounded from above: $t_1 < \pi a$ and $s_{\mathrm{C}} < \pi a$; correspondingly a radial geodesic consisting of one outgoing or ingoing segment from r_0 to r_{M} has $s(r_0, r_{\mathrm{M}}) < \pi a/2$. These upper limits correspond to $r_0 \to 0$

independently of r_{M}. Yet the spatial distance (the length of a radial spacelike geodesic at $t = \text{const}$) from any finite r_0 to $r = \infty$ is infinite. Furthermore, it has been shown that in the 2-surface (t, r) there exist points p and q which are chronologically related (q lies inside the future null cone of p) and such that there is no timelike (and necessarily radial) geodesic joining them [13] (Chap. 6), [15–17].

The twins B and C start from the same place, yet they will not meet again after making one circle and one radial flight, respectively, since $T = 2\pi a > t_1$. There are, however, two cases in which they can reunion.

1. Let C make a number of radial flights back and forth in such a way that at $r = r_0$ it bounces, *i.e.* rapidly alters its radial velocity from $-u$ to $+u$. Its worldline consists of a number of smooth geodesic segments which are non-smoothly joined at r_0, it forms a broken geodesic. Moreover, let the energy k of C be suitably tuned (for fixed r_0) so that

$$\frac{kr_0}{\left(k^2-1\right)^{1/2}\left(a^2+r_0^2\right)^{1/2}} = \cos\left(\frac{m}{n}\pi\right) ,$$

where m and n are positive integers and $m < n$. Then, duration of n consecutive radial flights is $nt_1 = 2m\pi a = mT$, the duration of orbiting m full circles by B. When they meet at r_0 at $t = nt_1$ their proper times are ns_{C} and ms_{B} and their difference is

$$ns_{\mathrm{C}} - ms_{\mathrm{B}} = 2na\left(\arccos\frac{r_0}{r_{\mathrm{M}}} - \frac{m}{n}\pi\right) .$$

We compare the two angles by taking the ratio of their cosines,

$$\cos\left(\arccos\frac{r_0}{r_{\mathrm{M}}}\right)\left[\cos\left(\frac{m}{n}\pi\right)\right]^{-1} = \left(\frac{r_0^2+a^2}{a^2k^2}\right)^{1/2} < 1$$

according to (47) and one infers that $m\pi/n$ is the smaller angle and $ns_{\mathrm{C}} - ms_{\mathrm{B}} > 0$.

2. A physically more interesting opportunity is that the falling down twin C is not stopped at r_0 and is allowed to freely move farther. Then the whole sequence of events is following:
C starts from r_0 at $P_0(t = 0, \eta = -\alpha/2)$, reaches r_{M} at t_{M} and falls down, comes back to r_0 at $P_1(t = t_1, \eta = +\alpha/2)$, arrives at the centre at $P_2(t = t_2, \eta = \pi/2)$, crosses it and radially flies upwards at $\phi = \phi_0 + \pi$, passes by the opposite point r_0 at $P_3(t = t_3, \eta = \pi - \alpha/2)$, gets to the highest point r_{M} at $P_4(t = t_4, \eta = \pi)$ and turns downwards, falls down to r_0 at $P_5(t = t_5, \eta = \pi + \alpha/2)$, comes back to the centre $r = 0$ at $P_6(t = t_6, \eta = 3\pi/2)$ and finally returns to the starting place $r = r_0$ and $\phi = \phi_0$ at $P_7(t = t_7, \eta = 2\pi - \alpha/2)$ with the initial velocity $+u$.

The staticity of the metric gives rise to the symmetry properties of the segments of this worldline. Employing (52), (55) and (56), one gets the coordinate time intervals:

$$t_2 - t_1 \;=\; t_3 - t_2 = t_6 - t_5 = t_7 - t_6 = a \arctan\left(\frac{kr_0}{\sqrt{r_{\rm M}^2 - r_0^2}}\right), \quad (57)$$

$$t_4 - t_3 \;=\; t_5 - t_4 = t_{\rm M} \quad (58)$$

and the lengths of the corresponding geodesic segments,

$$s(P_1P_2) \;=\; s(P_2P_3) = s(P_5P_6) = s(P_6P_7) = \frac{\pi}{2}a - a \arccos\left(\frac{r_0}{r_{\rm M}}\right), \quad (59)$$

$$s(P_0P_1) \;=\; s_{\rm C} = 2s_{\rm M}\,, \qquad s(P_3P_4) = s(P_4P_5) = s_{\rm M}\,. \quad (60)$$

Adding these seven segments, one finds the time duration of the full cycle and the length of the geodesic,

$$\Delta t\left(-\frac{\alpha}{2}, 2\pi - \frac{\alpha}{2}\right) = t_7 = 2\pi a\,, \qquad s(P_0P_7) = 2\pi a\,. \quad (61)$$

Both the circular geodesic B and the radial geodesic C emanating from any point $r_0 > 0$ reconverge at P_7 at the coordinate time $T = t_7 = 2\pi a$ having the same length $2\pi a$ and then at $t = 4\pi a, 6\pi a, \ldots$ [14, 15] independently of the initial velocity of C. Once again, we emphasize that this is a trace of the original anti-de Sitter spacetime which is periodic in time, *i.e.* events $(t = 0, r, \theta, \phi)$ and $(t = 2\pi a, r, \theta, \phi)$ are identified. What is even more interesting, here is that all timelike geodesics emanating from P_0 actually intersect at $P_3(r = r_0,\ \phi = \phi_0 + \pi)$ which is spatially the antipodal point (with respect to the centre) to P_0. In fact, from (57) and (56) one gets $t(P_3) = t_3 = \pi a$ and from (60), (59) and (53) it follows $s(P_0P_3) = \pi a$. By continuity, it follows that the same holds for the radial geodesic which falls down from rest at P_0, *i.e.* $\dot r(0) = 0$. The circular timelike geodesic B also intersects all the radial geodesics at P_3 since from (42) one finds that for $t = t_3 = \pi a$ its angular coordinate is $\phi = \phi_0 + \pi$ and $s_{\rm B}(\pi) = \pi a$. Thus, we have shown analytically that the circular and all radial geodesics (which cross $r = 0$) emanating from P_0 do meet again after $\Delta t = \pi a$ at the antipodal point P_3 and all have the same length. P_3 is the future cut point of P_0 lying on all radial and circular timelike geodesics.

After one radial travel upwards and downwards, the twin C meets A at P_1 and they compare their proper times,

$$\frac{s_{\rm C}}{s_{\rm A}(t_1)} = \frac{1}{\sqrt{\left(\frac{r_0}{a}\right)^2 + 1}} \frac{\arccos\left[\frac{r_0}{a(k^2-1)^{1/2}}\right]}{\arccos\left[kr_0\left(k^2-1\right)^{-1/2}\left(r_0^2 + a^2\right)^{-1/2}\right]}\,. \quad (62)$$

It is not easy to analytically prove that $s_C > s_A$. We do it numerically and in Table I we give the ratio s_C/s_A for $r_0 = a$ and 5 values of k; it follows from (47) that $k^2 > 2$.

TABLE I

The ratio s_C/s_A for $r_0 = a$ as a function of energy k.

k	s_C/s_A
$\frac{10}{9}\sqrt{2}$	1,0331
2	1,0975
$\frac{5}{3}\sqrt{2}$	1,1351
$2\sqrt{2}$	1,1722
$10\sqrt{2}$	1,3547

For $r_M \gg r_0$, *i.e.* for $k \to \infty$ the ratio s_C/s_A tends to $\sqrt{2}$.

4.1. Conjugate points on timelike geodesics

In CAdS spacetime the necessary conditions $R_{\mu\alpha\nu\beta}\, u^\alpha u^\beta = \frac{1}{a^2}(g_{\mu\nu} - u_\mu u_\nu) \neq 0$ and $R_{\alpha\beta}u^\alpha u^\beta = 3/a^2 > 0$ imply that each timelike geodesic contains conjugate points provided it is sufficiently extended. To determine conjugate points on a given geodesic, one does not need to know the Jacobi vector fields associated with this geodesic. This is due to the fact that in CAdS the right-hand side of Eq. (3) is universal (as is the case of de Sitter spacetime): is independent of the form of the tangent vector u^α and the spacelike basis fields $e_b{}^\mu(s)$, $b = 1, 2, 3$ and is determined solely by the curvature tensor and relations (2). This means, in turn, that the Jacobi scalars $Z_b(s)$ are universal and for all timelike geodesics they satisfy

$$\frac{d^2}{ds^2} Z_b + \frac{1}{a^2}\, Z_b = 0 \tag{63}$$

with the general solution (the change of sign in Eq. (30) results in replacing exponential functions by trigonometric ones)

$$Z_b(s) = C_{b1} \sin\frac{s}{a} + C_{b2} \cos\frac{s}{a}\,, \tag{64}$$

C_{b1}, C_{b2} arbitrary constants. Let P_0 be any point on the given geodesic chosen as the initial point ($s = 0$), one seeks for points conjugate to P_0. The triad components Z_b must vanish at P_0 and under this condition they reduce to $Z_b(s) = C_{b1} \sin\frac{s}{a}$. These scalars have an infinite sequence of zeros at $s_n = n\pi a$, $n = 1, 2, \ldots$ In other words, each point on each timelike

geodesic has a point conjugate to it at a geodesic distance $\Delta s = \pi a$ and the sequence of conjugate points is infinite. Each of the three Jacobi vector fields corresponding to $Z_b(s)$ generates the sequence.

In the case of the circular curve B, the first conjugate point to P_0 is that lying in the middle of the geodesic segment corresponding to one full revolution, *i.e.* half way between P_0 and P_7; it coincides with the first future cut point P_3. Further conjugate points (the second one is P_7) at $s_n = n\pi a$ are identical with the subsequent future cut points. In [3] it was shown that if a static spherically symmetric spacetime admits stable timelike circular geodesics, then, in general, there exist on them three distinct infinite sequences of conjugate points. Due to the maximal symmetry of CAdS space, one expects that these sequences should coincide and, in fact, applying appropriate formulae from [3] one easily checks that this is the case.

The same holds for radial timelike geodesics which oscillate between spatially antipodal highest points $r = r_{\rm M}$: the subsequent conjugate points to P_0 coincide with their future cut points. Yet the radial geodesic which does not cross the centre $r = 0$ is free of conjugate points. CAdS spacetime is not globally hyperbolic and the theorems quoted as Theorems 2 to 6 in Section 3 of [3] do not apply. By symmetry considerations one expects that the geodesic C has no future cut points of P_0 earlier than P_3 and is maximal on the segment P_0P_1 whose length is $s_{\rm C} < \pi a$ and this implies that $s_{\rm C} > s_{\rm A}(t_1)$.

4.2. Jacobi fields on timelike radial and circular geodesics

According to (64) for each timelike geodesic the general Jacobi field has the same form

$$Z^\alpha(s) = \sum_{b=1}^{3} \left(C_{b1} \sin\frac{s}{a} + C_{b2} \cos\frac{s}{a} \right) {e_b}^\alpha \,, \tag{65}$$

only the basis vectors ${e_b}^\alpha(s)$ depend on the given curve.

The basis of spacelike vector fields on the radial geodesic C which satisfy (2) may be chosen as

$$\begin{aligned} {e_1}^\alpha &= \left[\frac{\varepsilon a}{r^2 + a^2} \left(a^2 \left(k^2 - 1 \right) - r^2 \right)^{1/2}, k, 0, 0 \right] , \\ {e_2}^\alpha &= \left[0, 0, \frac{1}{r}, 0 \right] , \qquad {e_3}^\alpha = \left[0, 0, 0, \frac{1}{r} \right] , \end{aligned} \tag{66}$$

where $\varepsilon = +1$ on the outgoing segment $(-\alpha/2 \le \eta < 0)$ and $\varepsilon = -1$ on the ingoing one $(0 < \eta \le \pi/2)$ and (48) holds. The component ${e_1}^0$ is continuous at $r = r_{\rm M}$ where it changes its sign since it vanishes there. The Jacobi

vector field $Z_2 {e_2}^\alpha + Z_3 {e_3}^\alpha$ connecting C to a nearby geodesic is directed off the 2-surface $t - r$ given by $\theta = \pi/2$ and the field $Z_1 {e_1}^\alpha$ lies in the surface.

In [3] it was shown that the basis triad of vectors satisfying (2) on the circular geodesic B has a universal form common to all SSS spacetimes, depending on four constants, whose values, in turn, depend, for the given value of r_0, on the metric functions g_{00} and g_{11}. In the present case, these read

$$
\begin{aligned}
{e_1}^\alpha &= \left[-\frac{r_0}{\left(r_0^2+a^2\right)^{1/2}} \sin\frac{s}{a}, \frac{1}{a}\left(r_0^2+a^2\right)^{1/2} \cos\frac{s}{a}, 0, -\frac{1}{ar_0}\left(r_0^2+a^2\right)^{1/2} \sin\frac{s}{a} \right] , \\
{e_2}^\alpha &= \left[0, 0, \frac{1}{r_0}, 0 \right] , \qquad {e_3}^\alpha = -a\,\frac{d}{ds}{e_1}^\alpha . \qquad (67)
\end{aligned}
$$

All the three Jacobi fields are directed off the 2-surface $t - \phi$ where B lies.

5. Bertotti–Robinson spacetime

This spacetime, first discovered by T. Levi-Civita (1917), was independently rediscovered by Bertotti [18] and Robinson [19]. The spacetime is homogeneous and spatially homogeneous, static, spherically symmetric (it admits a 6-dimensional isometry group) and conformally flat and it is a unique spacetime generated by a homogeneous non-null electromagnetic field; it also arises as a near-horizon limit of the non-extremal Reissner–Nordström black hole [20]. It is geodesically complete and it is conjectured that this spacetime and the Melvin solution are the only geodesically complete static Einstein–Maxwell spacetimes [21]; topologically it is $\mathrm{AdS}_2 \times S^2$, thus it is not globally hyperbolic (for a fuller description see [22] par. 12.3 and [8] par. 7.1). We investigate it in the chart

$$
ds^2 = a^2 \left(\sinh^2 x\, dt^2 - dx^2 - d\theta^2 - \sin^2\theta\, d\phi^2 \right) , \qquad (68)
$$

where a has dimension of length, $t \in (-\infty, +\infty)$, $x \in (0, \infty)$, all the coordinates are dimensionless and $x = 0$ is a coordinate singularity. The timelike Killing vector chosen as $K^\alpha = \frac{1}{a}\,\delta_0^\alpha$ becomes null on the hypersurface $x = 0$ which has topology $\mathrm{R}^1 \times S^2$. The conserved energy for a geodesic motion generated by K^α is as usual $K^\alpha p_\alpha = E/c$ and the standard definition $k = E/(mc^2)$ yields the first integral,

$$
\dot{t} \equiv \frac{dt}{ds} = \frac{k}{a \sinh^2 x} , \qquad (69)
$$

we assume $\dot{f} \equiv df/ds$ throughout the paper. Geodesic motions are 'flat', $\theta = \pi/2$ and the angular momentum is conserved too, giving rise to $\dot{\phi} =$

const $\equiv h$, hence $\phi = hs + \phi_0$. In this spacetime each 2-sphere has its area equal to $4\pi a^2$. Yet the metric radius of a sphere $t = \text{const}$ and $x = x_0$, *i.e.* the length of the spatial geodesic (in the 3-space $t = \text{const}$) from the centre to any point of the sphere is ax_0. In this sense, the variable x is interpreted as a radial coordinate and by a circular worldline one means a curve with $x = x_0 > 0$. The geodesic equation for the variable x,

$$\ddot{x} + \dot{t}^2 \sinh x \cosh x = 0\,, \tag{70}$$

excludes the existence of circular geodesics (with $\theta = \pi/2$ and $\dot{\phi} \neq 0$), hence a body in a free fall must approach the centre or recede from it. We shall not consider circular worldlines.

In the case of the twin A staying at $x = x_0 > 0$, $\theta = \pi/2$ and $\phi = \phi_0$, the universal integral of motion (for $\theta = \pi/2$)

$$g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = a^2 \left(\dot{t}^2 \sinh^2 x - \dot{x}^2 - \dot{\phi}^2\right) = 1 \tag{71}$$

implies

$$t(s) - t_0 = \frac{s}{a \sinh x_0}\,. \tag{72}$$

As in Schwarzschild and CAdS spacetimes, gravitation here is attractive and the twin C moves on a radial ($h = 0$) geodesic as in those cases: at $P_0(t = t_0,$ $x = x_0 > 0,\ \phi = \phi_0)$ it flies away outwards with the initial velocity $\dot{x} = u > 0$, reaches maximal height $x = x_{\mathrm{M}}$ at $t = t_{\mathrm{M}}$, falls back and returns to the starting place at $P_1(t_1 = 2t_{\mathrm{M}} - t_0,\ x = x_0)$. For the geodesic C the integral of motion (71) reads, taking into account (69),

$$\dot{x}^2 = \frac{1}{a^2}\left(\frac{k^2}{\sinh^2 x} - 1\right) \tag{73}$$

and

$$k = \left(a^2u^2 + 1\right)^{1/2} \sinh x_0\,. \tag{74}$$

The highest point of the flight is $\sinh x_{\mathrm{M}} = k$, hence $k > \sinh x_0 > 0$. Notice that as in anti-de Sitter spacetime ([8] (par. 5.2), [15]) the spatial infinity is inaccessible for a massive particle. In fact, to reach $x = \infty$ it should have infinite energy k. Integrating (73), one gets the dependence $s(x)$, the expressions are similar to those for CAdS and denoting $\kappa^2 \equiv k^2 + 1$ they read

$$s(x) = a\left[\arcsin\left(\frac{1}{\kappa}\cosh x\right) - \arcsin\left(\frac{1}{\kappa}\cosh x_0\right)\right] \tag{75}$$

for the outgoing segment (x grows from x_0 to x_{M}) and

$$s(x) = \pi a - a\left[\arcsin\left(\frac{1}{\kappa}\cosh x\right) + \arcsin\left(\frac{1}{\kappa}\cosh x_0\right)\right] \tag{76}$$

for x decreasing from $x_{\rm M}$ to x_0 and farther to $x = 0$. The length of C from P_0 to P_1 is

$$s_{\rm C} = 2s(x_{\rm M}) = \pi a - 2a \arcsin\left(\frac{1}{\kappa}\cosh x_0\right) < \pi a\,. \tag{77}$$

For $k \to \infty$ one gets $s_{\rm C} \to \pi a$ for any finite x_0. From $dt/dx = \dot{t}/\dot{x}$ and relations (69) and (73), one finds the time of flight from P_0 to P_1,

$$\begin{aligned}\Delta t \equiv t_1 - t_0 = 2(t_{\rm M} - t_0) \;=\; & 2\ln\left(k\cosh x_0 + \sqrt{k^2 - \sinh^2 x_0}\right) \\ & -2\ln\sinh x_0 - \ln\left(k^2+1\right)\,.\end{aligned} \tag{78}$$

We now compare the lengths of worldlines A and C. From (72) the length $s_{\rm A}$ in the time interval Δt is

$$s_{\rm A} = a\,\Delta t\,\sinh x_0\,. \tag{79}$$

A numerical example. For $x_0 = 5$ and $k = 1000$ one gets $x_{\rm M} = 7.60090$, $s_{\rm C} = 2.99304a$ and $s_{\rm A} = 1.99443a$, then $s_{\rm C}/s_{\rm A} = 1.50007$; in general, there is no doubt that the twin C is older than A at the reunion.

5.1. Jacobi fields and conjugate points on timelike radial geodesics

In the (t, x, θ, ϕ) chart the nonvanishing components of the curvature tensor are $R_{0101} = -a^2\sinh^2 x$ and $R_{2323} = -a^2\sin^2\theta$ and for the Ricci tensor these are $R_{00} = \sinh^2 x$, $R_{11} = -1$, $R_{22} = +1$, $R_{33} = \sin^2\theta$ and $R = 0$. With the aid of the two tensors and the vector $\dot{x}^\alpha$ tangent to timelike radial ($\theta = \pi/2$, $\phi = \phi_0$) geodesic curves, which is determined by (69) and (73), one finds that these lines contain conjugate points. On the geodesic C, a triad of spacelike vector fields satisfying (2) is conveniently chosen as

$$\begin{aligned} {e_1}^\alpha \;&=\; \left[\frac{\varepsilon\left(k^2 - \sinh^2 x\right)^{1/2}}{a\sinh^2 x}, \frac{k}{a\sinh x}, 0, 0\right]\,, \\ {e_2}^\alpha \;&=\; \left[0, 0, \frac{1}{a}, 0\right]\,, \qquad {e_3}^\alpha = \left[0, 0, 0, \frac{1}{a}\right]\,, \end{aligned} \tag{80}$$

where $\varepsilon = +1$ on the outgoing segment and $\varepsilon = -1$ on the ingoing one. Employing this basis one expands a general Jacobi vector field $Z^\mu(s) = \sum_b Z_b(s) {e_b}^\mu(s)$, $b = 1, 2, 3$ and the geodesic deviation equation for the Jacobi scalars $Z_b(s)$ takes on the following form:

$$\frac{d^2}{ds^2}Z_1 + \frac{1}{a^2}\,Z_1 = 0\,, \qquad \frac{d^2}{ds^2}Z_2 = 0\,, \quad \text{and} \quad \frac{d^2}{ds^2}Z_3 = 0\,. \tag{81}$$

These immediately give the generic Jacobi field along the geodesic

$$Z^{\mu}(s) = \left(C_{11}\sin\frac{s}{a} + C_{12}\cos\frac{s}{a}\right) e_1{}^{\mu} + (C_{21}s + C_{22})\, e_2{}^{\mu} + (C_{31}s + C_{32})\, e_3{}^{\mu}\,. \tag{82}$$

The special Jacobi scalars vanishing at a given initial point $s = 0$ are $Z_1 = C_1 \sin s/a$, $Z_2 = C_2\, s$ and $Z_3 = C_3\, s$. Conjugate points are determined by the special Jacobi field for which $C_2 = C_3 = 0$, then the deviation vector $Z^{\mu} = C_1 e_1{}^{\mu} \sin\frac{s}{a}$ lies in the 2-surface (t, x). Assuming that the geodesic infinitely oscillates between the outermost spatial points (as in the CAdS spacetime), one finds an infinite sequence of conjugate points Q_n at distances $s_n = n\pi a$, $n = 1, 2, \ldots$, from the initial point. One infers from the spherical symmetry that these points, being the cut points, are the only cut points on these curves and there are no other cut points on them. In the case of the geodesic C, the nearest point Q_1 conjugate to P_0 is at the distance $s = \pi a$. Since the length (77) of the segment P_0P_1 is $s_{\mathrm{C}} < \pi a$, point Q_1 is beyond this arc and the geodesic C is the longest curve among nearby curves joining P_0 and P_1, *i.e.* it attains the local maximum of length. B–R spacetime is not globally hyperbolic and most theorems on maximal curves in the space of all curves joining two given points (see [3]) do not apply. By a direct calculation we now show that the ingoing timelike radial geodesics are the maximal curves (their length is equal to the distance function) between any pair of points on the segment from the initial point to a point infinitesimally close to $x = 0$ of each geodesic of the class.

To this end, we transform from the chart (68) to the Gaussian normal geodesic (GNG) one, *i.e.* comoving coordinates in which the lines of the time coordinate τ are the radial geodesics. For the Reader's convenience, we briefly present here the derivation from [3] adapted to the B–R spacetime. Usually the GNG chart in a given spacetime is constructed in terms of worldlines of massive particles freely falling down from rest at spatial infinity. In B–R spacetime a particle with finite energy k cannot escape to infinity and according to (73) we assume that a swarm of particles radially falls down from the rest at $x = x_{\mathrm{M}}$, where $\sinh x_{\mathrm{M}} = k$ and $k > 0$. Then in the GNG coordinates (τ, R, θ, ϕ) the velocity field of the radial geodesics is $u^{\alpha} = (1, 0, 0, 0)$ and is the gradient of their common proper time, $u_{\alpha} = (1, 0, 0, 0) = \partial_{\alpha}\tau$. On the other hand, the velocity field in the chart (68) has components

$$u^{\alpha} = \left[\frac{k}{a\sinh^2 x}, -\frac{1}{a}\left(\frac{k^2}{\sinh^2 x} - 1\right)^{1/2}, 0, 0\right]$$

and the transformation law for the contravariant components of the field yields

$$\tau = akt + a \int \left(\frac{k^2}{\sinh^2 x} - 1 \right)^{1/2} dx \,. \tag{83}$$

In a similar way, one gets

$$R = at + ak \int \frac{dx}{\sinh x \left(k^2 - \sinh^2 x \right)^{1/2}} \,. \tag{84}$$

The inverse transformation is

$$\cosh x = \sqrt{k^2 + 1} \, \sin \left(\frac{kR - \tau}{a} \right) \,.$$

In the comoving coordinates the B–R metric is

$$ds^2 = d\tau^2 - \left(k^2 - \sinh^2 x \right) dR^2 - a^2 \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \,, \tag{85}$$

here $0 \leq \sinh x < k$ and alternatively $-g_{11} = k^2 - \sinh^2 x = (k^2 + 1) \cos^2 \left(\frac{kR - \tau}{a} \right)$ with $-\pi a/2 < kR - \tau < \pi a/2$, what implies that the comoving time coordinate is in the interval $kR - \pi a/2 < \tau < kR + \pi a/2$. Now take any radial geodesic in the domain of the GNG chart, $R = R_0$, $\theta = \theta_0$, $\phi = \phi_0$; along it there is $ds = d\tau$. Its length between two points on it, $S_1(\tau_1, R_0, \theta_0, \phi_0)$ and $S_2(\tau_2, R_0, \theta_0, \phi_0)$, where τ_1 and τ_2 are in the allowed interval, is $\tau_2 - \tau_1 < \pi a$. Let any other timelike curve with the endpoints S_1 and S_2 be parametrized by τ. Then its length is

$$\int_{\tau_1}^{\tau_2} \left[1 - \left(k^2 - \sinh^2 x \right) \left(\frac{dR}{d\tau} \right)^2 - a^2 \left(\frac{d\theta}{d\tau} \right)^2 - a^2 \sin^2 \theta \left(\frac{d\phi}{d\tau} \right)^2 \right]^{1/2} d\tau$$
$$< \tau_2 - \tau_1 < \pi a \,. \tag{86}$$

Thus, in the domain of the comoving chart the radial ingoing geodesics are maximal. From (75) one sees that the length of any radial outgoing (or ingoing) timelike geodesic from x_0 to x_{M} is less than $\pi a/2$ and tends to this upper limit for x_{M} and k tending to infinity. This implies that the ingoing geodesic from x_{M} to x_0 for any $0 < x_0 < x_{\mathrm{M}} < \infty$ entirely lies in the chart domain and is globally maximal. Since the metric (68) is time symmetric, the same theorem applies to outgoing radial geodesics.

6. Conclusions

The sample of the three spacetimes considered in this paper as applications of general methods developed in [3] do not allow one to formulate a general rule concerning properties of timelike worldlines which may be

used in various versions of the twin paradox. On the contrary, even in the maximally symmetric spacetimes, de Sitter and CAdS, one encounters a multitude of possibilities. The physical paradox is reduced to a purely geometrical problem of finding the (possibly unique) longest timelike curve joining two given points. This is a problem in global Lorentzian geometry and it is well known that in globally hyperbolic spacetimes it always has a well defined solution in the form of the maximal timelike geodesic segment whose length is, by definition, equal to the Lorentzian distance function between its endpoints. In principle, to find out the maximal geodesic, one must investigate all geodesics between given endpoints. High symmetry of the spacetime is helpful in these investigations to a limited extent. The four spacetimes (including Schwarzschild metric studied in [2]) are spherically symmetric, yet the differences in their global properties are at least as important as their spherical symmetry. Our current study of a general spherically symmetric static spacetime indicates that some common properties of timelike geodesics are accompanied by a diversity of distinct features in various metrics.

We choose three physically interesting worldlines: staying at rest and circular and radial motions and solve the twin problem by comparing their lengths. Then, we go further and for geodesic worldlines (radial and possibly circular), we determine the geodesic deviation vector fields and conjugate points and in this way we find the locally longest geodesic segments. Finally, in the three spacetimes studied here, de Sitter, CAdS and Bertotti–Robinson, we are able to determine all cut points on the radial and circular geodesics and show that they coincide with the conjugate points.

Circular geodesics in covering anti-de Sitter spacetime contain infinite number of conjugate points. While these geodesics lie in the two-surfaces $t-\phi$ ($r=r_0$, $\theta=\pi/2$), the nearby geodesics intersecting them at the conjugate points, lie (besides these points) outside these surfaces.

In CAdS space the radial geodesics (infinitely oscillating between spatial points maximally distant from the centre) contain an infinite sequence of conjugate points equally separated by $\Delta s=\pi a$ and their segments of this length are locally the longest curves between their endpoints. One Jacobi vector field lies in the $t-r$ surface ($\theta=\pi/2$) of the radial geodesics, while the other two fields are directed off it.

Similarly, in Bertotti–Robinson spacetime the oscillating radial timelike geodesics contain infinite number of equally separate conjugate points and these are the only cut points on these curves. Unlike the CAdS case, these points are determined by one deviation vector field, that lying in the two-surface $t-x$.

These few examples clearly show that in dealing with the geodesic deviation vectors and conjugate points one must study case by case.

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