GINZBURG–LANDAU MODEL IN THREE DIMENSIONAL LOBACHEVSKY SPACE

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We consider the Ginzburg–Landau model of a superconductor in three dimensional Lobachevsky space. Generally, covariant Ginzburg–Landau equations were derived and two types of solutions of these equations were obtained: with a flat and cylindrically symmetric boarders of superconductors. The first case is considered in the quasi-Cartesian coordinates system and it is shown that when the radius of curvature of Lobachevsky space ρ is less than a double quantity of the London penetration depth λ, magnetic field might increase with penetration depth. In the second case, which studies cylindrically symmetric superconductor, it is shown that magnetic field depends on two coordinates: depth penetration and coordinate along the surface of the superconductor. When radius of curvature ρ of Lobachevsky space goes to infinity, derived solutions and equations will go to the usual Ginzburg–Landau model in flat space.

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1. Introduction

The Meissner effect describes expulsion of magnetic field from a superconductor [1, 2] and dual effect to this phenomenon probably could help to explain confinement [3–5]. At the same time, model of Ginzburg–Landau is an example of the non-relativistic Higgs mechanism. In consideration of these facts and since some quantum-mechanical models in non-Euclidean space are used for construction of phenomenological models of component particles and nanoparticles [6–8], we suppose that nonlinear model could help to improve some of them. In addition, non-trivial curvature, which is not originated by geometry of space, have an effect on some physical processes [9]. Moreover, some models of a superconductor in a non-Euclidean space

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play significant role for field-theoretic models with a dynamically break symmetry [10, 11] and for models that are constructed on holographic methods for condensed matter [12].

The current paper is dedicated to formulation of the Ginzburg–Landau model in Lobachevsky space and investigation of influence of the additional parameter — radius of curvature of Lobachevsky space.

2. Formulation of equations

Let us consider the free energy of Ginzburg–Landau model in generally covariant form

$$
\begin{aligned}
F \;=\; & F_{\mathrm{n}} + \int \left(\alpha|\Psi|^2 + \frac{\beta|\Psi|^4}{2} + \frac{1}{4m}\left| -i\hbar\nabla_\mu \Psi - \frac{2e}{c}A_\mu \Psi \right|^2 \right. \\
& \left. + \frac{\left(\varepsilon^{\mu\nu\lambda}\nabla_\nu A_\lambda\right)^2}{8\pi} \right) \sqrt{|g|}dV \,,
\end{aligned} \tag{1}
$$

where F_{n} is free energy of superconductor in the normal state, m and e is a mass of electron and its charge correspondingly, A_λ is a vector potential, $\sqrt{|g|}dV$ is an invariant volume, α, β are constants of the system, Ψ is a wave function of superconducting electrons, Greek indices μ, ν, *etc.*, run over three coordinate labels, ∇_ν is a covariant derivative, $\varepsilon^{\mu\nu\lambda} = \frac{e^{\mu\nu\lambda}}{\sqrt{|g|}}$ and $e^{\mu\nu\lambda}$ is the Levi–Civita symbol.

Normalization of the wave function is the following $|\Psi|^2 = \frac{n_s}{2}$ — density of superconducting pairs. Minimizing of variation of free energy (1) with respect to Ψ^* yields the first Ginzburg–Landau equation

$$
\alpha\Psi + \beta|\Psi|^2\Psi + \frac{1}{4m}\left(-i\hbar\nabla_\mu - \frac{2e}{c}A_\mu \right)^2 \Psi = 0 \,. \tag{2}
$$

There was used Stokes's theorem $\int \nabla_\nu \left(\delta\Psi^* A^\nu\right) \sqrt{|g|}dV = \oint \delta\Psi^* A^\nu n_\nu \sqrt{|\gamma|}dS$, where γ is a determinant of the induced metric on a surface S, which bounds a volume V. The second Ginzburg–Landau equation can be obtained from a variation of free energy (1) with respect to the vector potential A_λ

$$
\frac{ei\hbar}{2m}\left(\Psi^*\partial_\nu\Psi - \Psi\partial_\nu\Psi^*\right) + \frac{2e^2}{mc}|\Psi|^2 A_\nu + \frac{c}{4\pi}\varepsilon^{\epsilon\mu\eta} g_{\nu\eta}\partial_\mu \left(\left(\varepsilon^{\lambda\tau\sigma}\partial_\tau A_\sigma \right) g_{\epsilon\lambda} \right) = 0 \,. \tag{3}
$$

For these equations the condition was used that variation δA_λ on the surface of superconductor S equals zero, since external magnetic field is fixed. In this paper, only simple connected superconductors are considered, hence, the wave function of superconductive electrons can be led to a real function due to gauge transformation. In this case, the first term in equation (3) is null. One can show that the calculation of the rotor of equation (3) gives the following result

$$\lambda^2 \varepsilon^{\nu\mu\delta} \partial_\mu \left(g_{\eta\delta} \varepsilon^{\sigma\eta\rho} \partial_\sigma \left(H_\rho \right) \right) - H^\nu = 0 \,, \tag{4}$$

where $\lambda^2 = \frac{c^2 m}{4\pi n_s e^2}$ is the square of the London penetration depth. When space is flat, equations (4) go to common London equations [2] that usually are used for description of the Meissner effect. Since it is important to have the form of external magnetic fields in Lobachevsky space, superconductors is considered in two special coordinate systems where features of some magnetic fields have been studied [13, 14].

3. Solutions of London equations in Lobachevsky space

3.1. Superconductor with a flat boarder

Let us consider equations (4) in the quasi-Cartesian system

$$dl^2 = e^{-\frac{2z}{\rho}} \left(dx^2 + dy^2 \right) + dz^2 \,, \tag{5}$$

where ρ is a curvature of Lobachevsky space. We choose coordinate system in the following way: axis z is a normal to the surface of superconductor and when $z > 0$, the medium is superconductive. The external magnetic field is directed along the superconductor and along the axis x. In this case, equation (4) has the form

$$\lambda^2 e^{\frac{2z}{\rho}} \partial_z \partial_z \left(e^{-\frac{2z}{\rho}} H^x(z) \right) - H^x(z) = 0 \,. \tag{6}$$

General solution of the equation (6) is

$$H^x(z) = C_1 e^{z\left(\frac{2}{\rho} - \frac{1}{\lambda}\right)} + C_2 e^{z\left(\frac{2}{\rho} + \frac{1}{\lambda}\right)} \,, \tag{7}$$

where C_1 and C_2 are constants.

Boundary conditions are chosen so that magnitudes of the magnetic field in flat space ($\rho \to \infty$) are $H^x(0) = H_0$ and $H^x(\infty) = 0$. Hence, the solution for this system has the form

$$H^x(z) = H_0 e^{z\left(\frac{2}{\rho} - \frac{1}{\lambda}\right)} \,. \tag{8}$$

The graphic for this solution with different values of the curvature of Lobachevsky space is shown in figure 1.

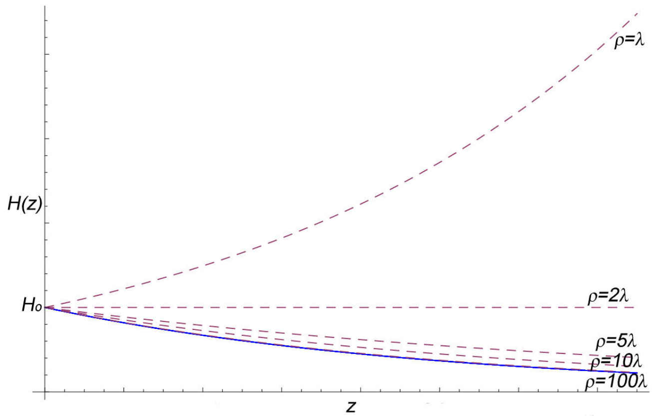

Fig. 1. The magnetic field dependence of the penetration depth with different values of curvature of Lobachevsky space. The solid line describes solution in flat space, the dashed line — Lobachevsky space with different value of curvature.

Figure 1 shows that when $\rho < 2\lambda$, the magnetic field will increase with the penetration depth. The ratio of magnetic field penetration depths into a superconductor in curved and flat space equals to $e^{\frac{2}{\rho}}$. Note that according to the Maxwell equations, the current of superconductive electrons in this system depends on value of curvature $j_s = \frac{c}{4\pi} H_0 \left(\frac{2}{\rho} - \frac{1}{\lambda}\right)$. Consequently, the ratio of currents in curved and flat space differs from unity only because of the component $\frac{2\lambda}{\rho}$.

3.2. Cylindrically symmetric superconductor

By analogy with the previous case, let us consider a cylindrically symmetric superconductor in the horospheric coordinate system

$$dl^2 = e^{-\frac{2z}{\rho}} \left(dr^2 + r^2 d\varphi^2\right) + dz^2 \,. \tag{9}$$

For this metric, equation (4) has the form

$$\lambda^2 \frac{e^{\frac{2z}{\rho}}}{r} \partial_r \left(r \partial_r H^z(r,z)\right) - H^z(r,z) = 0 \,. \tag{10}$$

General solution for this equation is

$$H^z(z,r) = C_3 J_0 \left(\frac{i e^{-\frac{z}{\rho}} r}{\lambda}\right) + C_4 Y_0 \left(-\frac{i e^{-\frac{z}{\rho}} r}{\lambda}\right) , \tag{11}$$

where J_0 — Bessel function of the first kind, Y_0 — Bessel function of the second kind, C_3 and C_4 are constants.

Let us consider cylindrically symmetric superconductor with the following boundary conditions: on the boarder R_{b} of its surface magnetic field equals to H_0 and when $\rho \to \infty$, we will have decaying solution for cylindrically symmetric superconductor in flat space. Let us choose the direction of z axis along the surface of the superconductor. In this case, solution for magnetic field in the cylindrically symmetric superconductor in Lobachevsky space has form

$$H^z(z,r) = H_0 J_0\left(\frac{ie^{-\frac{z}{\rho}}r}{\lambda}\right)/J_0\left(\frac{ie^{-\frac{z}{\rho}}R_{\mathrm{b}}}{\lambda}\right). \tag{12}$$

The magnetic field dependence of the penetration depth with different values of z for cylindrically symmetric superconductor in Lobachevsky space with curvature $\rho = 10\lambda$ is shown in figure 2.

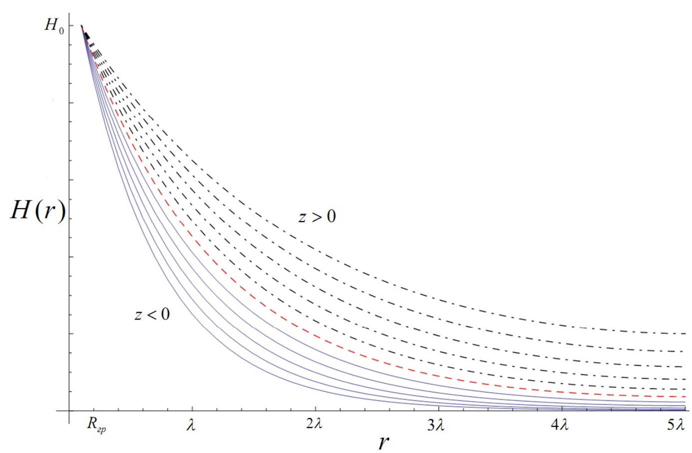

Fig. 2. The magnetic field dependence of the penetration depth with different values of z in Lobachevsky space with curvature $\rho = 10\lambda$. The dashed line describes solution in flat space, dot-dashed line for the case when $z > 0$ and solid line for the case when $z < 0$.

4. Conclusion

In this paper, we consider the Ginzburg–Landau model in Lobachevsky space and the generalized London equation was obtained in space with a constant curvature. Two types of solution of magnetic field for superconductors in Lobachevsky space were found: superconductor with a flat boarder and cylindrically symmetric superconductor. For the first case, it was shown that in Lobachevsky space magnetic field can grow with the penetration depth and the second case shows that in Lobachevsky space there can occur a situation when magnetic field depends on two coordinates: depth penetration and coordinate along the surface of the superconductor. These effects are not significant if we consider curvature as a cosmological parameter, but it can play a crucial role if we model compound particles or nanoparticles.

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