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MODIFICATION OF $\pi\pi$ AMPLITUDES AND POSITION OF THE σ POLE*

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Meson–meson interaction amplitudes especially $\pi\pi$ scattering amplitudes are often used incorrectly. It also applies to parameters of the σ meson. This causes two significant problems: threshold behavior and position of the σ pole which is suspicious. We modified multichannel S- and P-wave amplitudes for the $\pi\pi$ scattering, using dispersion relations with imposed crossing symmetry condition. The amplitudes are modified in the low-energy region to improve their consistency with experimental data and the dispersion relations. Agreement with data is achieved for the both amplitudes from the threshold up to 1.8 GeV and with dispersion relations up to 1.1 GeV. Consequences of the applied modifications, *e.g.* changes of the S-wave lowest-pole positions, are presented.

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1. Introduction

A model independent analysis of the $\pi\pi$ scattering is an important tool in getting information about the spectrum of light mesons. A reliable description of the process is, therefore, desirable to allow us to learn more on nature and parameters of the mesons.

The phenomenological multichannel amplitudes for the S- and P-waves in the $\pi\pi$ scattering were constructed without any specific assumptions about dynamics of the process, only requiring analyticity and unitarity of the S-matrix and applying the uniformization procedure [1]. This procedure can be applied exactly in the two-channel case. However, in the three-channel case, simplifying approximations have to be done resulting in a very poor description of experimental data in the threshold region.

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The crossing symmetry condition, which relates the S- and P-waves and which is an important below the inelastic threshold, was not taken into account in the construction of these amplitudes [1]. Since the crossing symmetry is properly included in the Roy-like dispersion relations [2], it is possible and desired to improve the low-energy behavior of the three-channel amplitudes of [1] and to check their consistency with the dispersion relations (DR).

Poles are singularities in the amplitudes and always related with resonances. There are many poles in the $\pi\pi$-interaction amplitudes and the most important one is the lightest one which is called σ and known as $f_0(500)$. In the multichannel uniformizing (MI) approach used in [1], a heavy and broad σ meson is predicted, $m = 829 \pm 10$ MeV and $\Gamma = 1108 \pm 22$ MeV, in disagreement (by many standard deviations) with results from DR [3] and values recommended by the Particle Data Group [4]. It is, therefore, interesting to show how much the modifications of the three-channel amplitudes affect position of the pole connected with the σ meson.

In this note, we present an example of using the dispersion relations for improving the low-energy behavior of a phenomenological three-channel S- and P-wave $\pi\pi$ amplitudes and for imposing the crossing symmetry condition on the amplitudes below 1.1 GeV.

2. Multichannel amplitudes and dispersion relations

Two channels coupled to the $\pi\pi$ one: $K\bar{K}$ and $\eta\eta\prime$ for S-wave and $\rho 2\pi$ and $\rho\sigma$ for P-wave, were explicitly considered in construction of the three-channel amplitudes [1]. The eight-sheeted Riemann surface was transformed into a uniformization plane using a variable w in which the left-hand branch point connected with the crossed channels was not taken into account. Also the crossing symmetry condition was not considered in the construction. A contribution of the left-hand cut was included in the background part of the amplitude (note that in Ref. [5] the left-hand branch point in w was already included in the S-wave analysis).

In the uniformization plane an influence of the $\pi\pi$-threshold branching point was neglected, keeping however the unitarity on the $\pi\pi$ cut, which was a necessary approximation in the three-channel case [1]. Therefore, there is a four-sheeted model of the initial Riemann surface in which the near-threshold data are not described properly. Note that in the two-channel case this approximation is not needed and the threshold data are described correctly [6].

The resonance part of the matrix element S_{ij} ($i, j = 1$ for pions) of the S-matrix is generated by clusters of complex-conjugate poles and zeros on the Riemann surface, which represent resonances [1]. For example, the

$f_0(500)$ (formerly was $f_0(600)$) resonance is represented by a cluster which possesses zero only in the S_{11} matrix element on the physical sheet. Location of poles on the unphysical sheets is given by the analytic continuation of the matrix elements [5]. The background and resonant parts of the S-matrix are separated and expressed via the Le Couteur–Newton relations with the Jost matrix determinant $d(w)$, where w is conformal energy variable [1]. In the model of the Riemann surface, only the semi-sheets of initial Riemann surface nearest to the physical region are considered. S-matrix elements are represented as

$$S = S^{\mathrm{bgr}} S^{\mathrm{res}} \,, \tag{1}$$

therefore [1],

$$S_{11} = S_{11}^{\mathrm{bgr}} S_{11}^{\mathrm{res}} = \frac{d_{\mathrm{bgr}}(-k_1, k_2, k_3)}{d_{\mathrm{bgr}}(k_1, k_2, k_3)} \frac{d_{\mathrm{res}}^*(-w^*)}{d_{\mathrm{res}}(w)} \,, \tag{2}$$

where S^{bgr} describes the background, k_j are the channel momenta and S^{res} the resonance contributions defined as

$$d_{\mathrm{res}}(w) = w^{-\frac{M}{2}} \prod_{r=1}^{M} (w + w_r^*) \,, \tag{3}$$

where the product includes all zeros w_{r} of the chosen resonances and M is a number of resonance zeros. The background part is modeled via complex energy-dependent phases $\alpha_j(s)$, $j = 1, 2, 3$ representing mainly an influence of other channels and the neglected left-hand cut

$$d_{\mathrm{bgr}}(k_j) = \exp\left[-i \sum \alpha_j(s)\right] \,. \tag{4}$$

The resonance zeros w_{r} and background parameters were obtained from fitting the phase shifts and inelasticity parameters in the assumed channels to experimental data [1]. Having matrix element S_{11}, one easily can calculate the full amplitudes $f_l^I(s)^{\mathrm{in}}$ for given isospin I and spin l

$$f_l^I(s)^{\mathrm{in}} = \frac{\sqrt{s}}{2k} \frac{S_{11} - 1}{2i} \,, \tag{5}$$

where $k = \sqrt{s/4 - M_\pi^2}$ and M_π^2 is the pion mass.

In order to obtain a precise description of $\pi\pi$ amplitudes, we have applied the Roy-like GKPY dispersion relations equations [2] with imposed crossing symmetry for the S- and P-wave amplitudes. These new dispersion relations impose quite strong constraints in our fits to the data on the analyzed $\pi\pi$ interactions. They read

$$\mathrm{Re} f_l^I(s)^{\mathrm{out}} = \sum_{I'=0}^{2} C_{\mathrm{st}}^{II'} a_0^{I'} + \sum_{I'=0}^{2}\sum_{l'=0}^{3} \int_{4m_\pi^2}^{S_{\max}} ds' K_{ll'}^{II'}\left(s,s'\right) \mathrm{Im} f_{l'}^{I'}\left(s'\right)^{\mathrm{in}} + d_l^I(s)\,, \tag{6}$$

where the first term is called "subtracting term" (ST_l^I) which is a linear combination of scattering lengths $a_0^{I'}$. In contrast to the standard Roy equations, the subtracting term in GKPY is constant and does not depend on the s.

In the second term the $K_{ll'}^{II'}(s,s')$ are known kernels and thus we will refer to integral terms as "kernel term" or $\mathrm{KT}(s)$. The input amplitudes and upper limit of the integration is up to $S_{\max} = 1420$ MeV [1, 2]. The summation includes also D- and F-waves described by phenomenological expressions [2]. In fact, subtracting term is smaller than kernel term which is clearly the dominant one. Therefore, no big cancellations between any two terms are needed in order to reconstruct the total real part of the amplitude.

The third term $d_l^I(s)$, which is called "driving term" has the same structure as the kernel term, but is not related to the phenomenological input amplitudes $f_l^I(s)^{\mathrm{in}}$. Its s and t dependence is given by Regge parameterization. As shown in Fig. 1 for both channels $\boldsymbol{s}$ as well as $\boldsymbol{t}$ which are the Mandelstam variables, output amplitudes should be similar and differ only by a crossing matrix factor $\hat{C_{\mathrm{st}}}$

$$T_s(s,t) = \hat{C_{\mathrm{st}}} T_t(t,s)\,. \tag{7}$$

In the dispersion relation Eq. (6), crossing symmetry is imposed under the assumption that Eq. (7) is valid.

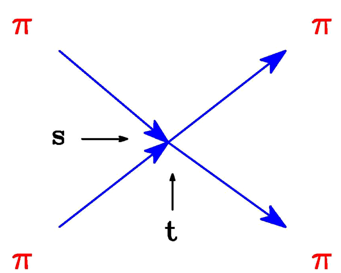

Fig. 1. Crossing symmetry.

Here $f_l^I(s)^{\mathrm{out}}$ and $f_l^I(s)^{\mathrm{in}}$ are the output and input amplitudes. The difference between $\mathrm{Re} f_l^I(s)^{\mathrm{out}}$ and $\mathrm{Re} f_l^I(s)^{\mathrm{in}}$ demonstrates a consistency of the amplitudes with the dispersion relations (*i.e.* with crossing symmetry). The smaller the difference, the better consistency with crossing symmetry (see the last term in Eq. (9) below).

3. Improvement of the amplitudes at low energies

Looking at results of the original amplitudes [1] from $\pi\pi$ threshold up to 900 MeV, in Fig. 2 (a), one sees that there is a significant problem with their threshold behavior. We replaced low-energy part of the amplitudes for S- as well as P-wave by polynomial functions from $\pi\pi$ threshold up to the so-called "matching point" (350–650 MeV). Value of polynomial functions and their derivatives at the matching points for S- and P-waves are matched with the original amplitude functions.

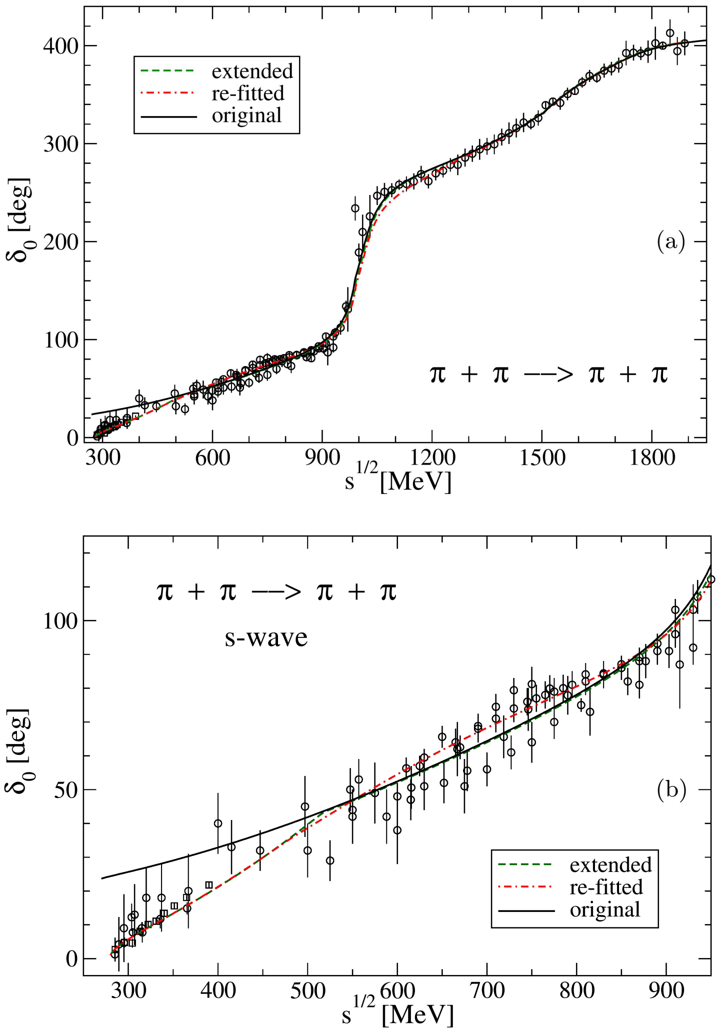

Fig. 2. Phase shift *versus* energy for S-wave.

This polynomial is a near threshold expansion of the S- and P-wave amplitudes and is determined by a generalized expansion in power of the pion momentum $k = \sqrt{s/4 - M_\pi^2}$

$$\mathrm{Re} f_l^I(s) = \frac{\sqrt{s}}{4k} \sin 2\delta_l^I = m_\pi k^{2l} \left[a_l^I + b_l^I k^2 + c_l^I k^4 + d_l^I k^6 + \mathcal{O}\left(k^8\right) \right] . \quad (8)$$

The amplitudes given by this expansion are matched with those for higher energies from the *original* amplitudes fitted to the data. In Eq. (8) a_l^I is the scattering length and b_l^I is the slope parameter with values: $a_0^0 = 0.211 m_\pi^{-1}$, $b_0^0 = 0.278 m_\pi^{-3}$, $a_1^1 = 0.0333 m_\pi^{-3}$, $b_1^1 = 0.00523 m_\pi^{-5}$ [2]. Coefficients c_l^I and d_l^I are calculated from the continuity conditions for the phase shift and its first derivative at the matching point.

Figure 2 (a) shows the phase shifts for $\pi\pi$ interaction in the S-wave *versus* energy from $\pi\pi$ threshold up to 1.8 GeV. The low-energy corrected original amplitudes are denoted as *extended* amplitudes. Above the matching point, the *original* and *extended* amplitudes are equivalent. Figure 2 (b) is focused on the threshold region which shows how the *extended* amplitude solved the threshold behavior problem of low-energy S-wave $\pi\pi$ interaction amplitude.

Parameters of the extended amplitude, which strongly influence the low-energy behavior of the amplitudes, were optimized (re-fitted) to fit the experimental data and to achieve a better consistency with the dispersion relations. It was performed by minimization of the χ^2 function

$$\begin{aligned} \chi^2 \; = & \sum_{I=0}^{1} \left(\sum_i \left(\frac{\delta_i^{\mathrm{exp}} - \delta_i^{\mathrm{th}}}{\Delta\delta_i^{\mathrm{exp}}} \right)^2 + \sum_i \left(\frac{\eta_i^{\mathrm{exp}} - \eta_i^{\mathrm{th}}}{\Delta\eta_i^{\mathrm{exp}}} \right)^2 \right) \\ & + \sum_{I=0}^{2} \sum_i \left(\frac{\mathrm{Re} f_i^{\mathrm{out}} - \mathrm{Re} f_i^{\mathrm{in}}}{\varDelta_{\mathrm{DR}}} \right)^2 . \end{aligned} \quad (9)$$

Symbols δ_i and η_i denote experimental and calculated values of the phase-shift and inelasticity parameter in the assumed channels of the S- and P-waves. The summation, therefore, runs also over the channels and partial waves. $\mathrm{Re} f_i^{\mathrm{out}}$ is calculated using the dispersion relations (6), and $\mathrm{Re} f_i^{\mathrm{in}}$ is the real part of the input amplitude on the right-hand side in Eq. (6). The number of experimental points is 492 and number of points in DR is 26 (note that we use data from S0- and P1-waves while for DR part the output amplitudes for the S0-, S2- and P1-waves). The scale parameter $\varDelta_{\mathrm{DR}} = 0.01$ makes a reasonable weight of the DR contribution to χ^2. Note that the last term in Eq. (9) provides a coupling between the S- and P-waves which would be otherwise independent in the analysis.

The re-fitted parameters are zeros of the lowest poles, $f_0(500)$, $f_0(980)$, $f_0(1500)$ and $\rho(770)$, the background parameters, the matching points and scattering lengths for S- and P-wave in the $\pi\pi$ channel. Experimental data used in this analysis are from Ref. [1] supplemented near the threshold with phases from the dispersive analysis [2] and data from the NA48 Collaboration [8].

4. Results

Applying the modifications, we achieved the S- and P-wave amplitudes for the $\pi\pi$ scattering. These modified amplitudes provided $\chi^2/\mathrm{n.d.f.} = 18.5$ (n.d.f. $= 483$) for the extended amplitudes, but the re-fitted parameters significantly improved the result, $\chi^2/\mathrm{n.d.f.} = 1.3$. The biggest and more important improvement was that for the DR contribution. The last term in Eq. (9) changed from 1573.2 to 58.7 (see Table I), which suggests a significant improvement of consistency of the amplitudes with the crossing symmetry. The re-fitted amplitudes provided also proper values of the phase shifts and inelasticity parameters in the assumed coupled channels as the original amplitudes.

TABLE I

Values of χ^2 for S-wave before and after fitting.

	Data	DR	Total
χ^2 initial	7727.6	1573.2	8920.5
χ^2 final	567.3	58.7	626.0

Figures 3 and 4 illustrate the $\mathrm{Re}f_0^0(s)$ before and after fitting. Black and gray points represent value of output and input respectively for specified energies. Comparison of these two figures shows how significantly the difference between input $\mathrm{Re}f_0^0(s)^{\mathrm{in}}$ and output $\mathrm{Re}f_0^0(s)^{\mathrm{out}}$ amplitudes decreased after re-fitting the chosen parameters.

Positions of poles changed strongly for the $f_0(500)$ resonance, *e.g.* on the sheet II the pole shifted from $617 - i554.0$ MeV for the original amplitude to $458 - i289$ MeV in the input one ($449^{+15}_{-14} - i287^{+14}$ MeV in the output). Note that the new pole position accords well with the result from the analysis based on the ChPT and Roy-like equations ($441^{+16}_{-8} - i272^{+9}_{-12.5}$) [7] and the result from the analysis based only on the Roy-like equations ($445^{+25}_{-25} - i278^{+22}_{-18}$) [3].

The poles of $f_0(980)$ shifted slightly, *e.g.* on the sheet II from $1013 - i31$ MeV to $997 - i22$ MeV in the input amplitude ($997^{+2}_{-2} - i22^{+2}_{-2}$ MeV in the output one), what is more consistent with the values suggested by the Particle Data Group: $980^{+20}_{-20} - i(25\text{–}50)$ MeV [4].

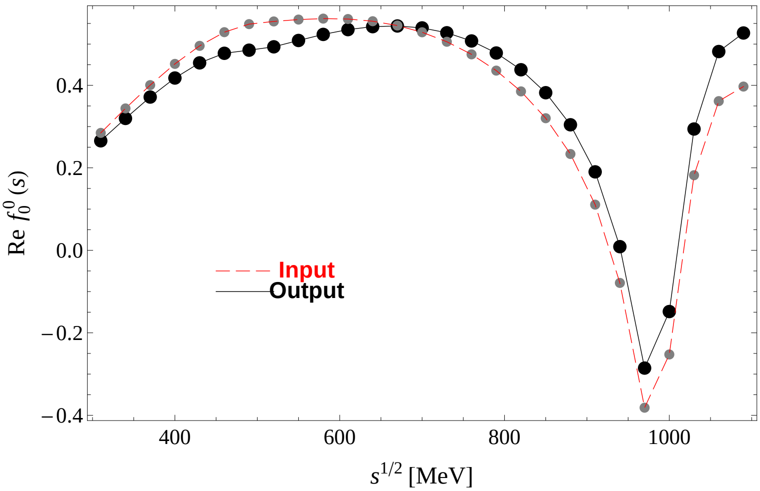

Fig. 3. Re$f_0^0(s)$ for input and output before fitting.

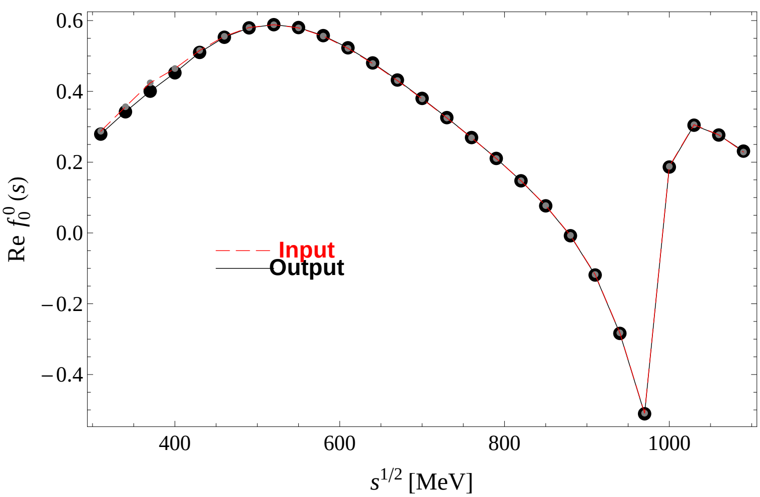

Fig. 4. Re$f_0^0(s)$ for input and output after fitting.

The poles of $f_0(1500)$ shifted, *e.g.* on the sheet II from $1502.4 - i236.8$ MeV to $1507.01 - i171.8$ MeV in the input amplitude ($1548.47^{+19.9}_{-19.9} - i222.778^{+21.6}_{-21.6}$ MeV in the output one), while the values suggested by the Particle Data Group are $1505^{+6}_{-6} - i218.778^{+7}_{-7}$ MeV [4].

The poles of $\rho(770)$ moved up by less than 1%, *e.g.* on the sheet II from $761^{+4}_{-3} - i71^{+1.9}_{-2.3}$ MeV to $766 - i73$ MeV in the input amplitude ($762^{+4.7}_{-4.7} - i73^{+4.9}_{-4.6}$ MeV in the output one).

Re-fitted scattering lengths for S-wave and P-wave become $a_0^0 = 0.221 m_\pi^{-1}$ and $a_1^1 = 0.0328 m_\pi^{-3}$ respectively. Re-fitted values of the background parameters are small suggesting that important part of dynamics is included in the resonant part of S-matrix. However, in the S-wave the background phase shift becomes negative starting at the $\pi\pi$ threshold which seems to be necessary for a good description of the data.

Figure 5 shows positions of the σ poles on the complex plane from various experiments and from the present analysis. PDG 2010 is the area that was expected to see the σ pole before 2012 (a broad region with $M = 400$–1200 MeV and $\Gamma = 2 \times (250$–$500)$ MeV). Since 2012 this area became much smaller (PDG 2012) due to improvement of our knowledge on the threshold parameters and $\pi\pi$ amplitudes. In our analysis, position of the pole moves from $617 - i554.0$ MeV for the original amplitude to $458 - i289$ MeV after fitting.

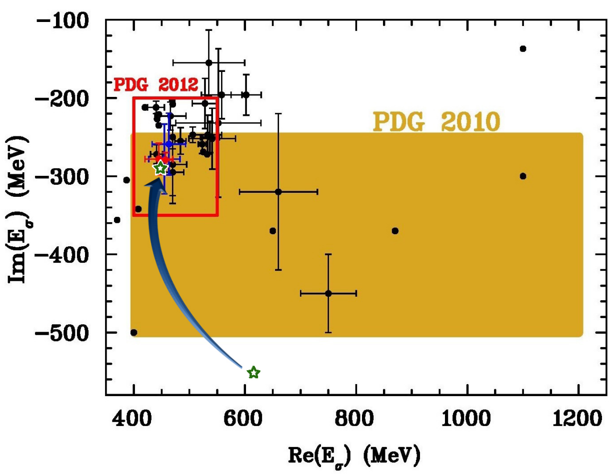

Fig. 5. Movement of the σ pole due to the fit to the GKPY equations.

5. Conclusions

To summarize, agreement of the phase shifts with low-energy data was improved for the new re-fitted S- and P-wave $\pi\pi$ scattering amplitudes. The amplitudes are calculated with the scattering lengths and slope (effective-range) parameters consistent with results of calculations based on DR and ChPT. Consistency of the three-channel amplitudes with the dispersion relations was improved significantly for the energies from the threshold up to 1.1 GeV which means that the amplitudes better fulfill the crossing symmetry condition. The lowest pole in S-wave is shifted to lower energy and nearer to the real axis which results in smaller values of the mass and width for the σ meson.

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