# GENERATION OF NEW EXACTLY SOLVABLE POTENTIALS OF POSITION-DEPENDENT MASS SCHRÖDINGER EQUATION BY EXTENDED TRANSFORMATION METHOD

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Exactly solvable potentials of the position-dependent mass Schrödinger equation are generated by taking 'Hulthén plus hyperbolic cotangent potential' as a parent system. We apply a simple transformation method that includes a co-ordinate transformation followed by a functional transformation of wave function, and also a set of plausible ansatze. The mass function of the parent system gets transformed to a new mass function of the generated system. The generated potentials are mostly Sturmian, which are energy dependent for non-power law potentials. Some of the generated Sturmian potentials can be converted into normal potentials by regrouping various potential parameters. The wave functions of generated systems are normalizable in most cases.

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## 1. Introduction

In quantum mechanics, there is a limited number of exactly solvable potentials (ESP) which yield exact analytic solutions of the Schrödinger equation. Therefore, for non-solvable potentials one has to depend on various approximation techniques such as perturbation theory, variational technique, WKB approximation method *etc.* However, the successful implementation of these approximation schemes for a given quantum system, largely depends on the nearness of the potential to some exactly solvable potential (ESP).

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This motivates us to find more and more ESPs which will facilitate practical quantum mechanical calculations. Besides, non-perturbative solutions of different potentials may lead to new physical ideas as well as new calculational techniques in quantum physics. Therefore, researchers are trying to device different methods to generate or construct exactly solvable potentials.

In the solid state physics, the concept of position-dependent effective mass has remarkably improved the understanding of several physical properties ranging from electronic conduction to complex optical phenomenon [1]. The envelope function of conduction band obeys the Schrödinger equation with position-dependent mass (PDM) [2]. In recent years, PDM Schrödinger equations have received considerable attention from a number of researchers. Special applications of PDM Schrödinger equations are found in the investigation of the electronic properties of semiconductors [2, 3], quantum dot and quantum well [4, 5], quantum liquid [6], nuclear many body problems [7] *etc.* Since mass is a function of position, the kinetic energy operator is non-commutative and the Hamiltonian operator remains non-Hermitian. Von Roos proposed [2] the kinetic energy operator in terms of three ambiguous parameters to make the effective Hamiltonian Hermitian where the effective potential is expressed as a sum of real potential profile and a modification emerging out of PDM.

Position-dependent mass Schrödinger equations are relatively more complex to solve unlike ordinary Schrödinger equations (with constant mass). Different techniques known for solving constant mass Schrödinger equations or extension of them, have been applied to solve PDM Schrödinger equations. For example, point canonical transformation (PCT) [8–12], Nikiforov–Uvarov (NU) method [13–15], supersymmetric (SUSY) approach [16, 17], quadratic algebra [18], analytical method [19], series solution method [20], Durboux transformation (DT) [21], *etc.* have been used in the literature. In the similar spirit of SUSY approach, extended transformation (ET) [22] method has been developed to generate new ESPs of constant mass Schrödinger equation from an already known exactly solvable potential. The ET method has been successfully exploited for generation or construction of ESPs from power law, non-power law and sometimes from multiterm ESPs [23–28] and also ring shape potentials [29].

Our main aim in this paper is to apply the ET method in PDM Schrödinger equations to generate exactly solvable potentials from Hulthén plus hyperbolic cotangent potential (which is a non-power law potential). The ET includes a co-ordinate transformation (CT) followed by a functional transformation (FT) and a set of plausible ansatze to mould the transformed equation in the form of Schrödinger equation. While applying the ET method in PDM Schrödinger equations, we had the advantage that the mass function of parent system gets transformed to new mass function for generated system.

The method provides us with a direct way of obtaining wave functions, energy eigenvalues and mass function for generated system from already solved system without actually solving the PDM Schrödinger equation. As in the case of constant mass system, the potentials of PDM Schrödinger equation generated (by ET) from non-power law potential are also in general Sturmian; some of which can be made normal by regrouping various potential parameters.

The organization of the paper is as follows: the ET method to solve PDM Schrödinger equation is described in Sec. 2, the method applied to generate new QSs from already solvable Hulthén plus hyperbolic cotangent potential is presented in Sec. 3, the normalizability property of the transformed wave functions is discussed in Sec. 4. Conclusions and important results are presented in Sec. 5.

## 2. Extended transformation (ET) to solve PDM Schrödinger equation

We start with one-dimensional position-dependent mass effective Hamiltonian [30] ($2m_0 = \hbar = 1$)

$$H_{\text{eff}} = -\frac{d}{dx}\left[\frac{1}{m(x)}\frac{d}{dx}\right] + V^{\text{eff}}(x)\,, \tag{1}$$

where $V^{\text{eff}}(x)$ is effective potential given by

$$V^{\text{eff}}(x) = V(x) + U_\alpha(x)\,. \tag{2}$$

Here $V(x)$ is real potential profile and $U_\alpha(x)$ is the modification of potential arising out of PDM. $U_\alpha(x)$ is given by

$$U_\alpha(x) = \frac{1}{2}(\beta+1)\frac{m''(x)}{m^2(x)} - [\alpha(\alpha+\beta+1)+\beta+1]\frac{m'^2(x)}{m^3(x)}\,, \tag{3}$$

where $\alpha$ and $\beta$ with $\gamma$ are ambiguous parameters, called Von Roos ambiguity [2] satisfying the relation $\alpha+\beta+\gamma = -1$ and $m(x)$ is the dimensionless mass function related to constant mass $m_0$ and PDM $M(x)$, by the relation $M(x) = m_0 m(x)$. The prime stands for derivative of the function with respect to its argument. Using equation (1), one dimensional position-dependent mass (1D-PDM) Schrödinger equation becomes

$$\left[-\frac{d}{dx}\left[\frac{1}{m(x)}\frac{d}{dx}\right] + V^{\text{eff}}(x)\right]\psi(x) = E_n\psi(x)\,.$$

The above equation finally takes the form

$$\psi''(x) - \frac{m'(x)}{m(x)}\psi'(x) + m(x)\left[E_n - V^{\text{eff}}(x)\right]\psi(x) = 0\,. \tag{4}$$

Let $V_A(x)$ be a known physical potential, hereafter termed as $A$-quantum system ($A$-QS). Equation (4), which is exactly solvable for $A$-QS is written as

$$\psi_A''(x) - \frac{m_A'(x)}{m_A(x)}\psi_A'(x) + m_A(x)\left[E_n^A - V_A^{\text{eff}}(x)\right]\psi_A(x) = 0\,, \tag{5}$$

where energy eigenvalues $E_n^A$ and corresponding wave functions $\psi_A(x)$ are known for given $V_A(x)$ and $m_A(x)$. We now invoke ET, which comprises of CT followed by FT as given below:

CT

$$x \to g_B(x)\,, \tag{6}$$

FT

$$\psi_B(x) = f_B^{-1}(x)\psi_A(g_B(x))\,, \tag{7}$$

where $\psi_B(x)$ stand for wave functions of transformed system, hereafter called the $B$-quantum system ($B$-QS). In constant mass systems, mass is the same for generated system and parent system, as mass is independent of position. For a system with PDM, mass is a function of position (co-ordinate). Therefore, mass function of a system with PDM should transform under a co-ordinate transformation. We, therefore, write the mass function of $B$-QS as

$$m_B(x) = m_A(g_B(x))\,. \tag{8}$$

The transformation function $g_B(x)$ must be a differentiable function of at least class $C^3$. The $g_B(x)$ and the modulated amplitude $f_B^{-1}(x)$ have to be specified within the framework of ET. Application of equations (6), (7) and (8) in equation (5) of the $A$-QS, gives

$$\psi_B''(x) + \frac{d}{dx}\ln\left(\frac{f_B^2}{g_B'(x)m_B(x)}\right)\psi_B'(x) + \left[\left\{\frac{d}{dx}\ln f_B(x)\right\} \times \left\{\frac{d}{dx}\ln\left(\frac{f_B'(x)}{g_B'(x)m_B(x)}\right)\right\} + g_B'^2(x)m_B(x)\left\{E_n^A - V_A^{\text{eff}}(g_B)\right\}\right]\psi_B(x) = 0\,. \tag{9}$$

Consistency demands that the coefficient of $\psi_B'(x)$ in equation (9) must be identical to $-\frac{m_B'(x)}{m_B(x)}$. Therefore, we put

$$\frac{d}{dx}\ln\left(\frac{f_B^2}{g_B'(x)m_B(x)}\right) = -\frac{m_B'(x)}{m_B(x)}$$

which fixes $f_B(x)$ as

$$f_B(x) = N_B g_B'^{\frac{1}{2}}(x)\,, \tag{10}$$

where $N_B$ is an integration constant and plays the role of normalization constant of the energy eigenfunctions of $B$-QS. Application of equation (10) in equation (9) gives

$$\psi_B''(x) - \frac{m_B'(x)}{m_B(x)}\psi_B'(x) + m_B(x)\Bigg[\frac{1}{2m_B(x)}\{g_B, x\} - \frac{1}{2}\frac{m_B'(x)}{m_B^2(x)}\frac{g_B''(x)}{g_B'(x)} + g_B'^2(x)\left\{E_n^A - V_A(g_B) - U_{\alpha A}(g_B)\right\}\Bigg]\psi_B(x) = 0\,, \tag{11}$$

where

$$\{g_B, x\} = \frac{g_B'''(x)}{g_B'(x)} - \frac{3}{2}\left(\frac{g_B''(x)}{g_B'(x)}\right)^2 \tag{12}$$

is the Schwartzian derivative symbol and $U_{\alpha A}(x)$ is the modification of potential for $A$-QS. If the potential $V_A(x)$ happens to be a multiterm potential having $n$-terms, one can choose various terms in $(2^n - 1)$ different ways as working potential. For simplicity, we choose single term potential only as working potential which is denoted by $V_A^w(x)$. In order to recast equation (11) in the standard form of 1D-PDM Schrödinger equation for $B$-QS, we have to make the following plausible ansatze which is an integral part of the method

$$g_B'^2(x)V_A^w(g_B(x)) = -E_n^B\,, \tag{13}$$

$$g_B'^2(x)E_n^A = -V_B^{(1)}(x)\,, \tag{14}$$

$$-g_B'^2(x)[V_A(g_B(x)) - V_A^w(g_B(x))] = -V_B^{(2)}(x)\,, \tag{15}$$

$$\frac{1}{2m_B(x)}\left[\{g_B, x\} - \frac{m_B'(x)}{m_B(x)}\frac{g_B''(x)}{g_B'(x)}\right] = -V_B^{(3)}(x)\,, \tag{16}$$

and

$$g_B'^2(x)U_{\alpha A}(g_B) = U_{\alpha B}(x)\,. \tag{17}$$

In equation (13), $V_A^w(g_B(x))$ is known in terms of $g_B(x)$. Therefore, solution of equation (13) specifies the functional form of $g_B(x)$. The $g_B(x)$ is used in calculating mass function $m_B(x)$ through equation (8). Again, $g_B(x)$ and its derivatives, $m_B(x)$ and its derivatives are utilized to calculate $V_B^{(1)}(x)$, $V_B^{(2)}(x)$, $V_B^{(3)}(x)$ and $U_{\alpha B}(x)$ (the modification of potential

for $B$-QS) through equations (14)–(17) respectively, which yield the $B$-QS potential $V_B(x)$ as

$$V_B(x) = V_B^{(1)}(x) + V_B^{(2)}(x) + V_B^{(3)}(x) \tag{18}$$

and effective potential $V_B^{\text{eff}}(x)$ as

$$V_B^{\text{eff}}(x) = V_B(x) + U_{\alpha B}(x)\,. \tag{19}$$

From equation (14), $V_B^{(1)}(x)$ will be in the form

$$V_B^{(1)}(x) = -C_B^2 v(x)\,,$$

where $v(x)$ is a function of $x$ and $C_B^2$ is a constant independent of $x$, called characteristic constant for $B$-QS, which is equal to the product of a function $f(E_n^B)$ and $E_n^A$ [23] *i.e.*

$$C_B^2 = f\left(E_n^B\right) E_n^A\,.$$

This equation gives $B$-QS energy eigenvalues as

$$E_n^B = f^{-1}\left(C_B^2\right) E_n^{A^{-1}}\,. \tag{20}$$

Finally, we get 1D-PDM Schrödinger equation for $B$-QS in the form

$$\psi_B''(x) - \frac{m_B'(x)}{m_B(x)}\psi_B'(x) + m_B(x)\left[E_n^B - V_B^{\text{eff}}(x)\right]\psi_B(x) = 0 \tag{21}$$

with exact energy eigenfunctions

$$\psi_B(x) = N_B {g_B'}^{-\frac{1}{2}} \psi_A(g_B(x))\,. \tag{22}$$

The normalizability condition for the $B$-QS wave functions is

$$I(-\infty,+\infty) = \int_{g_B(-\infty)}^{g_B(+\infty)} |\,\psi_B(x)\,|^2\,dx = \text{finite}\,. \tag{23}$$

## 3. Application of ET to generate ESPs of PDM Schrödinger equation

We have considered the Hulthén plus deformed type hyperbolic cotangent potential as $A$-QS, whose exact solution of the 1D-PDM Schrödinger equation is available [15]. The $A$-QS potential is given by

$$V_A(x) = -V_1 \frac{\exp(-2\lambda x)}{1 - q\exp(-2\lambda x)} + V_2 \coth_q(\lambda x)\,. \tag{24}$$

For

$$m_A(x) = \frac{1}{1 - q\exp(-2\lambda x)}\,, \tag{25}$$

the discrete energy eigenvalues are

$$E_n^A = -4\lambda^2\left[\delta - \left(n + \mu + \tfrac{1}{2}\right)\right]^2 + V_2 \tag{26}$$

and the corresponding wave functions are

$$\psi_A(x) = N_A S_A^{\mu+\frac{1}{2}}(1 - S_A)^{\lambda_A} P_n^{(2\mu,2\lambda_A)}(1 - 2S_A)\,, \tag{27}$$

where

$$\begin{aligned}
\lambda_A &= \left(\frac{V_2 - E_n^A}{4\lambda^2}\right)^{\frac{1}{2}}\,,\\
\mu &= \left(\frac{1}{4} - \frac{\beta+1}{2} - \alpha(\alpha+\beta+1)\right)^{\frac{1}{2}}\,,\\
\delta &= \pm\left(\frac{2qV_2 - V_1}{4q\lambda^2} - \alpha(\alpha+\beta+1)\right)^{\frac{1}{2}}\,,
\end{aligned}$$

and

$$S_A(x) = m_A(x) = \frac{1}{1 - q\exp(-2\lambda x)}\,.$$

$P_n^{(2\mu,2\lambda_A)}(1-2S_A)$ are well known Jacobi polynomials and $N_A$ is normalization constant. The parameters $\mu$, $\delta$ and $\lambda_A$ satisfy the relation

$$\mu + \lambda_A = \delta - n - \tfrac{1}{2}\,. \tag{28}$$

### *3.1. First order transformation*

From $A$-QS potential given by equation (24), we have chosen the following working potential

$$V_A^w(x) = -V_1\frac{\exp(-2\lambda x)}{1 - q\exp(-2\lambda x)}\,. \tag{29}$$

Application of equation (29) into equation (13), gives the transformation function $g_B(x)$ as

$$g_B(x) = \frac{1}{\lambda}\ln\sec p_n x\,, \tag{30}$$

where

$$p_n^2 = \frac{q\lambda^2 E_n^B}{V_1}\,. \tag{31}$$

Local property $g_B(0) = 0$ gives $q = 1$. The $g_B(x)$ given by equation (30), when applied in equation (8) gives mass function for $B$-QS as

$$m_B(x) = \csc^2 p_n x \,. \tag{32}$$

Again, application of equation (30) in equations (14)–(16) yields

$$V_B^{(1)}(x) = C_B^2 \tan^2 p_n x \,, \tag{33}$$

$$V_B^{(2)}(x) = \frac{p_n^2 V_2}{\lambda^2} \left(\sec^2 p_n x + 1\right) , \tag{34}$$

and

$$V_B^{(3)}(x) = -\frac{p_n^2}{4} \sec^2 p_n x \,, \tag{35}$$

respectively, where

$$C_B^2 = \frac{p_n^2}{\lambda^2} \left(-E_n^A\right) \tag{36}$$

is the characteristic constant of $B$-QS, obtained from the transformation of $A$-QS. Equation (36) subsequently provides us with energy eigenvalues of $B$-QS. Equation (36) can also be written as

$$C_B^2 = \frac{4\lambda^2 E_n^B}{V_1} \left[\left[\delta - \left(n + \mu + \frac{1}{2}\right)\right]^2 - \frac{V_2}{4\lambda^2}\right] . \tag{37}$$

Now equation (18), with the help of equations (33)–(35) yields the newly generated potential of $B$-QS as

$$V_B(x) = C_B^2 \tan^2 p_n x + \left[\frac{p_n^2 V_2}{\lambda^2} - \frac{p_n^2}{4}\right] \sec^2 p_n x + \frac{p_n^2 V_2}{\lambda^2} \,. \tag{38}$$

The potential $V_B(x)$ is $n$-dependent through the $n$-dependence of $p_n$. Therefore, $V_B(x)$ specifies a Sturmian QS. Each value of $n$ for $B$-QS represents an exact QS and $B$-QS consists of $n$ number of systems. The Sturmian $B$-QS can be converted into normal as well as physical by system specific regrouping method where we have to redefine the parameters of $A$-QS preserving the type of constraint equations. To make $p_n^2$ $n$-independent, we have made $V_1 \to V_n$ by setting $V_1 = \frac{q\lambda^2 E_n^B}{s^2}$, where a scale factor $s$ is introduced. This leads to $p_n \to p = s$, a constant. As a result, the Sturmian form of $B$-QS gets converted into the normal form as

$$V_B(x) = C_B^2 \tan^2 sx + s^2 V_0 \sec^2 sx + s^2 \left(V_0 + \tfrac{1}{4}\right) , \tag{39}$$

where

$$V_0 = \frac{V_2}{\lambda^2} - \frac{1}{4}\,. \tag{40}$$

Equation (17) leads modification $U_{\alpha B}(x)$ to

$$U_{\alpha B}(x) = 4s^2\left[\frac{\beta+1}{2}\left(1+\cos^2 sx\right) - \left[\alpha(\alpha+\beta+1)+\beta+1\right]\cos^2 sx\right]. \tag{41}$$

As a consequence, equation (37) gives the quantized energy eigenvalues of $B$-QS as

$$E_n^B = -4s^2\left[\left\{n+\mu+\lambda_B+\tfrac{1}{2}\right\}^2 - \tfrac{1}{2}\left(V_0+\tfrac{1}{4}\right)+\alpha(\alpha+\beta+1)\right], \tag{42}$$

where

$$\lambda_B^2 = \frac{1}{4}\left(\frac{C_B^2}{s^2}+V_0+\frac{1}{4}\right). \tag{43}$$

Parameters $\mu$ and $\lambda_B$ satisfy the relation

$$\mu+\lambda_B = \phi - n - \tfrac{1}{2}\,, \tag{44}$$

where

$$\phi^2 = \frac{1}{2}\left(V_0+\frac{1}{4}\right) - \frac{E_n^B}{4s^2} - \alpha(\alpha+\beta+1)\,. \tag{45}$$

From equation (22), we have got the exact energy eigenfunctions of the newly generated $B$-QS as

$$\psi_B(x) = N_B S_B^{\mu+\frac{1}{2}}(1-S_B)^{\lambda_B+\frac{1}{4}}P_n^{(2\mu,2\lambda_B)}(1-2S_B)\,, \tag{46}$$

where $N_B$ is normalization constant and

$$S_B(x) = m_B(x) = \csc^2 sx\,.$$

### *3.2. Second order transformation*

From $B$-QS potential as obtained from equation (39), we can choose any term as working potential to generate another new potential hereafter designated as $C$-QS by applying ET. We have chosen the following single term working potential for simplicity as in the previous case

$$V_B^w(x) = s^2 V_0 \sec^2 sx\,. \tag{47}$$

Applying the same procedure as in the generation of $B$-QS, we have got the transformation function for $C$-QS

$$g_C(x) = \frac{1}{s}\arccos\sec h(a_n x)\,, \tag{48}$$

where $a_n = s\eta_n$ and

$$\eta_n^2 = \frac{-E_n^C}{s^2 V_0}\,, \tag{49}$$

the mass function of $C$-QS as

$$m_C(x) = \coth^2(a_n x)\,, \tag{50}$$

the Sturmian form of potential for $C$-QS as

$$V_C(x) = C_C^2 \sec h^2(a_n x) + a_n^2\left(4\lambda_C^2 - A^2\right)\tanh^2(a_n x) + a_n^2 A^2 \sec h^2(a_n x)$$
$$+\frac{3}{2}a_n^2 \tanh^2(a_n x)\sec h^2(a_n x) + \frac{a_n^2}{4}\tanh^4(a_n x)\,, \tag{51}$$

and the modification term of potential as

$$U_{\alpha C} = 4a_n^2\left[\frac{\beta+1}{2}\sec h^2(a_n x)\left\{\sec h^2(a_n x)+1\right\}\right.$$
$$\left.-[\alpha(\alpha+\beta+1)+\beta+1]\sec h^4(a_n x)\right]\,, \tag{52}$$

where $A^2 = V_0 + \frac{1}{4}$ and $C_C^2$ is the characteristic constant of $C$-QS obtained from the transformation of $B$-QS, given by

$$C_C^2 = -\eta_n^2 E_n^B\,. \tag{53}$$

The $C$-QS potential, however is not possible to convert into normal by any system specific regrouping method. Equations (49) and (53) give the energy eigenvalues of $C$-QS as

$$E_n^c = -\frac{C_C^2 k_C^2}{4\left[\left\{n+\mu+\lambda_C+\frac{1}{2}\right\}^2 - \frac{A^2}{2} + \alpha(\alpha+\beta+1)\right]}\,, \tag{54}$$

where $k_C^2 = A^2 - \frac{1}{4}$ and $\lambda_C^2 = \frac{C_B^2}{4s^2} + \frac{A^2}{4}$. The parameters $\lambda_C$ and $\mu$ satisfy the relation

$$\mu + \lambda_C = \sigma - n - \tfrac{1}{2}\,, \tag{55}$$

where $\sigma^2 = \frac{A^2}{2} - \frac{C_C^2 k_C^2}{4E_n^C} - \alpha(\alpha+\beta+1)$. Again, we have got the exact energy eigenfunctions for $C$-QS as

$$\psi_C(x) = N_C S_C^{\mu+\frac{3}{4}}(1-S_C)^{\lambda_C} P_n^{(2\mu,2\lambda_C)}(1-2S_C)\,, \tag{56}$$

where $N_C$ is normalization constant and

$$S_C(x) = m_c(x) = \coth^2(a_n x)\,. \tag{57}$$

## 4. Normalizability of the transformed wave function

A very useful property of transformation method is that wave functions of the generated quantum systems are almost always normalizable. Equation (23) is the normalizability condition for wave functions of generated systems. Using equations (22) and (13) in equation (23), we have

$$\mid N_B \mid^2 \int_{g_B(-\infty)}^{g_B(+\infty)} \psi_A^*(x) \left[ -\frac{V_A^w(x)}{E_n^B} \right] \psi_A(x) dx = \mid N_B \mid^2 \frac{\langle V_A^w(x) \rangle}{-E_n^B} = \text{finite}\,. \tag{58}$$

All the wave functions $\psi_B(x)$ are normalizable for which $E_n^B \neq 0$ [23]. Since $\psi_A(x)$ is normalizable and $V_A^w(x)$ represents real physical system, the quantity $\langle V_A^w(x) \rangle$ necessarily exists, indicating that $\psi_B(x)$ is normalizable. Hence, $C$-QS wavefunctions $\psi_C(x)$ are also normalizable.

## 5. Conclusion

In this paper, we have generated two new exactly solvable potentials of the PDM Schrödinger equation from an already exactly known solvable potential in equations (39) and (51) respectively. We have applied a simple transformation method comprising of a coordinate transformation and a functional transformation of wave function. Mass function of the parent system ($A$-QS) has found to be got transformed to the new mass function for the generated system. The newly generated potentials are the non-power law Sturmian potentials of which $B$-QS has been made normal by system specific regrouping method. It is evident that the ET may be applied successively any number of times to generate new QSs, when we are considering a non-power law potential. In most cases, the wave functions of the generated systems are normalizable. The present formalism can be further generalized to $N$-dimensional PDM Schrödinger equation [31].

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