BOSE–EINSTEIN CORRELATIONS
AND THERMAL CLUSTER FORMATION
IN HIGH-ENERGY COLLISIONS[*]

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The blast wave model is generalized to include the production of thermal clusters, as suggested by the success of the statistical model of particle production at high energies. The formulae for the HBT correlation functions and the corresponding HBT radii are derived.

1. Introduction

The soft hadronic data collected in high energy collisions are frequently analyzed in the framework of *thermal* or *statistical models* (see *e.g.* [1–6])[1]. In the most popular applications, such models explain the relative abundances of hadrons, *i.e.*, the ratios of hadron multiplicities. Thermal models can be also used to analyze the hadron transverse-momentum spectra and correlations. In the latter case, we often refer to thermal models as to the *hydro-inspired models*. This name reflects the fact that such models do not include the full hydrodynamic evolution but use various hydrodynamics-motivated assumptions about the state of matter at the thermal (kinetic)

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[1] For a review and an extensive list of references, see [7].

freeze-out. One of the most popular hydro-inspired models is the *blast wave model* originally introduced in [8] and adapted to ultra-relativistic energies in [9], see also [10] and [11].

With growing beam energies, such as those presently available at the LHC, the final state hadron multiplicities also grow substantially, and hydrodynamic features of hadron production are expected to appear even in more elementary hadron+hadron and hadron+nucleus collisions, *e.g.*, see Ref. [12]. Quite recently, the blast wave model has been used in this context to analyze high-multiplicity pp collisions at the LHC [13]. The authors of [13] found indications of strong transverse radial flow in such events.

In the present paper, using as the starting point the blast wave model[2] featuring a boost-invariant, azimuthally symmetric fluid expanding in the transverse direction according to the Hubble law [15], we show how to include the possibility of the formation of thermal clusters as an intermediate step between freeze-out and particle emission. We are interested, in particular, in the consequences the production of such clusters may have on the measurements of the Bose–Einstein correlations (for a recent review, see [16]).

We note that similar studies have been performed earlier in Refs. [17, 18]. The approach presented in [17] is based on the assumption that the distribution of the particles emitted from a cluster is a Gaussian. Within our framework, the particle distribution within a cluster may be arbitrary and the HBT radii are expressed by the moments of the distribution. Moreover, the distribution of clusters assumed in our paper is different from that proposed in [17]. Our approach differs from that presented in Ref. [18] since we are using a different physical picture. The authors of Ref. [18] assume that the space-time evolution of each cluster/droplet is described by the hydrodynamic equations and the whole system consists of a set of such small hydrodynamic subsystems. In our approach, we model a physical process where a single and large hydrodynamic system breaks first into clusters and later into observed particles (pions).

A thermal cluster is characterized by the Boltzmann distribution of the momenta of its decay products

$$e^{-\beta E^*} = e^{-\beta p^\mu u_\mu}\,, \tag{1}$$

where E^* is the energy of the emitted particle in the cluster rest frame, p^μ is its four-momentum, and u^μ is the cluster four-velocity. $T = 1/\beta$ is the temperature of the cluster. The new point which we explicitly include in our analysis is the natural condition that the cluster is limited in space-time. This means that in the cluster rest frame the emission points of its

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[2] Our use of the blast wave model follows similar studies done earlier in the case of heavy-ion collisions [11, 14].

decay products are distributed in the region described by a positive function $g(x^*) = g(t^*, x^*, y^*, z^*)$ normalized to unity

$$\int d^4x^* g(t^*, x^*, y^*, z^*) = 1 \,, \tag{2}$$

where $[x^*, y^*, z^*]$ represent the distance from the center of the cluster to the particle emission point and t^* is the time elapsing from the moment the cluster appears in the system till the particle emission time. Our aim is to investigate how the finite size of the cluster influences the results and the interpretation of the HBT measurements.

In the next section, we define the model by introducing the source function embodying the formation and decay of clusters. It is based on the generalized Cooper–Frye formula and the Hubble-like expansion of the fluid. In Section 3 the momentum distribution of particles is evaluated. The HBT correlation functions are discussed in Sections 4 and 5, and in Section 6 the general formulae for the HBT radii are given. The results are summarized in the last section. Several appendices display some details of the algebra needed to obtain the results presented in the paper.

2. The generalized Cooper–Frye formula

2.1. Source function

Our approach is based on the Cooper–Frye formula [19], generalized to the case where matter created at an intermediate stage of the collision process consists of thermal clusters. The starting point is the following expression for the source/emission function

$$S(x,p) = \int d\Sigma_\mu(x_{\rm c})\, p^\mu f(x_{\rm c}) \int d^4x^* g(x^*) \delta^{(4)}\left(x - x_{\rm c} - L_{\rm c}\, x^*\right) e^{-\beta p^\mu u_\mu(x_{\rm c})} \,. \tag{3}$$

Here x and p are the spacetime position and four-momentum of the emitted particle, $x_{\rm c}$ and $u^\mu(x_{\rm c})$ are the spacetime position and the four-velocity of a cluster, $L_{\rm c}$ is the Lorentz transformation leading from the cluster rest frame to the frame where the measurements of the BE correlations are performed and which we shall call "the HBT frame". Finally, $d\Sigma_\mu(x_{\rm c})$ is an element of the freeze-out hypersurface which we take in the form

$$d\Sigma_\mu(x_{\rm c}) = S_0 \sigma_\mu(x_{\rm c})\, \delta(\tau_{\rm f} - \tau_{\rm c}) d^4x_{\rm c} = S_0 \sigma_\mu(x_{\rm c})\, \delta(\tau_{\rm f} - \tau_{\rm c}) \tau_{\rm c} d\tau_{\rm c} d\eta_{\rm c} d^2r_{\rm c} \,, \tag{4}$$

where S_0 is a normalization constant and the variables $\tau_{\rm c}$ and $\eta_{\rm c}$ are the longitudinal proper time and the space-time rapidity of the cluster

$$t_{\rm c} = \tau_{\rm c} \cosh \eta_{\rm c} \,, \qquad z_{\rm c} = \tau_{\rm c} \sinh \eta_{\rm c} \,. \tag{5}$$

In a similar way, we define the cluster radial distance from the collision axis and the azimuthal angle in the transverse plane

$$x_c = r_c \cos \phi_c \,, \qquad y_c = r_c \sin \phi_c \,. \tag{6}$$

The four-vector $\sigma_c^\mu = \sigma^\mu(x_c)$ defines the space-time orientation of an element of the freeze-out hypersurface

$$\sigma_c^\mu = (\cosh \eta_c, 0, 0, \sinh \eta_c) \,. \tag{7}$$

The function $f(x_c)$ in (3) describes the distribution of clusters in space, while the function $g(x^*)$ defines the distribution of the particle emission points in the cluster (in the cluster rest frame). The properties of the functions $f(x_c)$ and $g(x^*)$ will be discussed in more detail below. Here we only note that for small clusters, *i.e.*, for $x^* \to 0$, the source function (3) is reduced to the standard emission function [9]

$$S(x,p) = \int d\Sigma_\mu(x')p^\mu \delta^{(4)}(x' - x) \exp(-\beta p^\mu u_\mu(x'))f(x') \,. \tag{8}$$

Equations (3) and (4) allow to introduce a compact representation of the source function, which highlights its physical interpretation, namely[3]

$$S(x,p) = \int d^4 x_c \ S_c(x_c, u_c)S_\pi(x_c, u_c, x, p) \,, \tag{9}$$

where

$$S_c(x_c, u_c) = \delta(\tau_f - \tau_c)f(x_c) \tag{10}$$

and

$$S_\pi(x_c, u_c, x, p) = \int d^4 x^* \sigma_\mu(x_c)p^\mu e^{-\beta p_\mu u_c^\mu} \delta^{(4)} \left(x - x_c - L_c\, x^*\right) g(x^*) \,. \tag{11}$$

Function $S_c(x_c, u_c)$ is the distribution of the cluster four-velocity u_c and space-time position x_c, while $S_\pi(x_c, u_c, x, p)$ is the distribution of the final particles emerging from the cluster decay. Equation (9) shows that the source function can be represented as an integrated product of these two distributions.

We assume that function $f(x_c)$, defining the distribution of clusters in space, depends only on the transverse distance r_c. Hence, using Eqs. (9) and (10), the source function may be rewritten as

$$S(x,p) = \int r_c dr_c f(r_c) \int d\eta_c \int d\phi_c \, S_\pi \left(x_c, u_c, x, p\right) \,. \tag{12}$$

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[3] From now on, we shall omit all constant factors in the source function, since its normalization is irrelevant for the problems we are discussing in this paper.

2.2. Transverse Hubble expansion

Since the system is boost-invariant and cylindrically symmetric, the four-velocity of a cluster, $u_c = u(x_c)$, has the form [7]

$$u_c = (\cosh \eta_c \cosh \theta_c, \sinh \theta_c \cos \phi_c, \sinh \theta_c \sin \phi_c, \sinh \eta_c \cosh \theta_c) . \quad (13)$$

In addition, we assume that the transverse rapidity of the cluster θ_c and its position r_c are related by the condition of the radial Hubble-like flow [15]. This leads to the expressions

$$\sinh \theta_c = \omega r_c , \qquad \cosh \theta_c = \sqrt{1 + \omega^2 r_c^2} , \quad (14)$$

where ω is the parameter controlling the magnitude of the transverse flow.

The particle four-momentum is parameterized in the standard way in terms of rapidity, y, transverse momentum, $p_\perp$, transverse mass, $m_\perp$, and the azimuthal angle in the transverse plane, ϕ_p,

$$p = (m_\perp \cosh y, p_\perp \cos \phi_p, p_\perp \sin \phi_p, m_\perp \sinh y) . \quad (15)$$

Then, the scalar product of p and u_c is

$$p \cdot u_c = m_\perp \cosh(y - \eta_c) \cosh \theta_c - p_\perp \cos(\phi_p - \phi_c) \sinh \theta_c . \quad (16)$$

This form is used in the thermal Boltzmann distribution. In a similar way, we obtain the factor $p \cdot \sigma_c$ needed to define the element of the freeze-out hypersurface[4]

$$p \cdot \sigma_c = m_\perp \cosh(y - \eta_c) . \quad (17)$$

2.3. Distribution of the emitted particles in a thermal cluster

The decay distribution can be written as

$$S_\pi(x_c, u_c, x, p) = p_\mu \sigma_c^\mu \exp(-\beta p_\mu u_c^\mu) S^*(x_c, x, u_c) . \quad (18)$$

The first two factors in (18) describe the momentum distribution. We have

$$p_\mu \sigma_c^\mu \exp(-\beta p_\mu u_c^\mu) = m_\perp \cosh(y - \eta_c)$$
$$\times \exp\left[-\beta m_\perp \cosh \theta_c \cosh(\eta_c - y) + \beta p_\perp \sinh \theta_c \cos \phi\right] , \quad (19)$$

where $\phi = \phi_c - \phi_p$ is the angle in the transverse plane between $\vec{u}_{c,\perp}$ and $\vec{p}_\perp$.

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[4] The form of (17) follows directly from (15) and (16). Other forms are also possible here if one assumes different freeze-out conditions. Using (4) and (7), we follow the most popular version of the blast wave model.

The last factor in (18), *i.e.* the function $S^*(x_c, x, u_c)$, describes the distribution of the points of particle emission from the cluster, which is discussed in greater detail in Appendix A,

$$
S^*(x_c, x, u_c) = \int d^4 x^* g(x^*) \delta(t - t_c - T) \delta(x - x_c - X)
$$
$$
\times \delta(y - y_c - Y) \delta(z - z_c - Z) , \tag{20}
$$

where

$$
\begin{aligned}
T &= \cosh \eta_c \, (t^* \cosh \theta_c + x^* \sinh \theta_c) + z^* \sinh \eta_c , \\
X &= x^* \cos \phi_c \cosh \theta_c - y^* \sin \phi_c + t^* \cos \phi_c \sinh \theta_c , \\
Y &= y^* \cos \phi_c + x^* \cosh \theta_c \sin \phi_c + t^* \sin \phi_c \sinh \theta_c , \\
Z &= \sinh \eta_c \, (t^* \cosh \theta_c + x^* \sinh \theta_c) + z^* \cosh \eta_c .
\end{aligned} \tag{21}
$$

Integration over $d^4 x^*$ is easy and gives

$$
S^* = g(\hat{t}, \hat{x}, \hat{y}, \hat{z}) \tag{22}
$$

with

$$
\begin{aligned}
\hat{t} &= (T' \cosh \eta_c - Z' \sinh \eta_c) \cosh \theta_c - (Y' \sin \phi_c + X' \cos \phi_c) \sinh \theta_c , \\
\hat{x} &= -(T' \cosh \eta_c - Z' \sinh \eta_c) \sinh \theta_c + (Y' \sin \phi_c + X' \cos \phi_c) \cosh \theta_c , \\
\hat{y} &= Y' \cos \phi_c - X' \sin \phi_c , \\
\hat{z} &= -T' \sinh \eta_c + Z' \cosh \eta_c ,
\end{aligned} \tag{23}
$$

and $X'^\mu \equiv [T', X', Y', Z'] = (x - x_c)^\mu$.

3. Momentum distribution

By definition, the integral of the source function $S(x, p)$ over the space-time coordinates gives the momentum distribution

$$
\frac{dN}{dy d^2 p_\perp} = W(p) = \int d^4 x \, S(p, x) . \tag{24}
$$

The explicit calculation starting from Eq. (3) yields

$$
\begin{aligned}
W(p) &= \int d^4 x^* \int d\Sigma_\mu(x_c) p^\mu e^{-\beta p^\mu u_\mu(x_c)} f(x_c) g(x^*) \\
&= \int d\Sigma_\mu(x_c) p^\mu e^{-\beta p^\mu u_\mu(x_c)} f(x_c) .
\end{aligned} \tag{25}
$$

Thus, the particle momentum distribution is given by the same expression as that used in the standard Cooper–Frye formula (with the particle space-time coordinates replaced by the cluster coordinates). The integration over d^4x cancels the four Dirac delta functions appearing in (20) and leads to the formula

$$W(p) = m_\perp \int r_c dr_c f(r_c) \int d\eta_c \int d\phi_c \cosh(\eta_c - y) e^{-U \cosh(\eta_c - y) + V \cos \phi_c} ,$$ (26)

with

$$U = \beta m_\perp \cosh \theta_c , \qquad V = \beta p_\perp \sinh \theta_c .$$ (27)

Integration over η_c and ϕ_c gives

$$W(p_\perp) = m_\perp \int r_c dr_c f(r_c) K_1(U) I_0(V) ,$$ (28)

which agrees with a formula commonly used to interpret the transverse-momentum spectra [9].

4. HBT correlation function

Assuming that one can neglect correlations between the produced particles[5], the distribution of two identical bosons can be expressed in terms of the Fourier transform of the source function

$$W(p_1, p_2) = W(p_1) W(p_2) + |H(P, Q)|^2$$ (29)

with

$$H(P, Q) = \int d^4x e^{iQ \cdot x} S(x, P) .$$ (30)

Here $Q = p_1 - p_2$ and $\vec{P} = (\vec{p}_1 + \vec{p}_2)/2$. The time-component of the four-vector P is not uniquely defined. We shall adopt the convention $P_0 = \sqrt{m^2 + |\vec{P}|^2}$ [21]. In Appendix E, we discuss the consequences of another relation, $P_0 = (p_{01} + p_{02})/2$ [16].

The source function $S(x, P)$ appearing in (30) is given by our initial definition, see Eq. (3), with p replaced by P, namely

$$\begin{aligned} S(x, P) &= \int d^4x^* g(x^*) \int d\Sigma_\mu(x_c) P^\mu \exp\left(-\beta P^\mu u_\mu(x_c)\right) f(r_c) \\ &\quad \times \delta\left(t - t_c - T\right) \delta(z - z_c - Z) \delta\left(x - x_c - X\right) \delta\left(y - y_c - Y\right) \\ &= \int d\Sigma_\mu(x_c) P^\mu \exp\left(-\beta P^\mu u_\mu(x_c)\right) f(r_c) S^*(x_c, x, u_c) . \end{aligned}$$ (31)

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[5] Although, as pointed out in [20], this assumption may distort significantly the results for Q exceeding the inverse size of the system, it is not restrictive at small Q, the region which is of interest in this paper.

In the last line in (31) we used our definition of the function $S^*(x_c, x, u_c)$, see Eq. (20).

Equations (30) and (31) allow us to write the compact expression for the Fourier transform of the source function

$$H(P,Q) = \int d\Sigma_\mu(x_c) P^\mu \exp\left(-\beta P^\mu u_\mu(x_c)\right) f(r_c) e^{iQ \cdot x_c} \mathcal{G}(x_c, Q), \quad (32)$$

where

$$\mathcal{G}(x_c, Q) = \int d^4 x^* \exp\left[i(Q \cdot X)\right] g(x^*) \quad (33)$$

with X^μ given by Eq. (21).

5. Kinematics of the Fourier transform

We shall work in the so-called LCMS system (*i.e.* our HBT system is the LCMS system) in which $P_z = 0$, *i.e.* $p_{1z} = -p_{2z}$ and $y_{pair} = 0$. In this frame, the substitution $p \to P$ is simply realized by the change $m_\perp \to \sqrt{P_0^2 - P_z^2} = P_0$. Starting directly from (30) and (31), we find

$$H(P,Q) \;=\; P_0 \int d^4 x^* \int r_c \, dr_c f(r_c) \int d\phi_c$$
$$\times \int d\eta_c \cosh \eta_c e^{-U \cosh \eta_c + V \cos \phi_c - i\Phi} g(x^*), \quad (34)$$

where now $U = \beta P_0 \cosh\theta_c$, $V = \beta P_\perp \sinh\theta_c$, and the phase Φ is given by the formula

$$\Phi = -Q_0(t_c + T) + Q_z(z_c + Z) + Q_x(x_c + X) + Q_y(y_c + Y). \quad (35)$$

The phase Φ depends on the relative direction of $\vec{P} = (P_\perp, 0, 0)$ and $\vec{Q}$. One considers three regimes:

(i) *long* direction: $Q_x = Q_y = 0,$ $Q_z = q,$ and $Q_0 = 0,$

$$\Phi_{long} \;=\; q\chi_{long}, \quad \chi_{long} = \tau_f \sinh\eta_c + Z; \quad (36)$$

(ii) *side* direction: $Q_x = Q_z = 0,$ $Q_y = q,$ and $Q_0 = 0,$

$$\Phi_{side} \;=\; q\,\chi_{side}, \quad \chi_{side} = r_c \sin\phi_c + Y; \quad (37)$$

(iii) *out* direction: $Q_y = Q_z = 0,$ $Q_x = q$ and

$$Q_0 = \sqrt{m^2 + (P_\perp + q/2)^2} - \sqrt{m^2 + (P_\perp - q/2)^2}, \quad (38)$$

$$\Phi_{out} \;=\; q\,\chi_{out}, \quad \chi_{out} = -\frac{Q_0}{q}[\tau_f \cosh\eta_c + T] + r_c \cos\phi_c + X. \quad (39)$$

For small q, which is sufficient to obtain the HBT radii (as described in more detail below), we find $Q_0/q \approx P_\perp / \sqrt{m^2 + P_\perp^2} \equiv \zeta$. For arbitrary values of q, one should use explicitly formula (38).

With the help of the notation introduced above, the three desired versions of the Fourier transform may be written as one universal formula

$$
H_d(P_\perp, q) = P_0 \int d^4 x^* g(x^*) \int r_c \, dr_c f(r_c) \int d\phi_c e^{V \cos \phi_c}
$$
$$
\times \int d\eta_c \cosh \eta_c e^{-U \cosh \eta_c} e^{-iq\chi_d} \equiv H(P_\perp, q=0) \left\langle e^{-iq\chi_d} \right\rangle . \quad (40)
$$

The subscript d stands for "long", "side", and "out".

If clusters are not present, $H_d(P_\perp, q)$ can be explicitly expressed in terms of integrals of Bessel functions. The corresponding formulae are given in Appendix D.

Equation (40) is the basis for the consideration of the "numerator" contributions to the HBT radii discussed in Appendix B. The complementary "denominator" contributions are discussed in Appendix C. Let us note that sometimes it is assumed that the denominator does not contribute to the HBT radii [16].

6. The HBT radii

Experiments usually measure the correlation function defined as

$$
C(p_1, p_2) \equiv \frac{W(p_1, p_2)}{W(p_1)W(p_2)} - 1 = \frac{|H(P,Q)|^2}{W(p_1)W(p_2)} . \quad (41)
$$

The measured HBT radii are obtained from the fit to the correlation function in the Gaussian form separately for each of the directions *long*, *side* and *out*

$$
C(p_1, p_2) = e^{-R_{\text{HBT}}^2 q^2} \quad (42)
$$

with q given by (37). This means that they can be evaluated as the logarithmic derivative at $q = 0$

$$
R_{\text{HBT}}^2 = -\frac{d \log[C(p_1, p_2)]}{dq^2} \equiv R_H^2 - R_W^2 \quad (43)
$$

with

$$
R_H^2 = -\left\{ \frac{dH(P,q)/dq^2}{H(P,q)} + \frac{dH^*(P,q)/dq^2}{H^*(P,q)} \right\}_{q=0} ,
$$
$$
R_W^2 = -\left\{ \frac{dW(p_1)/dq^2}{W(p_1)} + \frac{dW(p_2)/dq^2}{W(p_2)} \right\}_{q=0} . \quad (44)
$$

Using the formulae of the previous section, it is thus possible to derive the expressions for the HBT radii for all the three configurations. In this section, we only give the final results. The algebra is outlined in Appendices B and C.

In Appendix B it is shown that

$$R_{Hd}^2 = \langle \chi_d^2 \rangle - \langle \chi_d \rangle^2 = \left\langle (\chi_d - \langle \chi_d \rangle)^2 \right\rangle, \tag{45}$$

where the average $\langle \varnothing \rangle$ is defined as

$$\langle \varnothing \rangle = \frac{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) d\phi_{\rm c} d\eta_{\rm c} \cosh \eta_{\rm c} e^{-U \cosh \eta_{\rm c}} e^{V \cos \phi_{\rm c}} \int d^4 x^* g(x^*) \varnothing}{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) d\phi_{\rm c} d\eta_{\rm c} \cosh \eta_{\rm c} e^{-U \cosh \eta_{\rm c}} e^{V \cos \phi_{\rm c}} \int d^4 x^* g(x^*)}. \tag{46}$$

These averages can be expressed in terms of integrals of Bessel functions as shown in Appendix D. The results are listed below. Denoting

$$\rho^2 \equiv \langle x^{*2} \rangle = \langle y^{*2} \rangle, \quad \rho_z^2 = \langle z^{*2} \rangle, \quad \rho_t^2 = \langle t^{*2} \rangle, \quad \delta_t = \langle t^* \rangle, \tag{47}$$

where $\langle \varnothing^* \rangle \equiv \int d^4 x^* \varnothing\, g(x^*)$, one obtains

$$\langle \chi_{\rm long} \rangle = \langle \chi_{\rm side} \rangle = 0,$$
$$\langle \chi_{\rm out} \rangle = \frac{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) \lambda_1 I_1(V) K_1(U)}{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) I_0(V) K_1(U)} - \frac{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) \lambda_2 I_0(V) K_0''(U)}{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) I_0(V) K_1(U)}, \tag{48}$$

where

$$\lambda_1 = r_{\rm c} + \delta_t \sinh \theta_{\rm c}, \qquad \lambda_2 = \zeta \left(\tau_{\rm f} + \delta_t \cosh \theta_{\rm c} \right). \tag{49}$$

The average $\left\langle \chi_{\rm long}^2 \right\rangle$ is given by the formula

$$\left\langle \chi_{\rm long}^2 \right\rangle = \frac{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) \lambda_2 I_0(V) \left[\kappa_{\rm l} [K_0''(U) - K_1(U)] + \rho_z^2 K_1''(U) \right]}{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) I_0(V) K_1(U)} \tag{50}$$

with

$$\kappa_{\rm l} = \tau_{\rm f}^2 + \rho_t^2 \cosh^2 \theta_{\rm c} + \rho^2 \sinh^2 \theta_{\rm c} + 2\tau_{\rm f} \delta_t \cosh \theta_{\rm c}. \tag{51}$$

For the *side* direction the result is

$$\left\langle \chi_{\rm side}^2 \right\rangle = \frac{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) \left[\kappa_{\rm s} [I_0(V) - I_1'(V)] + \rho^2 I_1'(V) \right] K_1(U)}{\int r_{\rm c} dr_{\rm c} f(r_{\rm c}) I_0(V) K_1(U)} \tag{52}$$

with

$$\kappa_{\rm s} = r_{\rm c}^2 + \rho^2 \cosh^2 \theta_{\rm c} + \rho_t^2 \sinh^2 \theta_{\rm c} + 2r_{\rm c} \delta_t \sinh \theta_{\rm c}, \tag{53}$$

while for the *out* direction we find

$$\langle \chi_{\text{out}}^2 \rangle = \zeta^2 \frac{\int r_{\text{c}} dr_{\text{c}} f(r_{\text{c}}) I_0(V) \left[\kappa_1 K_1''(U) + \rho_z^2 [K_1''(U) - K_1(U)] \right]}{\int r_{\text{c}} dr_{\text{c}} f(r_{\text{c}}) I_0(V) K_1(U)}$$

$$-2\zeta \frac{\int r_{\text{c}} dr_{\text{c}} f(r_{\text{c}}) \kappa_2 I_1(V) K_0''(U)}{\int r_{\text{c}} dr_{\text{c}} f(r_{\text{c}}) I_0(V) K_1(U)}$$

$$+ \frac{\int r_{\text{c}} dr_{\text{c}} f(r_{\text{c}}) [\kappa_3 I_0''(V) + \rho^2 [I_0(V) - I_0''(V)]] K_1(U)}{\int r_{\text{c}} dr_{\text{c}} f(r_{\text{c}}) I_0(V) K_1(U)} \tag{54}$$

with

$$\kappa_1 = \tau_{\text{f}}^2 + \rho_t^2 \cosh^2 \theta_{\text{c}} + \rho^2 \sinh^2 \theta_{\text{c}} + 2\tau_{\text{f}} \delta_t \cosh \theta_{\text{c}},$$

$$\kappa_2 = \tau_{\text{f}}(r_{\text{c}} + \delta_t \sinh \theta_{\text{c}}) + r_{\text{c}} \delta_t \cosh \theta_{\text{c}} + \left[\rho^2 + \rho_t^2 \right] \sinh \theta_{\text{c}} \cosh \theta_{\text{c}},$$

$$\kappa_3 = r_{\text{c}}^2 + \rho^2 \cosh^2 \theta_{\text{c}} + \rho_t^2 \sinh^2 \theta_{\text{c}} + 2r_{\text{c}} \delta_t \sinh \theta_{\text{c}}. \tag{55}$$

Evaluation of R_W^2 is given in Appendix C. The results for different directions are as follows

$$R_{W,\text{long}}^2 = 0, \tag{56}$$

$$R_{W,\text{side}}^2 = -\frac{1}{4M_\perp^2} \tag{57}$$

$$-\frac{\beta^2}{4} \frac{\int dr_{\text{c}}^2 f(r_{\text{c}}) \left[\sinh^2 \theta_{\text{c}} \left(I_0'(V)/V \right) K_1(U) + \cosh^2 \theta_{\text{c}} I_0(V) K_1'(U)/U \right]}{\int dr_{\text{c}}^2 f(r_{\text{c}}) I_0(V) K_1(U)},$$

$$R_{W,\text{out}}^2 = \frac{P_\perp^2 - m^2}{4M_\perp^4}$$

$$+ \frac{\beta^2}{4} \left[\frac{\int dr_{\text{c}}^2 f(r_{\text{c}}) \left[\sinh \theta_{\text{c}} I_0'(V) K_1(U) + \cosh \theta_{\text{c}} I_0(V) K_1'(U) \zeta \right]}{\int dr_{\text{c}}^2 f(r_{\text{c}}) I_0(V) K_1(U)} \right]^2$$

$$- \frac{\beta^2}{4} \frac{\int dr_{\text{c}}^2 f(r_{\text{c}}) [\sinh^2 \theta_{\text{c}} I_0''(V) K_1(U) + 2 \sinh \theta_{\text{c}} \cosh \theta_{\text{c}} I_0'(V) K_1'(U) \zeta]}{\int dr_{\text{c}}^2 f(r_{\text{c}}) I_0(V) K_1(U)}$$

$$- \frac{\beta^2}{4} \frac{\int dr_{\text{c}}^2 f(r_{\text{c}}) \cosh^2 \theta_{\text{c}} I_0(V) K_1''(U) \zeta^2}{\int dr_{\text{c}}^2 f(r_{\text{c}}) I_0(V) K_1(U)}$$

$$- \frac{\beta^2 m^2}{4M_\perp^2} \frac{\int dr_{\text{c}}^2 f(r_{\text{c}}) \cosh^2 \theta_{\text{c}} I_0(V) K_1'(U)/U}{\int dr_{\text{c}}^2 f(r_{\text{c}}) I_0(V) K_1(U)}, \tag{58}$$

where $\zeta = P_\perp / M_\perp$.

For the reader's convenience, we also include below all the needed relations for the Bessel functions:

$$
\begin{aligned}
K_1'(a) &= -K_0(a) - K_1(a)/a\,, \\
K_1''(a) &= K_0(a)/a + K_1(a) + 2K_1(a)/a^2\,, \\
K_2(a) &= K_0(a) + 2K_1(a)/a\,, \\
I_0'(a) &= I_1(a)\,, \\
I_0''(a) &= I_1'(a) = I_0(a) - I_1(a)/a\,.
\end{aligned}
\tag{59}
$$

7. Summary

The observed success of the statistical model in explaining many features of particle production processes in high-energy collisions suggests that particles are produced in form of "thermal clusters" which decay into the observed final state. In the present paper, we discussed how this mechanism can influence measurements of quantum interference. To this end, we have generalized the well-known blast wave model [9] to include the production of thermal clusters. The novel element of our approach is introducing the final size and life-time of a cluster which, as one may expect, modifies the interpretation of the HBT measurements and makes the model more flexible. The explicit formulae for the correlation functions and for the HBT radii have been derived in a form which is ready for direct application.

As the presence of thermal clusters is an almost unavoidable consequence of the success of the statistical model of particle production, we feel that our work provides the necessary tools which may serve to verify the statistical picture on a more fundamental level.

Furthermore, determination of the cluster parameters and verification if they reveal some universal features may be an important contribution to understanding of the statistical model.

In conclusion, we have shown that the presence of the thermal clusters does not invalidate the significance of the measurements of quantum interference but, on the contrary, allows to extract from them even more interesting information.

Appendix A

*Lorentz transformation connecting the cluster's rest frame
and the HBT frame*

The active Lorentz transformation L_c leading from the cluster rest frame (CRF), where its velocity is $u^* = (1, 0, 0, 0)$, to the HBT frame, where the velocity is u_c, may be represented as a composition of three Lorentz transformations: a Lorentz boost along the x-axis,

$$L_{(x)}(\theta_c) = \begin{pmatrix} \cosh\theta_c & \sinh\theta_c & 0 & 0 \\ \sinh\theta_c & \cosh\theta_c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.1}$$

a rotation around the z-axis,

$$R_{(xy)}(\phi_c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_c & -\sin\phi_c & 0 \\ 0 & \sin\phi_c & \cos\phi_c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.2}$$

and a boost along the z-axis,

$$L_{(z)}(\eta_c) = \begin{pmatrix} \cosh\eta_c & 0 & 0 & \sinh\eta_c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta_c & 0 & 0 & \cosh\eta_c \end{pmatrix}. \tag{A.3}$$

Indeed, by direct multiplication of the matrices one can check that

$$u_c = L_c\, u^* = L_{(z)}(\eta_c)R_{(xy)}(\phi_c)L_{(x)}(\theta_c)u^* . \tag{A.4}$$

In order to change from the HBT frame to CRF, we perform simply the inverse transformation

$$\begin{aligned} u^* = L_c^{-1} u_c &= L_{(x)}^{-1}(\theta_c)R_{(xy)}^{-1}(\phi_c)L_{(z)}^{-1}(\eta_c)u_c \\ &= L_{(x)}(-\theta_c)R_{(xy)}(-\phi_c)L_{(z)}(-\eta_c)u_c . \end{aligned} \tag{A.5}$$

In the HBT frame, the fluid element with four-velocity u_c is placed at the space-time point x_c with the coordinates

$$x_c = (\tau\cosh\eta_c, r_c\cos\phi_c, r_c\sin\phi_c, \tau\sinh\eta_c), \tag{A.6}$$

and a particle is emitted from the space-time point

$$x = (t, x, y, z). \tag{A.7}$$

Then, the Lorentz transformation of the coordinate difference $x^* \equiv [t^*, x^*, y^*, z^*]$ is

$$
\begin{aligned}
x^\mu - x_c^\mu &= L_c \, x^* \\
&= \begin{pmatrix}
\cosh \eta_c \left(t^* \cosh \theta_c + x^* \sinh \theta_c \right) + z^* \sinh \eta_c \\
x^* \cos \phi_c \cosh \theta_c - y^* \sin \phi_c + t^* \cos \phi_c \sinh \theta_c \\
y^* \cos \phi_c + x^* \cosh \theta_c \sin \phi_c + t^* \sin \phi_c \sinh \theta_c \\
\sinh \eta_c \left(t^* \cosh \theta_c + x^* \sinh \theta_c \right) + z^* \cosh \eta_c
\end{pmatrix}.
\end{aligned} \tag{A.8}
$$

Appendix B

HBT radii — numerator contributions

As demonstrated by Eq. (40), for small values of the momentum difference Q, the Fourier transform appearing in the numerator of the correlation function can be schematically written as

$$H = \int d\Omega s(\Omega, P) e^{iQ \cdot x}, \tag{B.1}$$

where $\Omega = [r_c, \phi_c, \eta_c; x^*]$ denotes, symbolically, all variables to be integrated over, $d\Omega = r_c f(r_c) dr_c d\phi_c d\eta_c \cosh \eta_c d^4 x^* g(x^*)$, and

$$s(\Omega, P) = P_0 e^{-U \cosh \eta_c} e^{V \cos \phi_c}. \tag{B.2}$$

For the three directions, we write $Q \cdot x = q\chi$, where χ is independent of q and the three relevant options for χ are given by Eqs. (36)–(39).

We need to evaluate the derivative $d \log H / dq^2$ at $q = 0$. To this end, we observe that, up to the second order in q,

$$
\begin{aligned}
\log H &= \log \left[\int d\Omega s(\Omega, P) \left(1 + iq\chi - q^2 \chi^2 / 2 \right) \right] \\
&= \log \left[\int d\Omega s(\Omega, P) \right] + \log \left[1 + iq\langle\chi\rangle - q^2 \left\langle \chi^2 \right\rangle / 2 \right], \tag{B.3}
\end{aligned}
$$

where

$$\langle \varnothing \rangle \equiv \frac{\int d\Omega s(\Omega, P) \varnothing}{\int d\Omega s(\Omega, P)}. \tag{B.4}$$

Consequently, one finds

$$R_H^2 = - \left[\frac{d \log H}{dq^2} + \frac{d \log H^*}{dq^2} \right]_{q=0} = \langle \chi^2 \rangle - \langle \chi \rangle^2 = \left\langle (\chi - \langle \chi \rangle)^2 \right\rangle , \qquad \text{(B.5)}$$

where the asterisk denotes complex conjugation. Using these formulae and the explicit expressions (36), (37), and (39), one can evaluate the radii for all directions. The symmetries of $s(\Omega, P)$ imply

$$\begin{aligned}
\langle \chi_{\text{long}} \rangle &= \langle \chi_{\text{side}} \rangle = 0 , \\
\langle \chi_{\text{out}} \rangle &= \langle (r_{\text{c}} + \delta_t \sinh \theta_{\text{c}}) \cos \phi_{\text{c}} \rangle - \zeta \langle (\tau_{\text{f}} + \delta_t \cosh \theta_{\text{c}}) \cosh \eta_{\text{c}} \rangle .
\end{aligned} \qquad \text{(B.6)}$$

Using the abbreviation (47), we have

$$\langle \chi_{\text{long}}^2 \rangle = \langle \kappa_l \sinh^2 \eta_{\text{c}} + \rho_z^2 \cosh^2 \eta_{\text{c}} \rangle , \qquad \text{(B.7)}$$

$$\langle \chi_{\text{side}}^2 \rangle = \langle \kappa_s \sin^2 \phi_{\text{c}} + \rho^2 \cos^2 \phi_{\text{c}} \rangle , \qquad \text{(B.8)}$$

$$\begin{aligned}
\langle \chi_{\text{out}}^2 \rangle &= \zeta^2 \langle \kappa_1 \cosh^2 \eta_{\text{c}} + \rho_z^2 \sinh^2 \eta_{\text{c}} - 2\kappa_2 \cosh \eta_{\text{c}} \cos \phi_{\text{c}} \rangle \\
&\quad + \langle \kappa_3 \cos^2 \phi_{\text{c}} + \rho^2 \sin^2 \phi_{\text{c}} \rangle
\end{aligned} \qquad \text{(B.9)}$$

with κ_l, κ_s and $\kappa_1, \kappa_2, \kappa_3$ given by (51), (53) and (55).

Observing that

$$\begin{aligned}
\int \frac{d\eta}{2} \cosh^n \eta \, e^{-U \cosh \eta} &= (-1)^n \frac{d^n K_0(U)}{dU^n} , \\
\int \frac{d\phi}{2\pi} \cos^n \phi \, e^{V \cos \phi} &= \frac{d^n I_0(V)}{dV^n} ,
\end{aligned} \qquad \text{(B.10)}$$

one can express all these averages in terms of Bessel functions. The resulting formulae are listed in Section 6.

Appendix C

HBT radii — denominator contributions

The contribution of the denominator to the HBT radii is given by the formula

$$R_W^2 = - \frac{d}{dq_\perp^2} \log \left[w(\vec{p}_+) w(\vec{p}_-) \right] \qquad \text{(C.1)}$$

with $\vec{p}_\pm = \vec{P}_\perp \pm \vec{q}_\perp / 2$ and where the derivative is evaluated at $q_\perp = 0$. Since in the *long* case $q_\perp \equiv 0$, we find immediately that $R_{W,\text{long}}^2 = 0$. For the other two cases, the functions w are defined by Eq. (26). To evaluate (C.1), one needs them only up to second order in $q_\perp$.

The contributions from the m_T factors are easily evaluated. The results are given as the first terms in (57) and (58). The other contributions are more involved.

Consider first the *side* direction. In this case, we have

$$U(\vec{p}_\pm) = U\left[1 + q_\perp^2/8m_\perp^2\right], \qquad V(\vec{p}_\pm) = V\left[1 + q_\perp^2/8P_\perp^2\right], \qquad \text{(C.2)}$$

where U and V are given by (27). Expanding $e^{-U(\vec{p}_\pm)\cosh\eta + V(\vec{p}_\pm)\cos\phi}$ in powers of $q_\perp^2$ and integrating term by term it is straightforward to obtain Eq. (57). For the *out* direction, we have

$$\begin{aligned} U(\vec{p}_\pm) &= U\left(\vec{P}\right)\left[1 \pm P_\perp q_\perp/2m_\perp^2 + m^2 q_\perp^2/8m_\perp^4\right], \\ V(\vec{p}_\pm) &= V\left(\vec{P}\right)\left[1 \pm q_\perp/2\right]. \end{aligned} \qquad \text{(C.3)}$$

Consequently, one finds

$$\begin{aligned} e^{-U(\vec{p}_\pm)\cosh\eta}&e^{V(\vec{p}_\pm)\cos\phi} = e^{-U\cosh\eta + V\cos\phi} \\ &\times\left[1 \mp \frac{\beta\cosh\theta q_\perp\zeta}{2}\cosh\eta - \frac{\beta\cosh\theta q_\perp^2 m^2}{8m_\perp^3}\cosh\eta + \frac{\beta^2\cosh^2\theta\zeta^2 q_\perp^2}{8}\cosh^2\eta\right] \\ &\times\left[1 \pm \frac{\beta\sinh\theta q_\perp}{2}\cos\phi + \frac{\beta^2\sinh^2\theta q_\perp^2}{8}\cos^2\phi\right]. \end{aligned} \qquad \text{(C.4)}$$

Observing that the terms linear in $q_\perp$ cancel when one considers the logarithm of the product $w(\vec{p}_+)w(\vec{p}_-)$, one obtains, after some algebra, Eq. (58).

Appendix D

Angle and space-time integrals

In our analysis, we frequently have to evaluate integrals of the form

$$G \equiv P_0 \int r_c dr_c f(r_c) \int d\phi\, d\eta\, \cosh\eta\, e^{-U'\cosh\eta + ia\sinh\eta} e^{V'\cos\phi - ib\sin\phi}. \quad \text{(D.1)}$$

In the case of the *long* direction, we have: $U' = U$, $V' = V$, $a = q\tau_f$, and $b = 0$. Hence, we may write

$$G_{\text{long}} = P_0 \int r_c dr_c f(r_c) \int d\phi\, e^{V\cos\phi} D_{\text{long}}(U, a), \qquad \text{(D.2)}$$

where

$$D_{\text{long}}(U, a) \equiv \int d\eta\, \cosh\eta\, e^{-U\cosh\eta} e^{ia\sinh\eta} = -\frac{d}{dU}\int d\eta\, e^{-U\cosh\eta} e^{ia\sinh\eta}.$$

Since

$$\int d\eta e^{-U\cosh\eta} e^{ia\sinh\eta} = \int d\eta e^{-\sqrt{U^2+a^2}\cosh(\eta-\eta')} = 2K_0\left[\sqrt{U^2+a^2}\right],$$
(D.3)

we finally obtain

$$G_{\text{long}} = 4\pi P_0\int r_{\text{c}}dr_{\text{c}}f(r_{\text{c}})I_0(V)UK_1(U_{\text{l}})/U_{\text{l}}, \qquad U_{\text{l}} = \sqrt{U^2+a^2}. \quad \text{(D.4)}$$

In the case of the *side* direction, we have: $U' = U$, $V' = V$, $a = 0$, and $b = qr_{\text{c}}$. This leads us to the expression

$$G_{\text{side}} = P_0\int r_{\text{c}}dr_{\text{c}}f(r_{\text{c}})\int d\eta\cosh\eta e^{-U\cosh\eta}D_{\text{side}}(V,b), \qquad \text{(D.5)}$$

where

$$D_{\text{side}}(V,b) \equiv \int d\phi e^{-ib\sin\phi}e^{V\cos\phi} = \int_0^{2\pi} d\phi e^{V_{\text{s}}\cos(\phi-\phi')} = 2\pi I_0(V_{\text{s}})$$

and $V_{\text{s}} = \sqrt{V^2 - b^2}$. Thus, finally

$$G_{\text{side}} = 4\pi P_0\int r_{\text{c}}dr_{\text{c}}f(r_{\text{c}})I_0(V_{\text{s}})K_1(U). \qquad \text{(D.6)}$$

If $V^2 < Q^2r^2$, V_{s} is imaginary and the function $I_0(V_{\text{s}})$ should be replaced by $J_0(|V_{\text{s}}|)$. In the case of the *out* direction, we use: $U' = U - iQ_0\tau_{\text{f}}$, $V' = V + iqr_{\text{c}}$, $a = b = 0$, and we get

$$G_{\text{out}} \equiv P_0\int r_{\text{c}}dr_{\text{c}}f(r_{\text{c}})\int d\phi e^{(V+iqr_{\text{c}})\cos\phi}\int d\eta\cosh\eta e^{-(U-iQ_0\tau_{\text{f}})\cosh\eta}$$

$$= 4\pi P_0\int r_{\text{c}}dr_{\text{c}}f(r_{\text{c}})I_0(V + iqr_{\text{c}})K_1(U - iQ_0\tau_{\text{f}}). \qquad \text{(D.7)}$$

Appendix E

The case where $P_0 = (p_{10} + p_{20})/2$

In this case, the formulae for R_H^2 are different from those given in Section 5 because the variable $U' \equiv \beta P_0\cosh\theta_{\text{c}}$ depends on Q. Indeed

$$P_0 \equiv \frac{1}{2}(p_{01} + p_{02}) = \frac{1}{2}\left[\sqrt{m^2 + \left(\vec{P} + \vec{Q}/2\right)^2} + \sqrt{m^2 + \left(\vec{P} - \vec{Q}/2\right)^2}\right].$$
(E.1)

Up to the second order in Q, we obtain

$$P_0 = \sqrt{m^2 + P^2} \left[1 + \frac{Q^2}{8(m^2 + P^2)} - \frac{\left(\vec{P} \cdot \vec{Q} \right)^2}{8 \left(m^2 + P^2 \right)^2} \right] . \qquad \text{(E.2)}$$

Consequently, up to the second order in Q, we have

$$e^{-U' \cosh \eta_\mathrm{c}} = e^{-U \cosh \eta_\mathrm{c}} \left[1 - q^2 (U/2) \xi \cosh \eta_\mathrm{c} \right] , \qquad \text{(E.3)}$$

where $U = \beta \sqrt{m^2 + P^2} \cosh \theta_\mathrm{c}$ is given by (27) and

$$\xi_\mathrm{long} = \frac{1}{4m^2} , \qquad \xi_\mathrm{side} = \frac{1}{4m_\perp^2} , \qquad \xi_\mathrm{out} = \frac{m^2}{4m_\perp^4} . \qquad \text{(E.4)}$$

Repeating the argument given in Appendix B, we thus obtain

$$R_H^2 = \left\langle \chi^2 \right\rangle - \left\langle \chi \right\rangle^2 + \xi \left\langle U \cosh \eta_\mathrm{c} \right\rangle - \xi \qquad \text{(E.5)}$$

with

$$\left\langle U \cosh \eta_\mathrm{c} \right\rangle = \beta m_\perp \frac{\int r_\mathrm{c} dr_\mathrm{c} f(r_\mathrm{c}) \cosh \theta_\mathrm{c} I_0(V) K_0''(U)}{\int r_\mathrm{c} dr_\mathrm{c} f(r_\mathrm{c}) I_0(V) K_1(U)} \qquad \text{(E.6)}$$

and where the last term represents the contribution from the factor P_0 in front of (40). Note that in this case the contribution from the denominator is always as calculated in the present paper and never put equal to zero [16].

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