SOLITARY WAVES, SHOCK WAVES AND SINGULAR SOLITONS OF GARDNER'S EQUATION FOR SHALLOW WATER DYNAMICS

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The paper addresses the dynamics of shallow water waves that are governed by the Gardner equation, that is a generalized version of the well-known Korteweg–de Vries equation. Exact solutions are obtained in presence of shoaling and advection terms with power law nonlinearity. The paper integrates the equation by the aid of G'/G-expansion method. This approach reveals singular soliton as well as shock wave solutions to the model. The solution existence criteria, also known as constraint conditions, are also displayed.

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1. Introduction

The dynamics of shallow water waves is an important area of research in oceanography [1–25]. There are several models that describe this kind of dynamics. A few of them are the Korteweg–de Vries (KdV) equation [2], modified KdV (mKdV) equation [2], Boussinesq equation [6], Perergrine equation [12], Kawahara equation [21], Benjamin–Bona–Mahoney equation [22], Rosenau–KdV equation [7, 10], Rosenau–RLW equation [18] and several others. However, for two-layered shallow water waves, the models that are commonly studied are the Gear–Grimshaw model [4], Bona–Chen equation [5], coupled Boussinesq equation [14] and many others. Another model that is also considered and studied at times is the Gardner equation (GE) it is a combination of KdV and mKdV equation [1, 3, 8, 16, 20]. Therefore, occasionally, GE is referred to as the KdV–mKdV equation. This paper will address the GE on a generalized setting when the nonlinear terms are generalized to an arbitrary power law nonlinearity. The relevance of the equations enumerated here is that all of these models are studied with weak nonlinearity.

The importance of studying GE is the strong nonlinearity that cannot be modeled with KdV equation or its types as mentioned before, since these equations are valid for small nonlinearity. This fact was experimentally observed in 1995 in Oregon Bay. In fact, it was concluded from these experimental observations that Gardner's equation models deep ocean waves rather than shallow water waves that is governed by KdV equation [2].

In order to keep it on a further generalized flavor, the coefficients of nonlinearity, dispersion, shoaling and advection are all taken to be time-dependent so that the scenario is as close to reality as possible. The integrability aspect of the equation will be the focus of this paper. The retrieval of the soliton solution will be a combination of the symbolic computation and the auxiliary equation approach. This combined approach will lead to solitary waves and singular soliton solutions. There are the necessary constraint conditions that will naturally fall out of these calculations.

2. Governing equation

The dimensionless form of the GE that is studied in ocean sciences is given by [3, 8, 16, 20]

$$u_t + 2auu_x - 3bu^2u_x + u_{xxx} = 0\,. \tag{1}$$

Our interest is focused on the GE with nonlinear terms of any order, most importantly, when all coefficients vary with respect to time

$$u_t + \alpha\,(t)\,u + \left[\beta(t) + \gamma_1(t)u^n + \gamma_2(t)u^{2n}\right]u_x + \delta(t)u_{xxx} = 0\,, \tag{2}$$

where $\alpha(t)$, $\beta(t)$, $\gamma_1(t)$, $\gamma_2(t)$ and $\delta(t)$ are arbitrary functions of the time variable t.

In (2), the first term represents the linear evolution term, the second term is the linear damping, also known as shoaling, while the third term is the advection term. The terms related to the coefficients $\gamma_1(t)$ and $\gamma_2(t)$ represent the nonlinear terms of any order, and the last term is the third order dispersion. Here, n is the index of power law nonlinearity.

The model equation (2) applies to the description of weakly nonlinear and weakly dispersive wave propagation in inhomogeneous media. Note that the simplest case when all coefficients in Eq. (2) are constant with $\alpha = \beta = 0$ and $\delta = 1$ has been well studied [1, 3, 8, 16, 20]. In this paper, we have studied temporally inhomogeneous GE. In general, (2) is not integrable, for any arbitrary n, by the standard method of integrability of the nonlinear evolution equations, namely the inverse scattering transform that is the nonlinear analog of Fourier transform. These special solutions may play an important role in the research of some physical phenomena arising in nonlinear systems described by GE.

3. Soliton solutions

First, we introduce the following transformation

$$u(x,t) = v^{\frac{1}{n}}(x,t)\,. \tag{3}$$

After substituting (3) into (2) and simplifying, (2) is reduced to

$$\begin{aligned} &n^2v^2v_t + \alpha n^3v^3 + \beta n^2v^2v_x + \gamma_1 n^2v^3v_x + \gamma_2 n^2v^4v_x \\ &+\delta\left\{(1-n)(1-2n)\,v_x^3 + 3n(1-n)\,vv_xv_{xx} + n^2v^2v_{xxx}\right\} = 0\,. \end{aligned} \tag{4}$$

Balancing v^4v_x with v_x^3 in (4) gives

$$4M + M + 1 = 3(M+1) \tag{5}$$

so that $M = 1$. Accordingly, we adopt the ansatz, with a modification for the solution to (4), as follows

$$v = f + g_1\varphi(\xi)\,, \tag{6}$$

$$\xi = p(t)x + q(t)\,, \tag{7}$$

$$\left(\frac{d\varphi}{d\xi}\right)^2 = q_4\varphi^4 + q_3\varphi^3 + q_2\varphi^2 + q_1\,, \tag{8}$$

where q_1, q_2, q_3 and q_4 are constants, and $f = f(t)$, $g_1 = g_1(t)$, $p = p(t)$ and $q = q(t)$ are functions of t, which are unknown and to be further determined.

Substituting (6) into (4) along with (8), collecting the coefficients of φ^0, φ^1, φ^2, φ^3 and $\varphi^i\varphi'$ to zero, where $i=0,1,2,3,4$, and setting them to zero we get the following set of coupled differential equations for f, g_1, p and q:

$$\begin{aligned}
\varphi^0 &: f^2\left(f'+\alpha n f\right)=0\,, && (9)\\
\varphi^1 &: \left(f^2 g_1\right)'+3\alpha n f^2 g_1=0\,, && (10)\\
\varphi^2 &: \left(f g_1^2\right)'+3\alpha n f g_1^2=0\,, && (11)\\
\varphi^3 &: g_1^2\left(g_1'+\alpha n g_1\right)=0\,, && (12)
\end{aligned}$$

$$\begin{aligned}
\varphi' :\ & n^2 f^2 g_1\left(p'x+q'\right)+\beta n^2 f^2 g_1 p+\gamma_1 n^2 f^3 g_1 p+\gamma_2 n^2 f^4 g_1 p\\
&+\delta\left(1-n\right)\left(1-2n\right) g_1^3 p^3 q_1+\delta n^2 f^2 g_1 p^3 q_2=0\,, \qquad (13)
\end{aligned}$$

$$\begin{aligned}
\varphi\varphi' :\ & 2n^2 f g_1^2\left(p'x+q'\right)+2\beta n^2 f g_1^2 p+3\gamma_1 n^2 f^2 g_1^2 p+4\gamma_2 n^2 f^3 g_1^2 p\\
&+3\delta n\left(1-n\right) f g_1^2 p^3 q_2+3\delta q_3 n^2 f^2 g_1 p^3+2\delta q_2 n^2 f g_1^2 p^3=0\,, \qquad (14)
\end{aligned}$$

$$\begin{aligned}
\varphi^2\varphi' :\ & n^2 g_1^3\left(p'x+q'\right)+\beta n^2 g_1^3 p+3\gamma_1 n^2 f g_1^3 p+6\gamma_2 n^2 f^2 g_1^3 p\,,\\
&+\delta\left(1-n\right)\left(1-2n\right) g_1^3 p^3 q_2+\tfrac{9}{2} q_3\delta n(1-n) f g_1^2 p^3\\
&+3q_2\delta n(1-n) g_1^3 p^3+6q_4\delta n^2 p^3 f^2 g_1\\
&+6q_3\delta n^2 f g_1^2 p^3+q_1 g_1^3\delta n^2 p^3=0\,, \qquad (15)
\end{aligned}$$

$$\begin{aligned}
\varphi^3\varphi' :\ & \gamma_1 n^2 g_1^4 p+4\gamma_2 n^2 f g_1^4 p+\delta\left(1-n\right)\left(1-2n\right) q_3 g_1^3 p^3\\
&+6n(1-n)\delta q_4 f g_1^2 p^3+\tfrac{9}{2} n(1-n)\delta q_3 g_1^3 p^3\\
&+3\delta n^2 q_3 g_1^3 p^3+12 q_4\delta n^2 f g_1^2 p^3=0\,, \qquad (16)
\end{aligned}$$

$$\varphi^4\varphi' : \gamma_2 n^2 g_1^5 p+\delta g_1^3 p^3 q_4\left(n+1\right)\left(2n+1\right)=0\,, \qquad (17)$$

where prime denotes the differential with respect to the variable t. These nine equations can be solved for seven unknowns to give for $q_1=0$:

$$\begin{aligned}
f(t) &= k_1 e^{-n\int\alpha(t)dt}\,, && (18)\\
g_1(t) &= k_2 e^{-n\int\alpha(t)dt}\,, && (19)\\
p(t) &= \sqrt{\frac{\gamma_2(t)}{\delta(t)}}\,k_3 e^{-n\int\alpha(t)dt}\,, && (20)
\end{aligned}$$

$$\begin{aligned}
q_2 &= \frac{n k_1^2}{12 k_3^2\left(2n+1\right)}\,, \qquad q_3=\frac{-n^2 k_1 k_2}{2k_3^2\left(5n+2\right)\left(2n+1\right)}\,,\\
q_4 &= \frac{-n^2 k_2^2}{k_3^2\left(n+1\right)\left(2n+1\right)}\,, \qquad (21)
\end{aligned}$$

and

$$\begin{aligned} q(t) \;=\; & \int dt \left\{ n\alpha(t)x - \beta(t) - \frac{4k_1}{3}\gamma_1(t)e^{-n\int \alpha(t)dt} \right. \\ & -\frac{5k_1^2}{3}\gamma_2(t)e^{-2n\int \alpha(t)dt} - \frac{k_1^2}{12(2n+1)}\gamma_2(t)e^{-2n\int \alpha(t)dt} \\ & \left. + \frac{n^2k_1^2}{2(5n+2)(2n+1)}\gamma_2(t)e^{-2n\int \alpha(t)dt} \right\} \sqrt{\frac{\gamma_2(t)}{\delta(t)}}k_3 e^{-n\int \alpha(t)dt}, \end{aligned} \tag{22}$$

where k_1, k_2 and k_3 are arbitrary constants with $k_3 \neq 0$, along with a constraining relation

$$f(t) = \frac{q_3}{4q_4}g_1(t) - \frac{(2n+1)\gamma_1(t)}{2(n+2)\gamma_2(t)}. \tag{23}$$

Given the results for $f(t)$ and $g_1(t)$, the constrained relation yields

$$\frac{\gamma_1(t)e^{n\int \alpha(t)dt}}{\gamma_2(t)} = \frac{(q_3k_2 - 4q_4k_1)(n+2)}{2q_4(2n+1)} \tag{24}$$

which means that the time-varying parameters $\alpha(t)$, $\gamma_1(t)$ and $\gamma_2(t)$ are not independent and the existing solutions are obtained in the framework of this relationship.

Returning to the auxiliary ordinary differential equation (8), one can see that it has the following exact solutions for $q_1 = 0$:

Case 1: If $q_2 > 0$:

$$\varphi(\xi) = \frac{-q_2q_3\,\mathrm{sech}^2\left(\pm\frac{\sqrt{q_2}}{2}\xi\right)}{q_3^2 - q_2q_4\left(1 - \tanh\left(\pm\frac{\sqrt{q_2}}{2}\xi\right)\right)^2}, \tag{25}$$

Case 2: If $q_3^2 - 4q_2q_4 > 0$, $q_2 > 0$:

$$\varphi(\xi) = \frac{2q_2\,\mathrm{sech}\left(\sqrt{q_2}\xi\right)}{\sqrt{q_3^2 - 4q_2q_4} - q_3\,\mathrm{sech}\left(\sqrt{q_2}\xi\right)}, \tag{26}$$

Case 3: If $q_2 = 4$, $q_3 = -\frac{4(2b+d)}{a}$, $q_4 = \frac{c^2+4b^2+4bd}{a^2}$:

$$\varphi(\xi) = \frac{a\,\mathrm{sech}^2\xi}{b\,\mathrm{sech}^2\xi + c\tanh\xi + d}, \tag{27}$$

where a, b, c and d being arbitrary constants. In addition to the above solutions, we have found other soliton-like solutions to Eq. (8) for $q_1 = 0$ as follows:

Case 4: If $4q_2q_4 - q_3^2 > 0$, $q_2 > 0$:

$$\varphi(\xi) = \frac{-2q_2}{q_3 + \sqrt{4q_2q_4 - q_3^2}\sinh\left(\sqrt{q_2}\xi\right)}\,, \tag{28}$$

Case 5: If $q_3^2 - 4q_2q_4 > 0$, $q_2 > 0$:

$$\varphi(\xi) = \frac{-2q_2\,\mathrm{sech}^2\left(\frac{\sqrt{q_2}}{2}\xi\right)}{2\sqrt{q_3^2 - 4q_2q_4} - \left(\sqrt{q_3^2 - 4q_2q_4} - q_3\right)\,\mathrm{sech}^2\left(\frac{\sqrt{q_2}}{2}\xi\right)}\,. \tag{29}$$

In order to construct the explicit soliton-like solutions for the generalized KdV–mKdV equation with variable coefficients (2), we substitute one of the solutions $\varphi(\xi)$ given in (25)–(29) into (6) and the result in (3) as follows:
Type 1: From (3), (6), (7) and Case 1, we can obtain the following soliton-like solutions for Eq. (2):

$$u = \left\{k_1e^{-n\int\alpha(t)dt} - k_2e^{-n\int\alpha(t)dt}\left[\frac{q_2q_3\,\mathrm{sech}^2\left(\pm\frac{\sqrt{q_2}}{2}\xi\right)}{q_3^2 - q_2q_4\left(1-\tanh\left(\pm\frac{\sqrt{q_2}}{2}\xi\right)\right)^2}\right]\right\}^{\frac{1}{n}}\,, \tag{30}$$

$$\xi = p(t)x + q(t)\,, \tag{31}$$

where the soliton parameters $p(t)$ and $q(t)$ are given by (20) and (22), k_1, k_2 and k_3 are arbitrary constants, $k_3 \neq 0$ and $q_2 > 0$.
Type 2: From (3), (6), (7) and Case 2, we get a soliton-like solution of the form:

$$u = \left\{k_1e^{-n\int\alpha(t)dt} + k_2e^{-n\int\alpha(t)dt}\left[\frac{2q_2\,\mathrm{sech}\left(\sqrt{q_2}\xi\right)}{\sqrt{q_3^2 - 4q_2q_4} - q_3\,\mathrm{sech}\left(\sqrt{q_2}\xi\right)}\right]\right\}^{\frac{1}{n}}\,, \tag{32}$$

$$\xi = p(t)x + q(t)\,, \tag{33}$$

where the soliton parameters $p(t)$ and $q(t)$ are given by (20) and (22), k_1, k_2 and k_3 are arbitrary constants, $k_3 \neq 0$, $q_3^2 - 4q_2q_4 > 0$ and $q_2 > 0$.
Type 3: From (3), (6), (7) and Case 3, we have a soliton-like solution for Eq. (2) given by:

$$u = \left\{k_1e^{-n\int\alpha(t)dt} + k_2e^{-n\int\alpha(t)dt}\left[\frac{a\,\mathrm{sech}^2\xi}{b\,\mathrm{sech}^2\xi + c\tanh\xi + d}\right]\right\}^{\frac{1}{n}}\,, \tag{34}$$

$$\xi = p(t)x + q(t)\,, \tag{35}$$

where the soliton parameters $p(t)$ and $q(t)$ are given by (20) and (22), a, b, c and d are arbitrary real constants. k_1, k_2 and k_3 are arbitrary constants, $k_3 \neq 0$.

Type 4: From (3), (6), (7) and Case 4, we can obtain the following soliton-like solutions for Eq. (2):

$$u = \left\{k_1 e^{-n\int \alpha(t)dt} - k_2 e^{-n\int \alpha(t)dt}\left[\frac{2q_2}{q_3 + \sqrt{4q_2q_4 - q_3^2}\sinh\left(\sqrt{q_2}\xi\right)}\right]\right\}^{\frac{1}{n}}, \quad (36)$$

$$\xi = p(t)x + q(t)\,, \quad (37)$$

where the soliton parameters $p(t)$ and $q(t)$ are given by (20) and (22), k_1, k_2 and k_3 are arbitrary constants, $k_3 \neq 0$, $4q_2q_4 - q_3^2 > 0$ and $q_2 > 0$.

Type 5: From (3), (6), (7) and Case 5, we can obtain the following soliton-like solutions for Eq. (2):

$$u = \left\{k_1 e^{-n\int \alpha(t)dt} - k_2 e^{-n\int \alpha(t)dt} \times\left[\frac{2q_2\,\mathrm{sech}^2\left(\frac{\sqrt{q_2}}{2}\xi\right)}{2\sqrt{q_3^2 - 4q_2q_4} - \left(\sqrt{q_3^2 - 4q_2q_4} - q_3\right)\,\mathrm{sech}^2\left(\frac{\sqrt{q_2}}{2}\xi\right)}\right]\right\}^{\frac{1}{n}}, \quad (38)$$

where

$$\xi = p(t)x + q(t) \quad (39)$$

and the soliton parameters $p(t)$ and $q(t)$ are given by (20) and (22), k_1, k_2 and k_3 are arbitrary constants, $k_3 \neq 0$, $q_3^2 - 4q_2q_4 > 0$ and $q_2 > 0$.

4. Application of the G'/G-expansion method to GE with time-dependent coefficients

In this section, we will apply the G'/G-expansion method [19, 23] to handle the generalized form of the Gardner equation with time-dependent coefficients. It must be noted that this is a powerful integration scheme that was first reported in the previous decade. A detailed description of this integration algorithm is given in details when it was first reported in 2008 [19, 23]. This method will be applied to GE with time-dependent coefficients given by

$$u_t + \left(a(t) + b(t)u^n + c(t)u^{2n}\right)u_x + u_{xxx} = 0\,. \quad (40)$$

The traveling wave transformation

$$u(x,t) = U(\xi)\,, \qquad \xi = x - v(t)t\,, \quad (41)$$

transforms the Eq. (40) to the following ODE

$$-\left(v(t)+t\frac{dv(t)}{dt}\right)U'+\left(a(t)+b(t)U^n+c(t)U^{2n}\right)U'+U'''=0\,, \tag{42}$$

where the prime denotes the differential with respect to ξ.

Integrating Eq. (42) once, and considering the zero constant for integration, we have

$$\left(a(t)-v(t)-t\frac{dv(t)}{dt}\right)U+\frac{b(t)}{n+1}U^{n+1}+\frac{c(t)}{2n+1}U^{2n+1}+U''=0\,. \tag{43}$$

Balancing U'' with U^{2n+1} gives

$$N+2=(2n+1)N\Leftrightarrow N+2=2nN+N\Leftrightarrow N=\frac{1}{n}\,.$$

We then assume that Eq. (43) has the following formal solutions:

$$U(\xi)=B\left(\frac{G'}{G}\right)^{\frac{1}{n}}\,,\qquad B\neq 0\,, \tag{44}$$

where B is a constant to be determined later and the function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi)+\lambda G'(\xi)+\mu G(\xi)=0\,, \tag{45}$$

where λ and μ are real constants to be determined. Therefore, we have

$$U' = -\frac{1}{n}B\left(\frac{G'}{G}\right)^{\frac{1}{n}+1}-\frac{1}{n}B\lambda\left(\frac{G'}{G}\right)^{\frac{1}{n}}-\frac{1}{n}B\mu\left(\frac{G'}{G}\right)^{\frac{1}{n}-1}\,, \tag{46}$$

$$\begin{aligned} U'' = {} & \left(\frac{1}{n^2}+\frac{1}{n}\right)B\left(\frac{G'}{G}\right)^{\frac{1}{n}+2}+\left(\frac{2}{n^2}+\frac{1}{n}\right)B\lambda\left(\frac{G'}{G}\right)^{\frac{1}{n}+1} \\ & +\left(\frac{2}{n^2}B\mu+\frac{1}{n^2}B\lambda^2\right)\left(\frac{G'}{G}\right)^{\frac{1}{n}}+\left(\frac{2}{n^2}-\frac{1}{n}\right)B\mu\lambda\left(\frac{G'}{G}\right)^{\frac{1}{n}-1} \\ & +\left(\frac{1}{n^2}-\frac{1}{n}\right)B\mu^2\left(\frac{G'}{G}\right)^{\frac{1}{n}-2}\,. \end{aligned} \tag{47}$$

Substituting Eqs. (44)–(47) into Eq. (43) and collecting all terms with the same order of G'/G together, we convert the left-hand side of Eq. (43) into a polynomial in G'/G. Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for λ, μ, $v(t)$ and B:

$\left(\frac{G'}{G}\right)^{\frac{1}{n}+2}$ coeff.:

$$\frac{c(t)}{2n+1}B^{2n+1}+\left(\frac{1}{n^2}+\frac{1}{n}\right)B=0\,,$$

$\left(\frac{G'}{G}\right)^{\frac{1}{n}+1}$ coeff.:

$$\frac{b(t)}{n+1}B^{n+1}+\left(\frac{2}{n^2}+\frac{1}{n}\right)B\lambda=0\,,$$

$\left(\frac{G'}{G}\right)^{\frac{1}{n}}$ coeff.:

$$\left(a(t)-v(t)-t\frac{dv(t)}{dt}\right)B+\left(\frac{2}{n^2}\mu+\frac{1}{n^2}\lambda^2\right)B=0\,,$$

$\left(\frac{G'}{G}\right)^{\frac{1}{n}-1}$ coeff.:

$$\left(\frac{2}{n^2}-\frac{1}{n}\right)B\mu\lambda=0\,,$$

$\left(\frac{G'}{G}\right)^{\frac{1}{n}-2}$ coeff.:

$$\left(\frac{1}{n^2}-\frac{1}{n}\right)B\mu^2=0\,. \tag{48}$$

The solution of the above system is

$$\begin{aligned} B &= \left(-\frac{(n+1)(2n+1)}{c(t)n^2}\right)^{\frac{1}{2n}}\,, \quad \mu=0\,, \quad \lambda=-\sqrt{-\frac{n^2b^2(t)(2n+1)}{c(t)(n+1)(n+2)^2}}\,,\\ v(t) &= \frac{1}{t}\int\left\{a(t')-\frac{(2n+1)b^2(t')}{(n+1)(n+2)^2c(t')}\right\}dt'\,. \end{aligned} \tag{49}$$

From Eqs. (41), (44), (45) and (49), we obtain the exact traveling wave solution of the generalized form of the Gardner equation with time-dependent coefficients

$$\begin{aligned} u(x,t) &= \left\{-\frac{b(t)(2n+1)}{c(t)(n+2)}\right.\\ &\left.\times\frac{c_2e^{\sqrt{-\frac{n^2b^2(t)(2n+1)}{c(t)(n+1)(n+2)^2}}\left(x-\int\left\{a(t')-\frac{(2n+1)b^2(t')}{(n+1)(n+2)^2c(t')}\right\}dt'\right)}}{c_1+c_2e^{\sqrt{-\frac{n^2b^2(t)(2n+1)}{c(t)(n+1)(n+2)^2}}\left(x-\int\left\{a(t')-\frac{(2n+1)b^2(t')}{(n+1)(n+2)^2c(t')}\right\}dt'\right)}}\right\}^{\frac{1}{n}}\,, \end{aligned} \tag{50}$$

where c_1 and c_2 are arbitrary constants.

Equation (50) is a new type of exact traveling wave solution to the generalized form of the Gardner equation with time-dependent coefficients. Especially, if we choose $c_1 = c_2$ in (50), we obtain the solitary wave solution of the generalized form of the Gardner equation with time-dependent coefficients, namely

$$\begin{aligned} u(x,t) &= \left\{ -\frac{b(t)(2n+1)}{2c(t)(n+2)} \left(1 + \tanh\left[\sqrt{-\frac{n^2 b^2(t)(2n+1)}{4c(t)(n+1)(n+2)^2}} \right.\right.\right. \\ &\quad \left.\left.\left. \times \left(x - \int \left\{ a\left(t'\right) - \frac{(2n+1)b^2\left(t'\right)}{(n+1)(n+2)^2 c(t')} \right\} dt' \right) \right] \right) \right\}^{\frac{1}{n}}, \end{aligned} \tag{51}$$

and

$$\begin{aligned} u(x,t) &= \left\{ -\frac{b(t)(2n+1)}{2c(t)(n+2)} \left(1 + \coth\left[\sqrt{-\frac{n^2 b^2(t)(2n+1)}{4c(t)(n+1)(n+2)^2}} \right.\right.\right. \\ &\quad \left.\left.\left. \times \left(x - \int \left\{ a\left(t'\right) - \frac{(2n+1)b^2(t')}{(n+1)(n+2)^2 c(t')} \right\} dt' \right) \right] \right) \right\}^{\frac{1}{n}}. \end{aligned} \tag{52}$$

These are shock wave and singular soliton solution respectively.

5. Conclusions

This paper thus addressed the soliton solutions of the GE with power law nonlinearity in presence of time-dependent coefficients of shoaling, advection, nonlinearity and dispersion. The solution structure in each of the cases clearly indicate that the shoaling is a purely dissipative term and, therefore, the solitons will dissipate for $\alpha(t) > 0$. These results are very promising for further research and stands on a strong footing for further extension. In future, there are several other perturbation terms that will be taken into consideration such as higher order dispersion, higher order stabilization, just to name a few. These perturbed GE models will be analyzed in future using this tool as well as various other mathematical tools, such as Lie symmetry analysis, variational iteration method, exp-function approach and several others. Additionally, the stochastic perturbation terms will be taken into consideration that will lead to the Langevin equation which will be integrated in order to obtain the mean free velocity of the solitary wave. These results will be all declared in future publications.

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