EIGENVALUES OF SUPERSYMMETRIC QUANTUM MATRIX MODELS

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Recently proposed by Korsch and Glück [*Eur. J. Phys.* **23**, 413 (2002)] an extremely simple method for computation of eigenvalues via direct representation of position and momentum operators in matrix form is successfully applied to the calculation of energies of the ground and excited states of the x^2y^2 Hamiltonian and its supersymmetric quantum matrix extensions.

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1. Introduction

The concept of supersymmetry or fermion–boson symmetry is one of the most fascinating topics in the quantum field theory, which provides a unified description of bosons and fermions [1–5]. Although it has not yet been observed in nature, thousands of papers have been written on this subject. Its validity in particle physics follows from the common belief in grand unification through the feasibility of incorporating quantum gravity. Supersymmetric quantum mechanics, introduced in 1981 by Witten [6] is based on the simplest superalgebra, in order to provide a simple non-relativistic model for the spontaneous SUSY breaking mechanism. Witten's formulation of non-relativistic SUSY quantum mechanics attracted considerable attention in the last decade and is still serving as a useful tool in quantum physics rest on the existence of two operators $\hat{Q}$ and $\hat{P}$. $\hat{Q}$ is called the supercharge and $\hat{P}$ is Witten parity operator with eigenvalues ± 1. The quantum system is supersymmetric if operators obey the following rules:

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$$
\begin{aligned}
\left\{\hat{Q}, \hat{P}\right\} &= \hat{Q}\hat{P} + \hat{P}\hat{Q} = 0\,, \\
\hat{H} &= \hat{Q}^2\,, \\
\hat{P}^2 &= 1\,.
\end{aligned} \tag{1}
$$

Supersymmetry transformations are generated as

$$
\hat{Q}\,|\text{fermion}\rangle = |\text{boson}\rangle\,, \qquad \hat{Q}\,|\text{boson}\rangle = |\text{fermion}\rangle \tag{2}
$$

and

$$
\hat{P}\,|\text{fermion}\rangle = -\,|\text{fermion}\rangle\,, \qquad \hat{P}\,|\text{boson}\rangle = |\text{boson}\rangle\,. \tag{3}
$$

An interesting example is supercharge

$$
\hat{Q} = (p_x - a_x)\,\sigma_x + (p_y - a_y)\,\sigma_y\,, \tag{4}
$$

where a_x, a_y are components of external vector potential, characterizing a magnetic field $B_z = \partial_x a_y - \partial_y a_x$, which is perpendicular to the x–y plane.

The corresponding supersymmetric Hamiltonian

$$
\hat{H} = \hat{Q}^2 = (p_x - a_x)^2 + (p_y - a_y)^2 - B_z\sigma_z \tag{5}
$$

is identical with the two-dimensional Pauli Hamiltonian. The Pauli matrices are formulated in the following way:

$$
\begin{aligned}
\sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}
$$

Ground state energy of this Hamiltonian is zero, as has been shown by Aharonov and Casher [7] using Atiyah–Singer index theorem.

2. Matrix representation and spectrum of x^2y^2 Hamiltonian

In 1983 B. Simon in his remarkable paper [8] solved the problem suggested by J. Goldstone and R. Jackiw on the existence of discrete spectrum for the Hamiltonian

$$
\hat{H} = \hat{p}_x^2 + \hat{p}_y^2 + \hat{x}^2\hat{y}^2\,. \tag{6}
$$

When looking at the x^2y^2 potential shown in Fig. 1, it is clear that the existence of the bound states is not trivial, especially when compared with potential of the archetypal two-dimensional harmonic oscillator shown in the left panel.

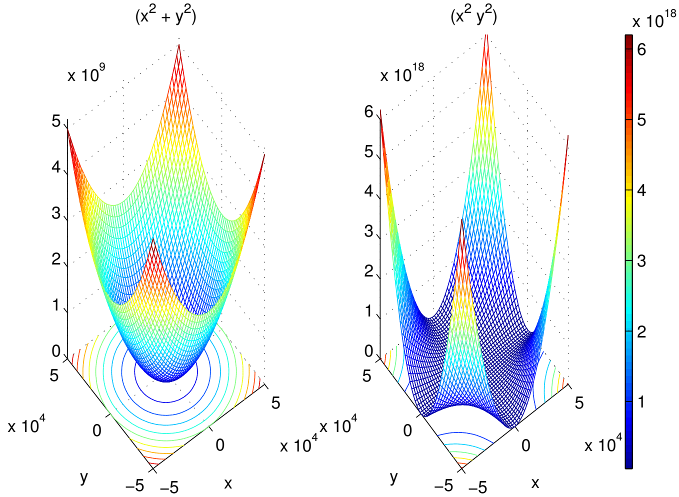

Fig. 1. Comparison of the 2D harmonic potential (left) with x^2y^2 potential (right).

It is naturally expected that systems with potentials equal to zero on an unbounded set, have continuous spectrum. Using five different methods, Simon demonstrated the discreetness of the spectra. The reason that the spectrum turns out to be exclusively discrete is due to the quantum fluctuations in the transverse directions, as has been only recently shown by Korcyl [9]. When a particle is moving in one of the valleys, the transverse potential is the potential of a harmonic oscillator with frequency proportional to the distance from the centre of valley. The zero-mode energy of such fluctuations increases when the particle is moving deep into the valley. Therefore, the particle is exposed to an effective potential barrier, which prevents it from escaping.

Recently, Korsch and Glück in [10] proposed a new, extremely simple method for computation of eigenvalues via representation of position and momentum operators in matrix form, and our first aim is to apply it for the calculation for the energies of the ground and excited states of the x^2y^2 Hamiltonian. First, we would rewrite Hamiltonian given by (6) in the mathematically more precise form of Kronecker tensor product. For this purpose, we introduce the definition of the Kronecker tensor product of two matrices. Let $A := [a_{ij}]$ be a matrix of the order of $m \times n$ and $B := [b_{kl}]$ be a matrix of the order of $r \times s$. Then, the Kronecker product of A and B is defined as

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} . \tag{7}$$

Position and momentum operators, $\hat{x}$ and $\hat{p}$ respectively, are represented (in a basis of normalized eigenstates of harmonic oscillator $|\mu\rangle$) by matrices x and p, respectively [10]

$$x = \frac{x_0}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{8}$$

$$p = \frac{i\,p_0}{\sqrt{2}} \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & \dots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{9}$$

Thus, the x^2y^2 Hamiltonian (6) is expressed in matrix form using the Kronecker product as

$$\hat{H} = p_x^2 \otimes I_N + I_N \otimes p_y^2 + x^2 \otimes y^2 , \tag{10}$$

where I_N is the unit matrix of dimension N.

The obtained eigenvalues for various dimensions of matrix representation of Hamiltonian (10) are given in Table I. As can be seen, the convergence of (non-degenerate) ground state (Fig. 2) as well as first 19 excited states (some of them are degenerate) is rather fast.

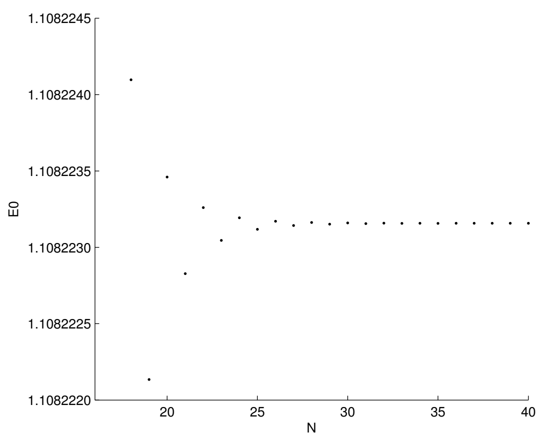

Fig. 2. Convergence of ground state eigenvalue for x^2y^2 Hamiltonian.

TABLE I

Eigenvalues of the x^2y^2 Hamiltonian (10) for various dimension N of x and p matrices.

Eig.	Dimension of x and p matrix			
n	$N = 20$	$N = 40$	$N = 70$	$N = 100$
0	1.1082234604	1.1082231576	1.1082231576	1.1082231576
1	2.3786369867	2.3786378305	2.3786378293	2.3786378293
2	2.3786369867	2.3786378305	2.3786378293	2.3786378293
3	3.0561855644	3.0560811678	3.0560811547	3.0560811547
4	3.5153412537	3.5149490976	3.5149490453	3.5149490453
5	4.0894727319	4.0934693830	4.0934692764	4.0934692763
6	4.0894727319	4.0934693830	4.0934692764	4.0934692764
7	4.7655349352	4.7527752560	4.7527724020	4.7527724018
8	5.0049162416	4.9849691677	4.9849635877	4.9849635875
9	5.0112765593	5.0112792815	5.0112792815	5.0112792816
10	5.2843297458	5.4989372186	5.4989795160	5.4989795149
11	5.2843297458	5.4989372186	5.4989795160	5.4989795149
12	6.1606466816	6.1451100151	6.1448192842	6.1448192749
13	6.1606466816	6.2374887302	6.2371281175	6.2371281062
14	6.3467376678	6.6695933096	6.6723500376	6.6723500394
15	6.4466426047	6.6695933096	6.6723500376	6.6723500394
16	7.0868715122	7.1748007456	7.1810981730	7.1810982462
17	7.0868715122	7.1748007456	7.1810981730	7.1810982462
18	8.0741850074	7.3857108935	7.3755740344	7.3755734790
19	8.1410850165	7.3914533591	7.3817605127	7.3817599769

3. Supersymmetric extension of x^2y^2 Hamiltonian

Supermembranes represent models of supersymmetric extended objects and can be obtained as a large N limit of supersymmetric quantum mechanical model of $N \times N$ matrices. To investigate potential instability of supermembranes, de Witt *et al.* [11] proposed a toy model of supermembrane as a supersymmetric extension of x^2y^2 Hamiltonian

$$\hat{H} = \hat{p}_x^2 + \hat{p}_y^2 + \hat{x}^2\hat{y}^2 + \hat{x}\hat{\sigma_x} - \hat{y}\hat{\sigma_y}\,. \tag{11}$$

The supercharge for this model is given by

$$\hat{Q} = \hat{Q}^\dagger = p_x\sigma_x + p_y\sigma_y - xy\sigma_z \tag{12}$$

and the Witten parity operator

$$\hat{P} = \frac{1}{\sqrt{2}}\left(\sigma_x + \sigma_z\right)\,. \tag{13}$$

Such a triple $\left(\hat{H}, \hat{P}, \hat{Q}\right)$ exhibits supersymmetry rules (2).

The supersymmetric x^2y^2 Hamiltonian (11) has a representation in matrix form

$$\begin{aligned}\hat{H} \;&=\; p_x^2 \otimes I_N \otimes \sigma_0 + I_N \otimes p_y^2 \otimes \sigma_0 + x^2 \otimes y^2 \otimes \sigma_0 \\ &\quad + x \otimes I_N \otimes \sigma_x - I_N \otimes y \otimes \sigma_y \,, \end{aligned} \tag{14}$$

where I_N is the unit matrix of dimension N.

In the previous section, we have shown that the spectrum of bosonic x^2y^2 Hamiltonian is discrete and particle is confined. Now, we have much more complicated situation, because previously system looked like harmonic oscillator in vicinity of the potential valley, but now inclusion of fermionic part may lower the energy barrier to zero value (ground state energy of supersymmetric system, with unbroken symmetry, equals exactly to zero). It means that the supermembrane would be unstable. To calculate eigenvalues of this interesting model, we have successfully used the Korsch–Glück matrix method [10].

Calculated two-fold degenerated eigenvalues of SUSY Hamiltonian (11) are listed in Table II.

The Korsch and Glück method works well also for supersymmetric extension of x^2y^2 Hamiltonian, but the convergence is slower (Fig. 3) as has been already shown on this and related supersymmetric models [12–20]. One can predict the energy of the ground state as a function of the number of basis functions with very high accuracy, but in any concrete numerical calculation (using a finite basis) it cannot be expected to get zero energy, unless the limiting case of infinite basis is considered [16].

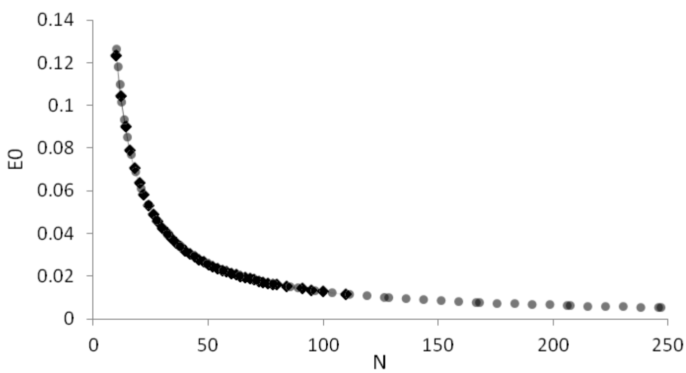

Fig. 3. Depicted convergence of ground state eigenvalue E_0 for the SUSY Hamiltonian (11), calculated for large dimensions of x and p matrices.

TABLE II

Eigenvalues of the of SUSY x^2y^2 Hamiltonian (14) for various dimension N of x and p matrices.

Eig.	Dimension of x and p matrix		
n	$N = 40$	$N = 84$	$N = 90$
0	0.032108991636366	0.015278224251210	0.014254035840192
1	0.032108991638310	0.015278224254877	0.014254035842510
2	0.128271168480141	0.061088549177454	0.056995604423391
3	0.128271168480540	0.061088549190081	0.056995604442740
4	0.287973528380799	0.137355630143736	0.128161269237406
5	0.287973528382465	0.137355630151015	0.128161269253507
6	0.510269431088999	0.243945648746561	0.227638829227932
7	0.510269431090023	0.243945648761428	0.227638829246742
8	0.793540541812128	0.380651789064673	0.355256071580868
9	0.793540541812173	0.380651789065249	0.355256071590181
10	1.134933765599259	0.547168882804748	0.510761460284285
11	1.134933765600634	0.547168882817118	0.510761460311879
12	1.529039903815373	0.743049901993007	0.693791653783149
13	1.529039903815540	0.743049901997125	0.693791653794402
14	1.964579809312500	0.967629627604936	0.903815893482729
15	1.964579809313219	0.967629627626407	0.903815893512385
16	2.416399135312330	1.219884623196185	1.140036978674562
17	2.416399135313741	1.219884623202547	1.140036978680519
18	2.838138042593580	1.498161352837602	1.401205525000713
19	2.838138042595983	1.498161352843226	1.401205525021434

This supersymmetric x^2y^2 model has two flat directions $(x, 0)$ and $(0, y)$. In a recent study [12], one flat direction has been eliminated using potential

$$V(x, y) = x^2y^2 + y^2 \,. \tag{15}$$

These authors studied also a third model with both flat directions removed using potential $x^2(y^2+1)^2+y^2$. This is also a supersymmetric model with supercharge

$$Q = p_x\sigma_x + p_y\sigma_y - (xy + x - iy)\sigma_z \tag{16}$$

and Hamiltonian

$$H = Q^2 = p_x^2 + p_y^2 + x^2(y+1)^2 + y^2 + x\sigma_x - (y+1)\sigma_z \,. \tag{17}$$

As has been shown [12], a model without flat directions (potential $x^2(y+1)^2+y^2$ in direction ($x \to \infty$, $y = -1$) remains equal to 1) has a non-empty continuous spectrum comprising the interval $[1, \infty)$ (Fig. 4), moreover, there is a ground state with energy ~ 0.8 below the bottom of the essential spectrum. Our results shown in Table III numerically support this hypothesis, with energy E_0 converged to value $E_0 = 0.88549$ (see Table III and Table IV for comparison). The convergence of eigenvalues is much faster compared with the model with flat directions (Fig. 5 and 6).

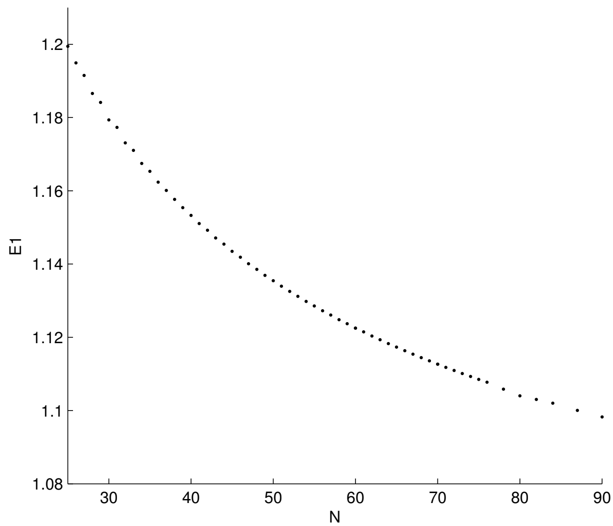

Fig. 4. Convergence of eigenvalue E_1 to value 1 for model Hamiltonian (17) with eigenvalue space spanned by interval $[1, \infty]$. There is also ground state eigenvalue below E_1 as reported in [12].

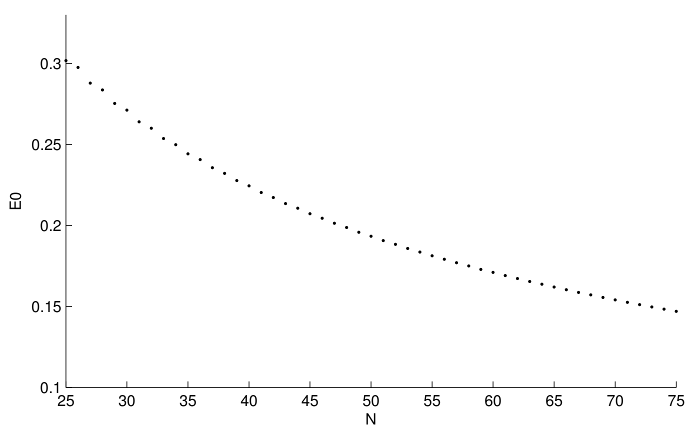

Fig. 5. Convergence of ground state eigenvalue for Hamiltonian with potential with one flat direction (15).

Convergence rates of ground state energy for all studied models are summarized in Fig. 7. The slowest convergence rate is observed for supersymmetric model with two flat directions, slightly better convergence is found for model with one flat direction, and for remaining two models with discrete levels, convergence rate is very fast, similar to those observed for one-dimensional cases studied in [10].

TABLE III

Eigenvalues of the of SUSY x^2y^2 Hamiltonian (17) without any flat direction, [12] for various dimension N of x and p matrices.

Eig.	Dimension of x and p matrix		
n	$N = 40$	$N = 84$	$N = 90$
0	0.885729458545192	0.885495272317939	0.885491967562919
1	1.153287608446259	1.102021888902019	1.098262566535757
2	1.220704996350386	1.124397173962305	1.118197728956234
3	1.438106824695513	1.248313539871256	1.235853920583860
4	1.623456524712892	1.328151959031734	1.309060873703431
5	1.929158045480484	1.498066014272517	1.469879889993195
6	2.216223640582740	1.637300148648031	1.599137825528064
7	2.596618648784355	1.852507452228366	1.802487518778423
8	2.933141275247389	2.045297865024240	1.982804523112304
9	3.273428318232698	2.304034610959533	2.227609875353356
10	3.539891290336405	2.539819440106451	2.450581928375242
11	3.792948995030348	2.829380297906724	2.728560831260849
12	4.239145877600699	3.075991534916736	2.970948860821363
13	4.508828281572678	3.325479164001877	3.232571388881452
14	5.048068693450596	3.544747326391260	3.446340850526223
15	5.078649991537276	3.748257451523354	3.638513458839009
16	5.268406333686308	4.075881969766556	3.927762094812896
17	5.732493504647465	4.325488912076878	4.175309359688333
18	6.066454362534508	4.744781398933587	4.564689259747438
19	6.183475393529313	4.795904888482301	4.679199650181006

TABLE IV

Eigenvalues of the of SUSY x^2y^2 Hamiltonian in [12] with potential with one flat direction (15) for various dimension N of x and p matrices.

Eig.	Dimension of x and p matrix		
n	$N = 40$	$N = 84$	$N = 90$
0	0.224412760874009	0.136136775321641	0.129894434282108
1	0.224412760875169	0.136136775333259	0.129894434287081
2	0.556922394071622	0.320636135507270	0.304192203244933
3	0.556922394074814	0.320636135520052	0.304192203271422
4	1.036071267537807	0.587757496466452	0.556436898019655
5	1.036071267538249	0.587757496481558	0.556436898042593
6	1.650992466645775	0.936461008004919	0.885987591458979
7	1.650992466647987	0.936461008022982	0.885987591460721
8	2.383991984538727	1.363349373730809	1.289969435717125
9	2.383991984539172	1.363349373736938	1.289969435727615
10	3.201460423287847	1.863562377306768	1.764241676624575
11	3.201460423291770	1.863562377307912	1.764241676645780
12	3.860339324000347	2.430044209505573	2.302940588334195
13	3.860339324002987	2.430044209524389	2.302940588347987
14	4.185160005257025	3.050457305236363	2.896473133160725
15	4.185160005257749	3.050457305236696	2.896473133185602
16	5.012290393530286	3.677817912548523	3.519191556602067
17	5.012290393531584	3.677817912552523	3.519191556619433
18	5.631283979631718	3.959824303413765	3.909691530520745
19	5.631283979633396	3.959824303421172	3.909691530529043

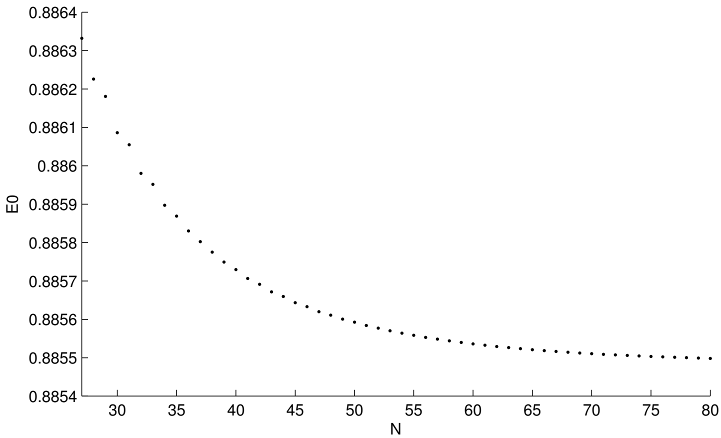

Fig. 6. Convergence of ground state eigenvalue for model Hamiltonian (17) with potential without flat directions [12].

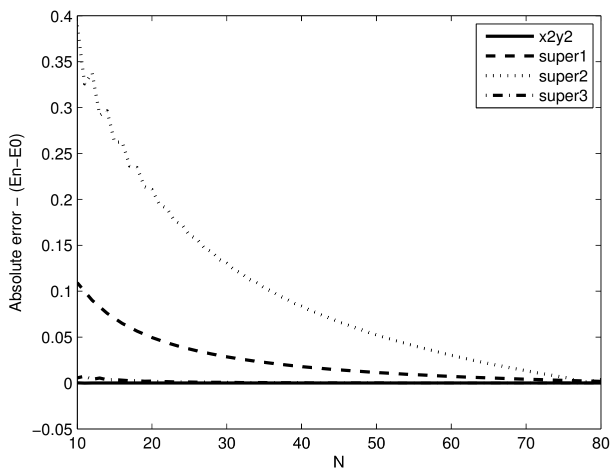

Fig. 7. Comparison of convergence rates of four above mentioned models (bosonic model x^2y^2, supersymmetric models with both flat directions (super1), with one flat direction (super2) and the toy model with a gap (super3) [12]) reported in previous sections. Absolute error $(E_N - E_0)$ is evaluated considering the reference E_0 calculated at dimension $N = 80$, whilst E_N corresponds to energy value at particular dimension N.

Further inspection of convergence patterns are given in following figures (Figs. 8 to 11). According to goodnesses of fits, the statement about nature of ground energy state can be suggested. In a particular case, the distinction between discrete and continuous energy state is acquired by considering

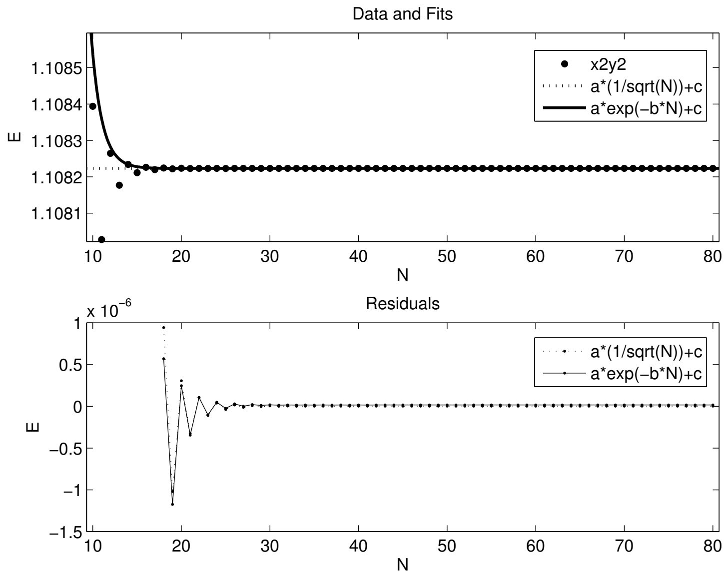

Fig. 8. Tight dependence of bosonic x^2y^2 model (6) ground state eigenvalue on dimension N of p and x matrices on exponential trend indicates and confirms [8] its discrete energy disposition. The convergence is reached at approx. $N = 20$. The $1/\sqrt{N}$ type fit with parameters $a = 3.368E - 6$ and $c = 1.1082$ matches up almost constant function. The first 10 cut-off eigenvalues are excluded from fitting procedure as the values at dimension are the prime matter of concern. The residuals for values of $E(N)$ *i.e.* the differences $E(N) - E(N)_{\rm fit}$ are given in the bottom subfigure.

suitable functional fit to calculated x and p matrices dimension dependent energy values. The dependences of eingenvalues on cut-off of basis functions involved in calculation have been demonstrated by [9, 17–19, 21]. As far as the particular set of cut-off or (N-dependent) energies resembles $1/N$ or slower functional form (as in our case $1/\sqrt{(N)}$), the state belongs to continuous energy spectrum. On the other hand, discrete energy spectrum is spanned by states which convergence is of $\exp(-N)$ fast type. Whether $1\sqrt{(N)}$ type or $1/N$ type fit is selected relies on comparison of their goodness (statistic measures how successful the fit is in description of the data variation in our cases R-square and root mean square error (RMSE) for particular cases are considered. R-square and RMSE are also reported for exponential fits in Figs. 8 to 11). The graphs of residuals of E values $(E(N) - E(N)_{\rm fit})$ are also attached as subfigures for detailed insight into the quality of fits.

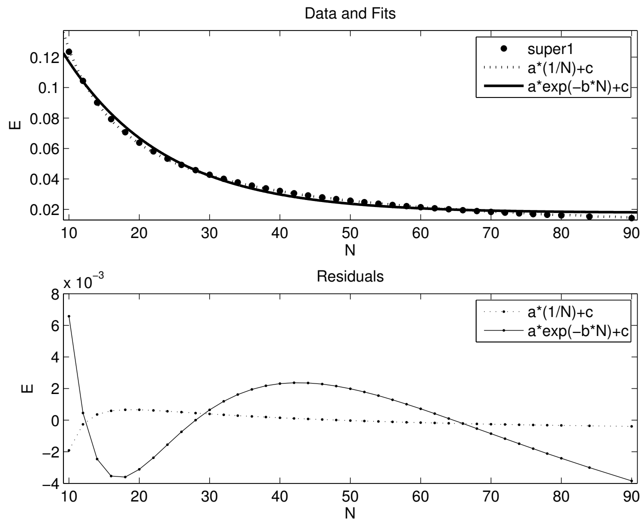

Fig. 9. Ground state energy of supersymmetric extension of bosonic model (11) (depicted as super1 dots) belongs to continuous energy spectrum as manifested by consistency of convergence with $1/(N)$ fit. The values of R-square $= 0.9997$ and RMSE $= 0.0004644$ for $1/(N)$ fit compared to R-square $= 0.9826$ and RMSE $=$ 0.002315 for exponential fit favour the linear inverse fit.

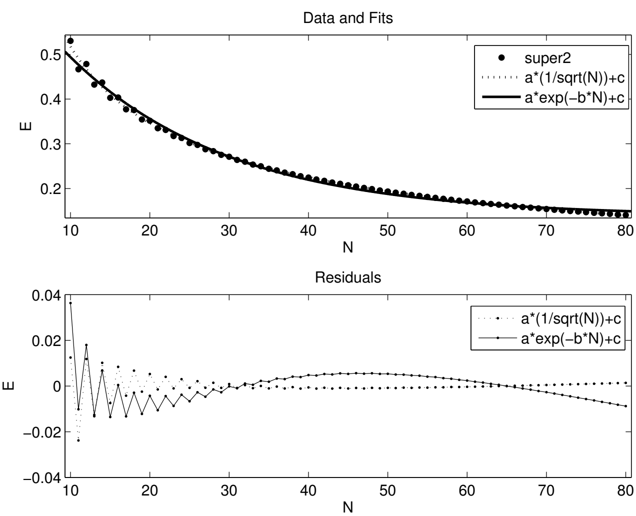

Fig. 10. Model with mass term embedded in one potential valley is characterized by continuous spectrum comprising the interval $[0, \infty]$. Goodness of fits: Exponential fit — R-square $= 0.994$; RMSE $= 0.007412$; $1/\sqrt{(N)}$ fit — R-square $= 0.9977$; RMSE $= 0.004553$.

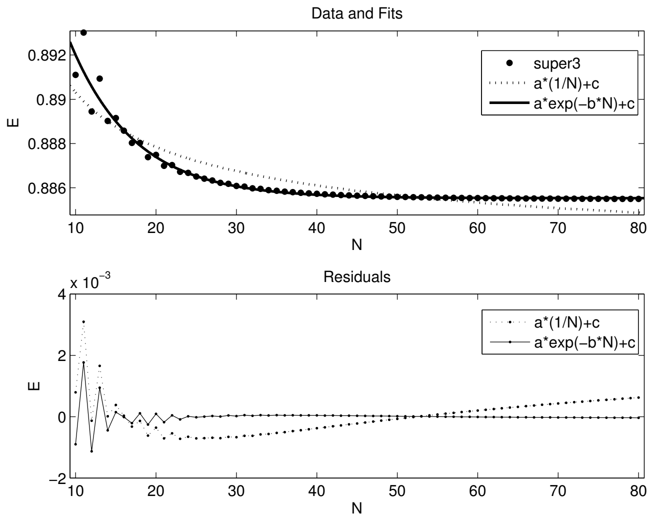

Fig. 11. Unlike the supersymmetric case in Fig. 9, the model with gap (15) exhibits isolated energy state below the essential spectrum $[1,\infty]$ $E = 0.8142$ [12]. Thus no wonder, the exponential fit reflects convergence tendency more appropriately compared to $1/N$ fit. Goodness of fits: Exponential fit — R-square $= 0.9577$; RMSE $= 0.00031$; $1/N$ fit — R-square $= 0.9081$; RMSE $= 0.004534$.

4. Conclusion

We have tested the method of Korsch and Glück [10] on several non-trivial examples of supersymmetric quantum models. The ground and a few excited states of them were determined and results are very accurate and very easy to obtain. As can be seen from Matlab code in Appendix, the realization of the method we have used, is very effective and gives the possibility to calculate the non-pertubative effects, therefore the method can be very useful in more realistic cases, including advanced topics of quantum field theory.

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Appendix

The Matlab code for the calculation of the first 20 eigenvalues of supersymmetric (SUSY) Hamiltonian (14) can be found using the following Matlab code:

```
format long
N = 50; s = 1;
```

```
n = 1:N -1;
m = sqrt(n);
x1 = s/sqrt(2) * (diag(m,-1) + diag(m,1));
p1 = i/s/sqrt(2) * (diag(m,-1) - diag(m,1));
x2 = s/sqrt(2) * (diag(m,-1) + diag(m,1));
p2 = i/s/sqrt(2) * (diag(m,-1) - diag(m,1));
Y=eye(N);
H1 = kron(p1^2,Y);
H2 = kron(Y,p2^2);
P1 = [0 1; 1 0];
P3 = [1 0; 0 -1];
P2 = [0 -i; i 0];
II = [1 0; 0 1];
INT=kron(x1*x1, x2*x2);
INTT=kron(INT+H1+H2,II);
INT1= kron(x1,Y);
INT11=kron(INT1,P1);
INT2= kron(Y,x2);
INT22=kron(INT2,P2);
H=INTT-INT22+ INT11;
[C,Eig] = eig(H);
EigSort = sort(eig(H));
EigSort(1:20)
```

Matlab code for the calculation of the first 20 eigenvalues of Hamiltonian with potential (15) with only on flat direction [12]

```
format long
N =50; s = 1;
n = 1:N -1;
m = sqrt(n);
x1 = s/sqrt(2) * (diag(m,-1) + diag(m,1));
p1 = i/s/sqrt(2) * (diag(m,-1) - diag(m,1));
x2 = s/sqrt(2) * (diag(m,-1) + diag(m,1));
p2 = i/s/sqrt(2) * (diag(m,-1) - diag(m,1));
Y=eye(N);
H1 = kron(p1^2,Y);
H2 = kron(Y,p2^2);
H3 = kron(Y,x2^2);
P1 = [0 1; 1 0];
P3 = [1 0; 0 -1];
P2 = [0 -i; i 0];
II = [1 0; 0 1];
```

```
INT=kron(x1*x1, x2*x2);
INTT=kron(INT+H1+H2+ H3,II);
INT1= kron(x1,Y);
INT11=kron(INT1,P1);
INT2= kron(Y,x2);
INT22=kron(INT2,P2);
H=INTT-INT22+ INT11;
[C,Eig] = eig(H);
EigSort = sort(eig(H));
EigSort(1:20)
```

Matlab code for the calculation of the first 20 eigenvalues of Hamiltonian with potential (17) without any flat direction [12]

```
format long
N = 50; s = 1;
n = 1:N -1;
m = sqrt(n);
x1 = s/sqrt(2) * (diag(m,-1) + diag(m,1));
p1 = i/s/sqrt(2) * (diag(m,-1) - diag(m,1));
x2 = s/sqrt(2) * (diag(m,-1) + diag(m,1));
p2 = i/s/sqrt(2) * (diag(m,-1) - diag(m,1));
Y=eye(N);
H1 = kron(p1^2,Y);
H2 = kron(Y,p2^2);
H3 = kron(Y,x2^2);
H4= kron(x1^2,Y);
H5= 2*kron(x1^2,x2);
P1 = [0 1; 1 0];
P3 = [1 0; 0 -1];
P2 = [0 -i; i 0];
II = [1 0; 0 1];
INT=kron(x1*x1, x2*x2);
INTT=kron(INT+H1+H2+ H3 +H4 + H5,II);
INT1= kron(x1,Y);
INT11=kron(INT1,P1);
INT2= kron(Y,x2);
INT3= kron(Y,Y);
INT22=kron(INT2+ INT3,P2);
H=INTT-INT22+ INT11;
eig(H);
EigSort = sort(eig(H));
EigSort(1:20)
```

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