GEOMETRY AND OFF-SHELL NILPOTENCY FOR $\mathcal{N} = 1$ SUPERSYMMETRIC YANG–MILLS THEORY*

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We show that for $\mathcal{N} = 1$ supersymmetric Yang–Mills theory, it is possible to build an off-shell nilpotent BRST and anti-BRST algebra in terms of a BRST superspace formalism. This is based on the introduction of the basic fields of the quantized theory together with an auxiliary real field via the lowest components of the superfield components of a superYang–Mills connection. Here, the associated supercurvature is constrained by horizontality conditions as in ordinary Yang–Mills theory. We also show how the off-shell BRST-invariant quantum action can be constructed starting from a gauge-fixed superaction.

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1. Introduction

It is well known that the quantization of globally supersymmetric gauge theories has been considerably studied a long time ago. Several approaches have been proposed with various methods in order to perform the quantization of such theories (for a review, see *e.g.* Ref. [1] and references therein).

In the component field formalism, the supersymmetry algebra achieved without auxiliary fields closes only on-shell. This can be explained by the fact that the supersymmetry transformations of the models are realized nonlinearly and, therefore, the main problem affecting such theories is linked to its algebraic structure which involves equations of motion and field-dependent gauge transformations. This gives rise to an infinite dimensional algebra, even if auxiliary fields can be introduced to put the formalism off-shell [2]. To avoid these difficulties, in Ref. [3] the construction of a generalized BRST operator has been proposed by collecting together all the symmetries forming the theory, namely ordinary BRST, supersymmetries and translations. According to this procedure, the role of the auxiliary

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fields is covered by the external sources coupled to the nonlinear variations of the quantum fields. This approach has been already successfully applied to supersymmetric [4, 5], ordinary [6] gauge field theories as well as to non-gauge field theories [7]. Let us mention that in the Wess–Zumino model, it is only with auxiliary fields that one can obtain a tensor calculus (for a review, see *e.g.* Ref. [8] and references therein).

Another possibility to solve the problem of the quantization of theories with on-shell algebra is to introduce the Batalin–Vilkovisky (BV) formalism. The BV formalism is a very general covariant Lagrangian approach which overcomes the need of closed classical algebra by a suitable construction of BRST operator. The construction is realized by introducing a set of the so-called antifields besides the fields occurring in the theory. The elimination of these antifields at the quantum level via a gauge-fixing procedure leads to the quantum theory in which effective BRST transformations are nilpotent only on-shell. Let us note, that the BV approach can be used to obtain the on-shell BRST, invariant gauge fixed action for $\mathcal{N} = 1$ supersymmetric Yang–Mills theory in four dimensions without requiring the set of auxiliary fields [10].

Another interesting approach with infinite number of auxiliary fields has been developed in the context of harmonic superspace [11]. In this framework, quantization of supersymmetric theories has been discussed for various supersymmetries [12].

On the other hand, it is also known that the extension of spacetime with two ordinary anticommuting coordinates to a $(4, 2)$-dimensional superspace [13] leads in Yang–Mills-type theories to incorporate the gauge fields, the ghost and anti-ghost fields into a natural gauge superconnection, see also Ref. [14] and references therein. In such a superspace formalism, the BRST and anti-BRST transformations are derived systematically from the horizontality conditions imposed on the supercurvature.

Let us note that the natural geometrical way to derive the BRST structure of general gauge theories is to work, in the same spirit as in Yang–Mills type theories, by using the superconnection formalism. Within this framework and in contrast to what is done in Yang–Mills theories, all superfield components of the supercurvature cannot be constrained through horizontality conditions. This is a consequence of the fact that the gauge theories we consider are reducible and/or open. It is the consistency of the Bianchi identities which is guaranteed by the remaining superfield components of the supercurvature. Their lowest components allow the introduction of auxiliary fields. These, together with the fields given by the lowest components of the superfield components of the superconnection, represent the basic fields of the quantized gauge theory. The off-shell nilpotency of the BRST and anti-BRST transformations of these fields is automatically ensured, thanks to the structure equations and the Bianchi identities. The BRST and anti-BRST operators are related, as usual, to the partial derivatives with respect to the two anticommuting coordinates of the superspace. Essentially, the introduction of the auxiliary fields gives rise to the construction of the off-shell BRST-invariant quantum action. As shown in Ref. [15] for the case of non-Abelian BF theory where the classical gauge algebra is reducible and in Ref. [16] for the case of the simple supergravity where the classical gauge algebra is open, the superspace formalism has been used in order to realize the BRST structure of such theories. It leads to recast all the fields in geometrical way and to introduce auxiliary fields ensuring the off-shell invariance of the quantum action.

Our main aim in this paper consists in applying this formalism for discussing the off-shell nilpotent version of the BRST and anti-BRST transformations for global $\mathcal{N}=1$, $4D$ supersymmetric Yang–Mills theory where the classical gauge algebra is open [8, 17]. Let us mention that in Refs. [15, 16] the superspace formalism has been applied successfully to theories with local symmetry, while in the present work, we are interested in applying this formalism to a global supersymmetry. The classical action for the $\mathcal{N}=1$ supersymmetric Yang–Mills in four dimensions is given by [17]

$$S_0 = \int dx^4 \mathrm{Tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda\gamma^\mu D_\mu\overline{\lambda}\right)\,, \tag{1}$$

where 'Tr' denotes the trace over the gauge algebra, (A_μ, λ) is the gauge multiplet, the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and $D_\mu = \partial_\mu + [A_\mu,]$ is the covariant derivative. By construction, in addition to the ordinary Yang–Mills symmetry, action (1) is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\xi A_\mu &= i\xi\gamma_\mu\overline{\lambda} + i\overline{\xi}\gamma_\mu\lambda\,, \\ \delta_\xi\lambda &= \sigma^{\mu\nu}F_{\mu\nu}\xi\,, \end{aligned} \tag{2}$$

where $\sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and ξ is a spin 1/2 valued infinitesimal supersymmetry parameter.

In the following, we shall call the superspace obtained by enlarging space-time with two ordinary anticommuting coordinates BRST superspace, in order to distinguish it from the superspace of supersymmetric theories. Let us recall that a full off-shell structure of any supersymmetric field theory most naturally exhibits itself in superspace, provided the superfield formulation of the theory in terms of unconstrained superfields, is available. The BRST superspace formalism presented here permits us to derive the off-shell nilpotent BRST and anti-BRST algebra of quantized $\mathcal{N}=1$, $4D$ supersymmetric Yang–Mills theory. In particular, it gives another possibility leading to the minimal set of auxiliary fields in such theory.

Our paper is organized as follows: In Section 2, the BRST superspace approach and horizontality conditions for $\mathcal{N}=1$, $4D$ supersymmetric Yang–Mills theory are discussed. We also show how the various fields of such theory and their off-shell nilpotent BRST and anti-BRST transformations can be determined via a BRST superspace formalism. The construction of the BRST-invariant quantum action for $\mathcal{N}=1$ super Yang–Mills theory in terms of this off-shell structure is described in Section 3. Section 4 is devoted to concluding remarks.

2. Off-shell nilpotent BRST algebra

Let Φ be a super Yang–Mills connection on a $(4,2)$-dimensional BRST superspace with coordinates $z=(z^M)=(x^\mu,\theta^\alpha)$, where $(x^\mu)_{\mu=1,\ldots,4}$ are the coordinates of the spacetime manifold and $(\theta^\alpha)_{\alpha_{=1,2}}$ are ordinary anticommuting variables. Acting the exterior covariant superdifferential D on Φ, we obtain the supercurvature Ω satisfying the structure equation, $\Omega = d\Phi + (1/2)[\Phi,\Phi]$, and the Bianchi identity, $d\Omega + [\Phi,\Omega] = 0$. The superconnection Φ as 1-superform on the BRST superspace can be written as

$$\Phi = dz^M \left(\Phi^i_M I_i + \Phi^\mu_M P_\mu + \Phi^a_M Q_a\right) , \tag{3}$$

where $\{I_i\}_{i=1,\ldots,d=\dim G}$ and $\{P_\mu, Q_a\}_{\mu=1,\ldots,4;a=1,\ldots,4}$ are the generators of the internal symmetry group (G) and the $\mathcal{N}=1$ supersymmetric group (SG) respectively. They satisfy the following commutation relations

$$\begin{aligned} [I_i, I_j] &= f^k_{ij} I_k \,, \\ [I_i, P_\mu] &= [Q_a, P_\mu] = [P_\mu, P_\nu] = 0 \,, \\ [Q_a, Q_b] &= 2(\gamma^\mu)_{ab} P_\mu \,, \\ [I_i, Q_a] &= b^*_i Q_a \,, \end{aligned} \tag{4}$$

where $\{\gamma^\mu\}_{\mu=1,\ldots,4}$ are the Dirac matrices in the Weyl basis, $b^*_i = b_i$ for $a=1,2$ and $b^*_i = -b_i$ for $a=3,4$ giving the representation of the internal symmetry of Q_a and $[,]$ the graded Lie bracket. Let us mention that the supersymmetric generators $\{Q_a\}$ are given in the Majorana representation [17, 18]. Note that the Grassmann degrees of the superfield components of Φ are given by $|\Phi^i_M| = |\Phi^\mu_M| = m$, $|\Phi^a_M| = m+1$ (mod 2), where $m = |z^M|$ ($m=0$ for $M=\mu$ and $m=1$ for $M=\alpha$), since Φ is an even 1-superform.

However, we assign to the anticommuting coordinates θ^1 and θ^2 the ghost numbers (-1) and $(+1)$ respectively, and ghost number zero for an even quantity: either a coordinate, a superform or a generator. These rules permit us to determine the ghost numbers of the superfields (Φ^i_μ, Φ^ν_μ, Φ^a_μ, Φ^i_1, Φ^i_2, Φ^ν_1, Φ^ν_2, Φ^a_1, Φ^a_2) which are given by $(0,0,0,+1,-1,+1,-1,+1,-1)$.

Upon expressing the supercurvature Ω as

$$\Omega = \tfrac{1}{2}dz^N \wedge dz^M \Omega_{MN} = \tfrac{1}{2}dz^N \wedge dz^M \left(\Omega^i_{MN} I_i + \Omega^\mu_{MN} P_\mu + \Omega^a_{MN} Q_a\right) , \quad (5)$$

we find from the structure equation

$$\Omega_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu + [\Phi_\mu, \Phi_\nu] , \quad (6a)$$
$$\Omega_{\mu\alpha} = \partial_\mu \Phi_\alpha - \partial_\alpha \Phi_\mu + [\Phi_\mu, \Phi_\alpha] , \quad (6b)$$
$$\Omega_{\alpha\beta} = \partial_\alpha \Phi_\beta + \partial_\beta \Phi_\alpha + [\Phi_\alpha, \Phi_\beta] . \quad (6c)$$

Similarly, the Bianchi identity becomes

$$D_\mu \Omega_{\nu\kappa} + D_\kappa \Omega_{\mu\nu} + D_\nu \Omega_{\kappa\mu} = 0 , \quad (7a)$$
$$D_\alpha \Omega_{\mu\nu} - D_\nu \Omega_{\mu\alpha} + D_\mu \Omega_{\nu\alpha} = 0 , \quad (7b)$$
$$D_\alpha \Omega_{\beta\gamma} + D_\beta \Omega_{\alpha\gamma} + D_\gamma \Omega_{\alpha\beta} = 0 , \quad (7c)$$
$$D_\mu \Omega_{\alpha\beta} - D_\alpha \Omega_{\mu\beta} - D_\beta \Omega_{\mu\alpha} = 0 , \quad (7d)$$

where $D_M = \partial_M + [\Phi_M, .]$ is the M covariant superderivative. Now, we shall search for the constraints to the supercurvature Ω in which the consistency with the Bianchi identities (7) is ensured. This requirement ensures then the off-shell nilpotency of the BRST and anti-BRST algebra. The full set of supercurvature constraints turns out to be given by

$$\Omega_{\mu\alpha} = 0 , \qquad \Omega_{\alpha\beta} = 0 . \quad (8)$$

It is easy to check the consistency of this set of supercurvature constraints through an analysis of the Bianchi identities. Indeed, we remark that identities (7c) and (7d) are automatically satisfied because of the constraints (8), while identities (7a) and (7b) yield further restrictions on supercurvature Ω

$$\Omega^\kappa_{\mu\nu} = 0 , \qquad \Omega^a_{\mu\nu} = 0 . \quad (9)$$

At this point, let us mention that the consistency of the horizontability conditions (8) and (9) with the Bianchi identities (7), as we will see later, guarantees automatically the off-shell nilpotency of the BRST and anti-BRST transformations on all the fields belonging to $\mathcal{N}=1$ super Yang–Mills theory.

Now, in order to derive the off-shell BRST structure of $\mathcal{N}=1$ super Yang–Mills theory using the above BRST superspace formalism, it is necessary to give the geometrical interpretation of the fields occurring in the quantization of such theory. Besides the gauge potential $\Phi^i_{\mu|} = A^i_\mu$, there exists the superpartner $\Phi^i_{a|} = \lambda^i_a$ of A_μ and an auxiliary real field $\Phi^i_| = \Lambda^i$

which are introduced via the field redefinitions $\Phi^{ai} = -\frac{1}{4}(\gamma^{\mu})^{ab}[Q_b, \Phi^i_{\mu}]$ and $\Phi^i = \frac{1}{4}\delta^a_b[Q_a, \Phi^{bi}]$, respectively. The components $\Phi^{\nu}_{\mu|}$ and $\Phi^a_{\mu|}$ are identified to zero. Note that the symbol "|" indicates that the superfield is evaluated at $\theta^{\alpha} = 0$.

Furthermore, we introduce the following: $\Phi^i_{1|} = c^i_1$ is the ghost for Yang–Mills symmetry, $\Phi^i_{2|} = \overline{c}^i_2$ is the anti-ghost of c^i_1, $B^i = \partial_1 \Phi^i_{2|}$ is the associated auxiliary field, $\Phi^a_{1|} = \chi^a_1$ is the supersymmetric ghost, $\Phi^a_{2|} = \overline{\chi}^a_2$, is the anti-ghost of χ^a_1, $G^a = \partial_1 \Phi^a_{2|}$ is the associated auxiliary field, $\Phi^{\mu}_{1|} = \xi^{\mu}_1$ is the translation symmetry ghost, $\Phi^{\mu}_{2|} = \overline{\xi}^{\mu}_2$, is the anti-ghost of ξ^{μ}_1 and $E^{\mu} = \partial_1 \Phi^{\mu}_{2|}$ is the associated auxiliary field. Let us mention that the symmetry ghosts and anti-ghosts χ^a_{α} are commuting fields, while the others c^i_{α} and ξ^{μ}_{α} are anticommuting.

The action of the $\mathcal{N} = 1$ supersymmetric generators $\{P_{\mu}, Q_a\}$ on these fields is given by

$$
\begin{aligned}
[P_{\mu}, X] &= \partial_{\mu} X \,, \\
\left[Q_a, A^i_{\mu}\right] &= -(\gamma_{\mu})_{ab} \lambda^{bi} \,, \\
\left[Q_a, \lambda^{bi}\right] &= -\tfrac{1}{2}(\sigma^{\mu\nu})^b_a F^i_{\mu\nu} + \delta^b_a \Lambda^i \,, \\
\left[Q_a, c^i_{\alpha}\right] &= (\gamma^{\mu})_{ab} \chi^b_{\alpha} A^i_{\mu} \,, \\
\left[Q_a, \Lambda^i\right] &= (\gamma^{\mu})_{ab} D_{\mu} \lambda^{bi} \,, \\
\left[Q_a, F^i_{\mu\nu}\right] &= (\gamma_{\mu})_{ab} (D_{\nu}\lambda)^{bi} - (\gamma_{\nu})_{ab} (D_{\mu}\lambda)^{bi} \,,
\end{aligned}
\tag{10}
$$

where X is any field of the theory.

It is worthwhile to mention that we are interested in our present investigation on the global supersymmetric transformations, so that the parameters of the $\mathcal{N} = 1$ supersymmetric and translation groups must be space-time constant, *i.e.*

$$
\partial_{\mu} \chi^a_{\alpha} = 0 \,, \qquad \partial_{\mu} \xi^{\nu}_{\alpha} = 0 \,. \tag{11}
$$

Using the above identifications with (11) and inserting the constraints (8) and (9) into Eqs. (6) and (7b), we obtain

$$
\begin{aligned}
\partial_{\alpha} \Phi^i_{\mu|} &= (D_{\mu} c_{\alpha})^i - \xi^{\nu}_{\alpha} \left[P_{\nu}, A^i_{\mu}\right] - \chi^a_{\alpha} \left[Q_a, A^i_{\mu}\right] \,, \\
\partial_{\alpha} \Phi^i_{\beta|} + \partial_{\beta} \Phi^i_{\alpha|} &= -\left[c_{\alpha}, c_{\beta}\right]^i - \xi^{\nu}_{\alpha} \left[P_{\nu}, c^i_{\beta}\right] - \chi^a_{\alpha} \left[Q_a, c^i_{\beta}\right] \\
&\quad - \xi^{\nu}_{\beta} \left[P_{\nu}, c^i_{\alpha}\right] - \chi^a_{\beta} \left[Q_a, c^i_{\alpha}\right] \,, \\
\partial_{\alpha} \Phi^a_{\beta|} + \partial_{\beta} \Phi^a_{\alpha|} &= 0 \,, \\
\partial_{\alpha} \Phi^{\mu}_{\beta|} + \partial_{\beta} \Phi^{\mu}_{\alpha|} &= -2 \chi^a_{\alpha} (\gamma^{\mu})_{ab} \chi^b_{\beta} \,, \\
\partial_{\alpha} \Omega^i_{\mu\nu|} &= -\left[c_{\alpha}, F^i_{\mu\nu}\right] - \xi^{\tau}_{\alpha} \left[P_{\tau}, F^i_{\mu\nu}\right] - \chi^a_{\alpha} \left[Q_a, F^i_{\mu\nu}\right] \,,
\end{aligned}
$$

$$\begin{aligned} \partial_\alpha \Phi^{ai}_{|} &= \tfrac{1}{4}(\gamma^\mu)^{ab}\left[Q_b, \partial_\alpha \Phi^i_{\mu|}\right], \\ \partial_\alpha \Phi^i_{|} &= -\tfrac{1}{4}(\delta^a_b)\left[Q_a, \partial_\alpha \Phi^{bi}_{|}\right]. \end{aligned} \tag{12}$$

We also realize the usual identifications: $Q_\alpha(X_|) = \partial_\alpha X_|$, where X is any superfield and $Q = Q_1$ ($\overline{Q} = Q_2$) is the BRST (anti-BRST) operator. Inserting Eq. (10) into (12), and evaluating these at $\theta^\alpha = 0$, we find the following BRST transformations

$$\begin{aligned} QA^i_\mu &= (D_\mu c)^i - \xi^\nu \partial_\nu A^i_\mu + \chi\gamma_\mu\lambda^i, \\ Q\lambda^i_a &= -f^i_{jk}c^j\lambda^k_a - \xi^\mu\partial_\mu\lambda^i_a + \chi_a\Lambda^i + \tfrac{1}{2}\left(\chi\sigma^{\mu\nu}\right)_a F^i_{\mu\nu}, \\ Qc^i &= -\tfrac{1}{2}f^i_{jk}c^jc^k - \xi^\mu\partial_\mu c^i + \chi\gamma^u\overline{\chi}A^i_\mu, \\ Q\xi^\mu &= -\chi\gamma^u\overline{\chi}, \\ Q\Lambda^i &= -f^i_{jk}c^k\Lambda^j - \xi^\rho\partial_\rho\Lambda^i - \chi\gamma^\mu D_\mu\lambda^i, \\ QF^i_{\mu\nu} &= -f^i_{jk}c^kF^j_{\mu\nu} - \xi^\rho\partial_\rho F^i_{\mu\nu} - \chi^a\left\{(\gamma_\mu)_{ab}(D_\nu\lambda)^{bi} - (\gamma_\nu)_{ab}(D_\mu\lambda)^{bi}\right\}, \\ Q\chi^a &= 0, \qquad Q\overline{c}^i = B^i, \qquad QB^i = 0, \qquad Q\overline{\xi}^\mu = E^\mu, \\ QE^\mu &= 0, \qquad Q\overline{\chi}^a = G^a, \qquad QG^a = 0, \end{aligned} \tag{13}$$

and also the anti-BRST transformations, which can be derived from (13) by the following mirror symmetry of the ghost numbers given by : $X \to X$ if $X = A^i_\mu, \lambda^i_a, \Lambda^i$; $X \to \overline{X}$ if $X = Q, c^i, B^i, \xi^\mu, E^\mu, \chi^a, G^a$ and $\overline{\overline{X}} = X$, where

$$B^i + \overline{B}^i = -f^i_{jk}c^k\overline{c}^j - \overline{\xi}^\nu\partial_\nu c^i - \xi^\mu\partial_\mu\overline{c}^i - \chi\gamma^\mu\overline{\chi}A^i_\mu - \overline{\chi}\gamma^\mu\chi A^i_\mu, \tag{14}$$

$$E^\mu + \overline{E}^\mu = -2\chi\gamma^\mu\overline{\chi}, \qquad G^a + \overline{G}^a = 0. \tag{15}$$

Let us note that the introduction of an auxiliary real field Λ^i besides the fields present in quantized $\mathcal{N}=1$ super Yang–Mills theory in four-dimensions, guarantees automatically the off-shell nilpotency of the $\{Q, \overline{Q}\}$ algebra and then makes easier, as we will see in the next section, the gauge-fixing process.

3. Quantum action

In the present section, we show how to construct in the context of our procedure a BRST-invariant quantum action for $\mathcal{N}=1$ super Yang–Mills theory as the lowest component of a quantum superaction. For this purpose, we choose the following gauge-fixing superaction

$$\begin{aligned} S_{\text{sgf}} &= \int d^4x L_{\text{sgf}}, \\ L_{\text{sgf}} &= (\partial_1\Phi_2)(\partial^\mu\Phi_\mu) + (\partial^\mu\Phi_2)(\partial_1\Phi_\mu) + (\partial_1\Phi_2)(\partial_1\Phi_2). \end{aligned} \tag{16}$$

Let us recall that similar gauge-fixing superaction was used in Refs. [19–21]. We note first that in the case of Yang–Mills theory, the superaction involves a Lorentz gauge [22] given by

$$\partial_\mu \Phi^\mu_| = 0\,. \tag{17}$$

In the case of super Yang–Mills theory, we shall choose a supersymmetric gauge-fixing which is the extension of the Lorentz gauge. This gauge fixing can be obtained from (17) by using the following substitution

$$\Phi_\mu \to \widetilde{\Phi_\mu} = \Phi_\mu + [\partial_\mu \Phi^a, Q_a]\ . \tag{18}$$

Now, it is easy to see that the gauge-fixing superaction (16) can be written in the following form

$$S_{\rm sgf} = \int d^4x \left[(\partial_1\Phi_2)\left(\partial^\mu\widetilde{\Phi_\mu}\right) + (\partial^\mu\Phi_2)\left(\partial_1\widetilde{\Phi_\mu}\right)\right]\ . \tag{19}$$

To determine the gauge-fixing action $S_{\rm gf}$ as the lowest component of the gauge-fixing superaction $S_{\rm gf} = S_{{\rm sgf}|}$, we impose the following rules

$$\begin{aligned} {\rm Tr}(I^m I_n) &= \delta^m_n\,, \\ {\rm Tr}\left([Q_a, Q_b]\right) &= 2(\gamma^\mu)_{ab}\partial_\mu\,, \\ {\rm Tr}\left(P^2\right) &= 0\,. \end{aligned} \tag{20}$$

These rules permit us to compute the trace of each term in (19). Indeed, from (20), it is easy to put the gauge-fixing action $S_{\rm gf}$ in the form

$$\begin{aligned} S_{\rm gf} &= S_{{\rm sgf}|} = \int d^4x \Big[B\partial^\mu A_\mu + 2b^*_j G\left(\gamma^\mu\partial_\mu\Box\lambda^j\right) \\ &+ (\partial^\mu\bar{c})\left\{D_\mu c + \xi^\nu\partial_\nu A_\mu + \chi\gamma_\mu\lambda\right\} \\ &-2b^*_j\left(\partial^\mu\overline{\chi}\right)\gamma^\nu\partial_\nu\partial_\mu\left\{f^j_{ik}\lambda^i c^k + \xi^\tau\partial_\tau\lambda^j - \tfrac{1}{2}\chi\sigma^{\tau\nu}F^j_{\tau\nu} - \chi\Lambda^j\right\}\Big]\,. \end{aligned} \tag{21}$$

On the other hand, the presence of the extrafield breaks the invariance of the classical action (1). In fact, the only terms which may contribute to the Q-variation of the classical action S_0 are those containing the extrafield Λ^i. This follows from the fact that the BRST transformations up to terms Λ^i represent the $\mathcal{N}=1$ super Yang–Mills transformations expressed *à la* BRST. A simple calculation with the help of the BRST transformations (13) leads to

$$QS_0 = \chi^a\Lambda^i\left(\gamma^\mu\right)_{ab}\left(D_\mu\lambda^b\right)_i\ . \tag{22}$$

Thus, the classical action S_0 is no BRST-invariant, and in order to find the BRST-invariant extension $S_{\rm inv}$ of the classical action, we shall add to S_0 a term $\widetilde{S_0}$ so that

$$Q\left(S_0+\widetilde{S_0}\right)=0\,. \tag{23}$$

We remark that $\widetilde{S_0}$ is the part of the extended classical action related to the auxiliary field Λ^i and is given by

$$\widetilde{S_0}=-\tfrac{1}{2}\Lambda^i\Lambda_i\,. \tag{24}$$

Then, it is quite easy to show that $Q(S_0)=-Q(\widetilde{S_0})$ by a direct calculation with the help of the transformations (13).

Having found the BRST-invariant extended action $S_{\rm inv}$, we now write the full off-shell BRST-invariant quantum action $S_{\rm q}$ by adding to the Q-invariant action, $S_{\rm inv}=S_0+\widetilde{S_0}$, the Q-invariant gauge-fixing action $S_{\rm gf}$

$$S_{\rm q}=S_0+\widetilde{S_0}+S_{\rm gf}\,. \tag{25}$$

It is worth noting that the quantum action (25) allows us to see that the auxiliary field Λ^i does not propagate, as its equation of motion is a constraint

$$\frac{\delta S_{\rm gf}}{\delta\Lambda^i}=-\Lambda_i+2b_i^*\left(\widetilde{\chi}\gamma^\mu\partial_\mu\Box\chi\right)=0\,. \tag{26}$$

Thus, the essential role of the nondynamical auxiliary field Λ^i is to close the BRST and anti-BRST algebra off-shell.

The elimination of the auxiliary field Λ^i by means of its equation of motion (26) leads to the same gauge-fixed theory with on-shell nilpotent BRST transformations obtained in the context of BV formalism [10] as well as in the framework of the superfibre bundle approach [19].

Moreover, in our formalism, we have also introduced an anti-BRST operator $\overline{Q}$ and it is important to realize that both the BRST symmetry and anti-BRST symmetry can be taken into account on an equal footing. To this end, we simply use the fact that there is a complete duality, with respect to the mirror symmetry of the ghost number, between the Q and $\overline{Q}$-transformations. So, the $\overline{Q}$-variation of the classical action S_0 is given by

$$\overline{Q}S_0=\overline{\chi}^a\Lambda^i\left(\gamma^\mu\right)_{ab}\left(D_\mu\lambda^b\right)_i\,. \tag{27}$$

Using, however, the $\overline{Q}$-transformations of the auxiliary field (see Eqs. (13) with the mirror symmetry), we obtain that the Q-invariant action $S_{\rm inv}=S_0+S_0$ is also $\overline{Q}$-invariant. Furthermore, the Q-gauge-fixing action can be also written as in Yang–Mills theories in $\overline{Q}$-form. Therefore, the full off-shell BRST-invariant quantum action $S_{\rm q}=S_0+\widetilde{S_0}+S_{\rm gf}$ is also an off-shell anti-BRST-invariant quantum action.

4. Conclusion

In the present paper, we have developed a BRST superspace formalism in order to perform the quantization of the four-dimensional $\mathcal{N} = 1$ supersymmetric Yang–Mills theory as a model where the classical gauge algebra is not closed. In this geometrical framework, our construction is entirely based on the possibility of introducing *ab initio* a set of fields through a super Yang–Mills connection. The latter represents the gauge fields and their associated ghost and anti-ghost fields occurring in such theory, whereas the extrafield coming from the superconnection via the supersymmetric transformations is required to achieve the off-shell nilpotency of the BRST and anti-BRST operators. Let us note that for a local symmetry, the minimal set of auxiliary fields is introduced by the supercurvature, while for our case of global symmetry, the auxiliary real field is introduced via the superYang–Mills connection.

Furthermore, we have performed a direct construction of the BRST invariant gauge fixed action for $\mathcal{N} = 1$, $4D$ supersymmetric Yang–Mills theory in analogy with what is realized in BF theories [15] and simple supergravity [16]. The obtained quantum action allows us to see that the extrafield enjoys the auxiliary freedom. The elimination of this auxiliary field using its equation of motion permits us to recover the standard quantum action with the on-shell nilpotent BRST symmetry [10]. By using the mirror symmetry between the BRST and anti-BRST transformations, we can see that the BRST-invariant extended classical action is also anti-BRST-invariant. Therefore, the full quantum action is BRST and anti-BRST invariant, since the gauge-fixing action can be written as in the Yang–Mills case in BRST as well as anti-BRST exact form, due to the off-shell nilpotency of the BRST—anti-BRST algebra.

Finally, we should mention that the BRST superspace formalism represents the natural arena where the fields and their off-shell nilpotent BRST and anti-BRST transformations for gauge theories can be found. This is not only the case of Yang–Mills-type theories, arbitrary gauge theories may be also treated in this framework. Indeed, such formalism was applied to several interesting theories with local symmetry such as non-Abelian BF theory [15] and simple supergravity [16]. In the present work, this formalism has been applied successfully to the theory with a global supersymmetry. The off-shell nilpotency is naturally implemented through the introduction of auxiliary field required for the consistency of the BRST superspace geometry. Thus, it would be a very nice endeavor to use this basic idea to study the structure of auxiliary field in other gauge theories.

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