STRICTLY FINITE RANGE FORCES FROM THE SIGNUM-GORDON FIELD: EXACT RESULTS IN TWO SPATIAL DIMENSIONS

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Exact formula for the force between two identical static point charges coupled to the nonlinear scalar field of two-dimensional signum-Gordon model is obtained. Pertinent solution of the field equation is found in the form of one-dimensional integral. The force exactly vanishes when the distance between charges exceeds certain critical value.

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1. Introduction

Nonlinearity of field equations can have pronounced manifestations in physical predictions from the theory. Examples are ubiquitous and well-known: multiple ground states, static solitons, solitary waves, long lived oscillons, blow-up of solutions at a finite time, and many more. Another important effect due to the nonlinearity is that the field can react to the presence of external charges in a rather nontrivial way. In consequence, the force between the charges essentially differs from naive expectations based on free field models. Very old, yet still interesting example of this phenomenon is provided by classical non-Abelian gauge fields in the presence of static external charges, see, *e.g.*, [1–3]. Recently, we have studied this aspect of the nonlinearity using the signum-Gordon model [4, 5]. The remarkable simplicity of the field equation in this model enables us to present exact nonperturbative results, in contradistinction to the case of Yang–Mills field. The most striking findings are as follows: there are no Yukawa or Coulomb tails of the field in the region far away from charges; the charges are totally screened by the field; the force between them vanishes exactly when they are separated by a distance that exceeds a certain finite value. In the present paper, we corroborate these results by solving the two-dimensional case.

Let us briefly remind that the signum-Gordon model involves a single real scalar field φ that evolves according to the signum-Gordon equation

$$\partial_\mu \partial^\mu \varphi + g \,\mathrm{sign}\, \varphi = 0\,,$$

where $g > 0$ is the self-coupling constant. The sign function takes the values $0, \pm 1$, in particular $\mathrm{sign}\, 0 = 0$. The Lagrangian has the form

$$L = \tfrac{1}{2} \partial_\mu \varphi\, \partial^\mu \varphi - g\, |\varphi|\,.$$

The field potential $U(\varphi) = g|\varphi|$ is V-shaped ($|\;|$ denotes the modulus). The model has been studied in several aspects, such as self-similar solutions [6], oscillons (or rather breathers) [7], and Q-balls or boson stars in the version with a complex scalar field [8]. Let us add that recently the signum-Gordon equation has been discussed in a much wider mathematical framework of the theory of partial differential equations with compressed solutions [9]. In general, the model together with its extensions has turned out to be very useful theoretical laboratory in which we can study various above mentioned aspects of nonlinear fields.

In the present paper, similarly as in [4, 5], we regard the signum-Gordon field as the mediating field, which generates a force between two point charges coupled to it. In [4], the field generated by the two charges, as well as the force, are exactly calculated in one spatial dimension, while in three dimensions approximate formulas are given under the assumption that the point charges are close to each other. In [5], we study three-body forces in the one-dimensional case. Because the signum-Gordon equation is nonlinear one, finding its solutions is generally a nontrivial task, especially in dimensions larger than one. In the present paper, we investigate the two-dimensional case. Using a formal mathematical connection with the planar electrostatics, we show that also in this case the point charges can be totally screened by a cloud of the scalar field. This screening is easily seen for a single charge. The case of two or more charges situated not too far from each other is much more difficult. The main problem is the determination of the shape of the screening cloud. We have found this shape, as well as an integral formula for the scalar field forming the cloud. These results are used in order to derive the exact formula for the force between the two identical charges. The force completely vanishes when the distance between the charges exceeds certain critical value (equal to $2R_0(q)$, see formula (4) below) that nonanalytically depends on the strength q of the charges and the self-coupling constant g.

The plan of our paper is as follows. In Section 2, we establish the presence of the total screening in the case of two identical charges in two spatial dimensions. We find the exact shape of the screening cloud formed by the scalar field, and we obtain the integral formula for the field. Section 3 is devoted to the calculation of the force between the charges. Section 4 contains a summary and remarks.

2. The total screening of charges

Let us begin from the simple case of a single point-like charge of strength $q > 0$ located at the origin in the two-dimensional space. The field equation for the static field has the form

$$\triangle\varphi = g \, \mathrm{sign}\, \varphi - q\delta(\vec{x}\,) \,, \tag{1}$$

where $\triangle$ is the two-dimensional Laplacian. The fundamental solution of the linear Poisson equation $\triangle G = -q\delta(\vec{x}\,)$ has the form

$$G(\vec{x}\,) = -\frac{q}{2\pi} \ln \frac{|\vec{x}\,|}{l_0} \,,$$

where l_0 is a constant. We expect that close to the charge the solution φ is well approximated by $G(\vec{x}\,)$, and therefore it is positive. Then, the term $g \, \mathrm{sign}\, \varphi$ in (1) has the constant value g, and the exact solution of Eq. (1) has the form

$$\varphi(\vec{x}\,) = G(\vec{x}\,) + \frac{g}{4}\vec{x}^{\,2} + c_0 \,. \tag{2}$$

Here, c_0 is another constant. Because it can be included into the constant l_0, we put $c_0 = 0$. The constant l_0 is determined from the requirement that our $\varphi(\vec{x}\,)$ matches the vacuum solution $\varphi = 0$ at a certain radius R_0. The matching conditions have the standard form

$$\varphi = 0 \,, \qquad \varphi' = 0$$

on the circle $|\vec{x}\,| = R_0$. Here, $'$ stands for the derivative with respect to $|\vec{x}\,|$. The final form of solution (2) reads

$$\varphi = -\frac{q}{2\pi} \ln \frac{|\vec{x}\,|}{R_0(q)} + \frac{g}{4}\vec{x}^{\,2} - \frac{q}{4\pi} \,, \tag{3}$$

where

$$R_0(q) = \sqrt{\frac{q}{\pi g}} \,. \tag{4}$$

The field φ given by formula (3) has positive values in the circle $|\vec{x}\,| < R_0(q)$, hence the assumption that sign$\varphi = +1$ is fulfilled. To summarize, the scalar field in the presence of the single charge has the form

$$\varphi_q(\vec{x}\,) = \begin{cases} -\frac{q}{2\pi} \ln \frac{|\vec{x}\,|}{R_0(q)} + \frac{g}{4}\vec{x}^{\,2} - \frac{q}{4\pi} & \text{for} \quad |\vec{x}\,| < R_0(q)\,, \\ 0 & \text{for} \quad |\vec{x}\,| \geq R_0(q)\,. \end{cases}$$

The field $\varphi_q(\vec{x}\,)$ is quite interesting for the following reasons. First, note that its dependence on the coupling constant g, as well as on the charge q, is nonanalytic at $g = 0$, $q = 0$. Thus, the solution is nonperturbative in these parameters. Second, the charge q is completely screened by the circular cloud of the field, because outside the circle $|\vec{x}\,| \leq R_0$ the field attains the vacuum value $\varphi = 0$ exactly. The cloud has the constant charge density g sign$\varphi = g$ of the opposite sign than q. Formula (4) for the radius $R_0(q)$ is equivalent to the statement that the total absolute charge of the circular cloud is equal to q, namely $\pi R_0^2(q) g = q$.

Now, let us turn to less trivial case of two identical point charges of strengths q located at the points $\pm\vec{b} = (\pm b, 0)$ on the x^1 axis in the two-dimensional space. The distance between charges is given by $d = 2b$. Instead of Eq. (1), we now have

$$\triangle\varphi = g \,\mathrm{sign}\, \varphi - q\delta\left(\vec{x} - \vec{b}\right) - q\delta\left(\vec{x} + \vec{b}\right) . \tag{5}$$

Because of the nonlinear sign φ term, the pertinent solution of this equation is not just the sum of appropriately shifted in space solutions (3), unless the distance d between the charges exceeds $2R_0(q)$, in which case the two circular screening clouds surrounding the charges do not intersect each other. If the charges are close to each other, *i.e.*, $d < 2R_0(q)$, it is not clear whether the total screening is still present, and if present, what is the shape of the screening cloud of the field. We address these questions in the remaining part of this section. Our findings are utilized in the next section, where we calculate the force exerted on the charge located at $\vec{x} = \vec{b}$.

It is clear that when $b = 0$, we again have a single point charge, screened by the circular cloud of the field as discussed above, except that now the strength of the charge equals $2q$ instead q. Let us assume for a moment that the total screening persists also when $b > 0$, that is that there exists a region Σ surrounding the two charges such that $\varphi(\vec{x}\,) = 0$ for points $\vec{x}$ lying outside Σ, or on the boundary $\partial\Sigma$, and that $\varphi(\vec{x}\,) > 0$ if $\vec{x}$ lies inside Σ. Inside Σ, the term g sign $\varphi = g$ in Eq. (5) provides the constant charge density that screens the two point charges. This cloud of charge contributes to $\varphi(\vec{x}\,)$ and the contribution is given by the term

$$\varphi_{\rm cloud}(\vec{x}\,) = \frac{g}{2\pi} \int\limits_{\Sigma} d^2y \;\ln \frac{|\vec{x}-\vec{y}\,|}{R_0(q)}\,. \tag{6}$$

The two point charges contribute

$$\varphi_0(\vec{x}\,) = -\frac{q}{2\pi} \ln \frac{|\vec{x}-\vec{b}\,|}{R_0(q)} - \frac{q}{2\pi} \ln \frac{|\vec{x}+\vec{b}\,|}{R_0(q)}\,. \tag{7}$$

Note that $\varphi_0(\vec{x}\,)$ is the solution of (5) when $g=0$. The shape of the region Σ can be found from the requirement that we have the total screening, *i.e.*,

$$\varphi_{\rm cloud}(\vec{x}\,) + \varphi_0(\vec{x}\,) = 0 \tag{8}$$

for all points $\vec{x}$ outside Σ.

As already noticed, Eq. (5) is similar to Poisson equation of ordinary electrostatics. Standard reasoning known from the electrostatics gives the boundary conditions for φ at $\partial\Sigma$: $\varphi(\vec{x}\,)=0$, $\partial_n\varphi(\vec{x}\,)=0$ for all $\vec{x}\in\partial\Sigma$, where ∂_n denotes the derivative in the direction perpendicular to $\partial\Sigma$. These conditions are checked numerically after we determine the region Σ.

Condition (8) is utilized as follows. First, we rewrite the formulas for $\varphi_{\rm cloud}(\vec{x}\,)$, $\varphi_0(\vec{x}\,)$ using the rescaled polar coordinates, introduced as follows:

$$\vec{x} = R_0(q)\, r \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \vec{y} = R_0(q)\, \rho \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}.$$

In particular, $r=|\vec{x}\,|/R_0(q)$ and $\rho=|\vec{y}\,|/R_0(q)$. Thus,

$$\varphi_{\rm cloud}(r,\theta) = \frac{q}{4\pi^2}\int\limits_0^{2\pi} d\alpha \int\limits_0^{r_0(\alpha)} d\rho\,\rho \left[2\ln r + \ln\left(1+\frac{\rho^2}{r^2}-\frac{2\rho}{r}\cos(\theta-\alpha)\right)\right],$$

$$\varphi_0(r,\theta) = -\frac{q}{\pi}\ln r - \frac{q}{4\pi}\ln\left(1+\frac{d^2}{r^2}-\frac{2d}{r}\cos\theta\right) - \frac{q}{4\pi}\ln\left(1+\frac{d^2}{r^2}+\frac{2d}{r}\cos\theta\right),$$

where

$$d = \frac{b}{R_0(q)}\,.$$

The function $r_0(\alpha)$ gives the radial coordinate of the boundary of Σ at the azimuthal angle α. The considered set of two point charges is symmetric with respect to reflections in the both axises, as well as in the origin. We expect that the shape of Σ reflects these symmetries. Therefore,

$$r_0(\alpha) = r_0(2\pi-\alpha) = r_0(\pi-\alpha) = r_0(\pi+\alpha)\,.$$

Our present task is to determine the function $r_0(\alpha)$. To this end, we first notice that

$$\ln\left(1+\frac{\rho^2}{r^2}-\frac{2\rho}{r}\cos(\theta-\alpha)\right)=\ln\left(1-\frac{\rho}{r}\exp[i(\theta-\alpha)]\right)+\ln\left(1-\frac{\rho}{r}\exp[-i(\theta-\alpha)]\right) .$$

Expanding the two logarithms on the r.h.s. with respect to ρ/r, integrating over ρ, and using the symmetry of $r_0(\alpha)$, we obtain

$$\varphi_{\mathrm{cloud}}(r,\theta) = \frac{q}{4\pi^2}\int\limits_0^{2\pi} d\alpha\, r_0^2(\alpha)\, \ln r - \frac{q}{2\pi^2}\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k(k+2)r^k}\int\limits_0^{2\pi} d\alpha\, r_0^{k+2}(\alpha)e^{ik\alpha} . \tag{9}$$

This formula is to be compared with the Fourier series in θ for $\varphi_0(r,\theta)$, which can be obtained by expanding the logarithms similarly as above, and reads

$$\varphi_0(r,\theta) = -\frac{q}{\pi}\ln r + \frac{q}{2\pi}\sum_{k=1}^{\infty}\frac{1+(-1)^k}{k}\frac{d^k}{r^k}\cos(k\theta) . \tag{10}$$

Condition (8) is satisfied if

$$\int\limits_0^{2\pi} d\alpha\, r_0^2(\alpha) = 4\pi , \tag{11}$$

and

$$\int\limits_0^{2\pi} d\alpha\, r_0^{k+2}(\alpha)\, e^{ik\alpha} = \left(1+(-1)^k\right)\,(k+2)\,\pi\, d^k , \tag{12}$$

where $k=1,2,\ldots$ Because $r_0(\alpha)=r_0(\pi-\alpha)$, conditions (12) are satisfied automatically for odd values of k. For even values of k, we put $k=2l$ in (10) and rewrite (11) and (12) as the following set of conditions for $r_0(\alpha)$

$$\int\limits_0^{2\pi} d\alpha\, r_0^{2(l+1)}(\alpha)\, e^{2il\alpha} = 4(l+1)\,\pi\, d^{2l} , \tag{13}$$

where $l=0,1,2,\ldots$

Because the r.h.s.'s of conditions (13) depend on d^2, we expect that $r_0^2(\alpha)$ contains only even powers of d,

$$r_0^2(\alpha) = f_0(\alpha) + d^2 f_2(\alpha) + d^4 f_4(\alpha) + \ldots$$

The functions $f_{2n}(\alpha)$, $n = 0, 1, 2, \ldots$, have Fourier representation of the general form

$$f_{2n}(\alpha) = \sum_{k=0}^{\infty} c_{2n;k} \cos(2k\alpha)\,, \tag{14}$$

in compliance with the symmetries of $r_0(\alpha)$. The first function, f_0, gives r_0^2 when $d^2 = 0$. Thus, we put $d^2 = 0$ and $r_0^2(\alpha) = f_0(\alpha)$ in conditions (13); $f_0(\alpha)$ has the Fourier form (14). Simple analysis based on the formulas $2\cos(2k\alpha) = \exp(2ik\alpha) + \exp(-2ik\alpha)$ and $\int_0^{2\pi} d\alpha \exp(in\alpha) = 2\pi\delta_{n0}$ shows that the conditions for $f_0(\alpha)$ are satisfied only if this function is constant, $f_0(\alpha) = 2$. This result is in agreement with the fact that for $d = 0$, we actually have the single point charge of the strength $2q$, which has the circular screening cloud of the radius $R_0^2(2q) = 2R_0^2(q)$, *i.e.*, $r_0^2 = 2$ for our rescaled radial coordinate.

In order to determine the function $f_2(\alpha)$, we differentiate conditions (13) with respect to d^2, and we put $d^2 = 0$, $f_0 = 2$. This gives the conditions

$$\int_0^{2\pi} d\alpha\ e^{2il\alpha} f_2(\alpha) = 2\pi\, \delta_{l1}\,,$$

where $l = 0, 1, 2, \ldots$ It follows that $f_2 = 2\cos(2\alpha)$. Taking higher derivatives of (13) with respect to d^2 and performing similar calculations as above, we have found that $f_4 = 0$, $f_6 = 0$, $f_8 = 0$. With such partial results, we have made the educated guess that $f_{2n} = 0$ for all $n \geq 2$, *i.e.*, that

$$r_0^2(\alpha) = 2\ \left[1 + d^2\cos(2\alpha)\right]\ . \tag{15}$$

It turns out that indeed, $r_0^2(\alpha)$ given by this formula satisfies all conditions (13). The pertinent integration on the l.h.s.'s. of (13) is elementary.

The form of formula (15) implies that it holds only if $d \leq 1$, because $r_0^2 \geq 0$. In the case of $d = 1$, *i.e.*, $|\vec{b}\,| = R_0(q)$, this formula gives two circles which touch each other at the origin, and have the point charges at their centers. For $d > 1$, each charge has its own circular screening cloud, separated from the other.

It remains to check whether all values of the total field $\varphi(\vec{x}\,) = \varphi_{\mathrm{cloud}}(\vec{x}\,) + \varphi_0(\vec{x}\,)$ are strictly positive inside the region Σ, as it has been assumed. We can do this only numerically. It is a rather straightforward computation since the integral (6) giving φ_{cloud} has the already known compact domain Σ. Moreover, the two-dimensional integral can be reduced to one-dimensional one over the boundary of Σ, see formula (19) below. The numerical results corroborate our assumption. They also show that φ and $\partial_n\varphi$ are continuous on $\partial\Sigma$.

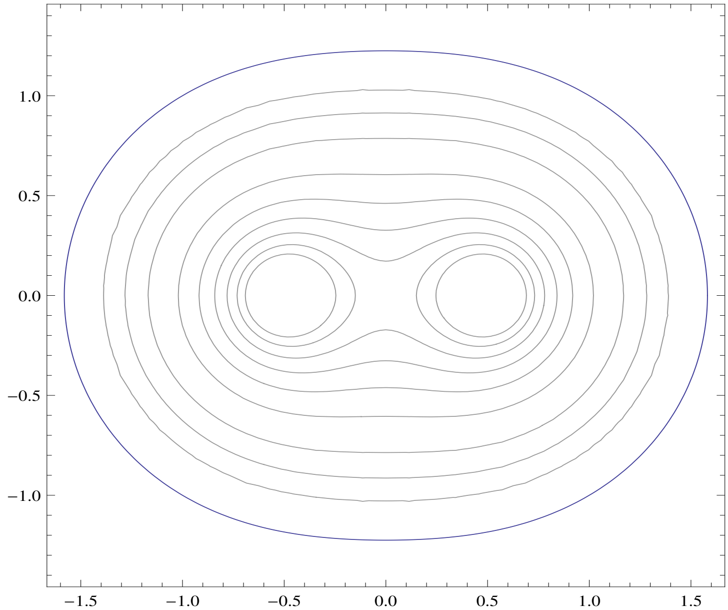

Fig. 1. The contour plot of φ_{qq} for $d = 1/2$. The outermost contour corresponds to $\varphi = 0$ and it is given by $r_0(\alpha)$, formula (15). This contour encircles the compact domain Σ. The horizontal and vertical axises correspond to $x^1/R_0(q)$, $x^2/R_0(q)$, respectively.

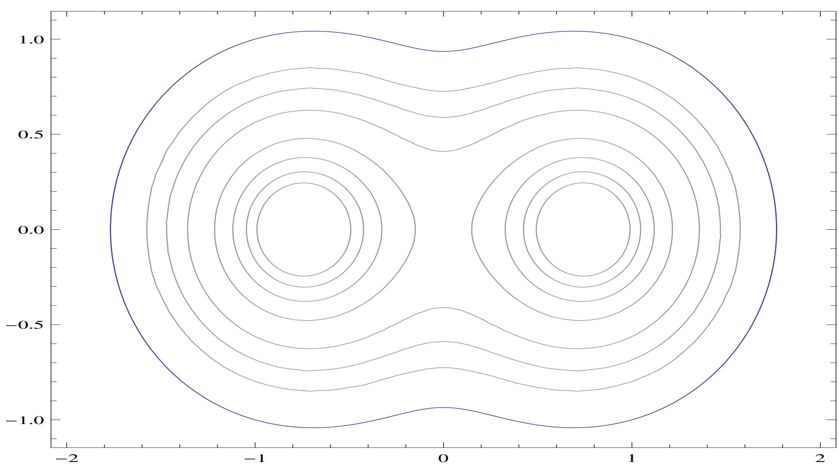

Fig. 2. The contour plot of φ_{qq} for $d = 3/4$. The meaning of the lines is the same as in Fig. 1. The picture shows the deformation of outer contours in vicinity of the vertical line $x^1 = 0$, that ultimately leads to the breakup of the cloud into two nonoverlapping circular clouds when $d = 1$.

Thus, we conclude that the pertinent solution of Eq. (5), denoted below as $\varphi_{qq}(\vec{x})$, has the following form

$$\varphi_{qq}(\vec{x}) = \begin{cases} \varphi_{\mathrm{cloud}}(\vec{x}) + \varphi_0(\vec{x}) & \text{for} \quad \vec{x} \in \Sigma\,, \\ 0 & \text{for} \quad \vec{x} \notin \Sigma\,, \end{cases} \tag{16}$$

where the compact region Σ has the boundary with the rescaled radial coordinate $r_0(\alpha)$ given by formula (15). Example plots of levels of $\varphi_{qq}(\vec{x})$ are presented in Figs. 1 and 2.

3. The force exerted on the charge located at $\vec{x} = \vec{b}$

The method of calculating the force exerted on an external charge coupled to a field is discussed in detail in [4]. Adapting it to the case at hand, we extract from the field φ_{qq} the logarithmically divergent proper field of the charge located at $\vec{x} = \vec{b}$

$$\varphi_{qq}(\vec{x}) = -\frac{q}{2\pi} \ln \frac{\left|\vec{x} - \vec{b}\right|}{R_0(q)} + u(\vec{x})\,.$$

Here,

$$u(\vec{x}) = -\frac{q}{2\pi} \ln \frac{\left|\vec{x} + \vec{b}\right|}{R_0(q)} + \varphi_{\mathrm{cloud}}(\vec{x})$$

is smooth at the point $\vec{x} = \vec{b}$. Next, we calculate the time rate of the transfer of momentum from the field to the charge located at $\vec{x} = \vec{b}$. It is equal to the force $\vec{F}$ exerted on that charge, see [4] for details and examples. It turns out that

$$\vec{F} = q \nabla u|_{\vec{x}=\vec{b}}\,, \tag{17}$$

where $\nabla u = (\partial u/\partial x^1, \partial u/\partial x^2)$. We see that the proper field of the charge does not contribute to the force exerted on that charge, as expected. The symmetry of the set of charges implies that $F^2 = 0$. The formula for the nonvanishing component F^1 can be rewritten in the polar coordinates as

$$F^1 = q \left. \frac{\partial u}{\partial |\vec{x}|} \right|_{\substack{\theta=0 \\ |\vec{x}|=b}} = \frac{q}{R_0(q)} \left. \frac{\partial u}{\partial r} \right|_{\substack{\theta=0 \\ r=d}}\,. \tag{18}$$

Computation of the force from formula (18) is significantly simplified when we rewrite φ_{cloud} inside the region Σ as a line integral over the boundary $\partial\Sigma$. First, we write φ_{cloud} in the form

$$\varphi_{\mathrm{cloud}}(\vec{x}) = \frac{g}{8\pi} \int_{\Sigma} d^2y \, \left(\partial_{y^1} W^2 - \partial_{y^2} W^1\right)\,,$$

where

$$W^i(\vec{x},\vec{y}) \;=\; \epsilon^{ik}\left(x^k - y^k\right)\left(\ln\frac{(\vec{x}-\vec{y})^2}{R_0^2(q)} - 1\right),$$
$$\partial_{y^1}W^2 - \partial_{y^2}W^1 \;=\; 2\ln\frac{(\vec{x}-\vec{y})^2}{R_0^2(q)}\,,$$

and ϵ^{ik} is the antisymmetric symbol, $i,k=1,2$. Next, we apply the Stokes theorem

$$\varphi_{\rm cloud}(\vec{x}) = \frac{g}{8\pi}\oint\limits_{\partial\Sigma} d\alpha\; \partial_\alpha\vec{y}_0(\alpha)\;\vec{W}\left(\vec{x},\vec{y}_0(\alpha)\right)\,. \qquad (19)$$

The curve $\vec{y}_0(\alpha)$, defined by

$$\vec{y}_0(\alpha) = R_0(q)\,r_0(\alpha)\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix},$$

represents $\partial\Sigma$. Using the contour representation (19), we obtain

$$\left.\frac{\partial\varphi_{\rm cloud}}{\partial r}\right|_{\substack{\theta=0\\ r=d}} = \frac{gR_0^2(q)}{4\pi}\left[\int\limits_0^{2\pi} d\alpha\; h_1(\alpha)\,h_2(\alpha) + \frac{1}{2}\int\limits_0^{2\pi} d\alpha\; h_3(\alpha)\,h_4(\alpha)\right], \qquad (20)$$

where

$$h_1(\alpha) \;=\; \frac{d - r_0(\alpha)\,\cos\alpha}{d^2 - 2\,d\,r_0(\alpha)\,\cos\alpha + r_0^2(\alpha)}\,,$$
$$h_2(\alpha) \;=\; r_0^2(\alpha) - d\,r_0(\alpha)\,\cos\alpha + \frac{2d^3\sin\alpha\,\sin(2\alpha)}{r_0(\alpha)}\,,$$
$$h_3(\alpha) \;=\; \ln\left[d^2 - 2\,d\,r_0(\alpha)\,\cos\alpha + r_0^2(\alpha)\right] - 1\,,$$
$$h_4(\alpha) \;=\; \frac{2\,d^2\sin\alpha\sin(2\alpha)}{r_0(\alpha)} - r_0(\alpha)\,\cos\alpha\,,$$

$r_0(\alpha)$ is given by formula (15), $0<d<1$.

In spite of their appearance, the integrals in (20) are elementary. First, notice that $h_4(\alpha) = -\partial_\alpha(r_0(\alpha)\,\sin\alpha)$. Therefore, in the second integral in (20), we may use integration by parts in order to eliminate the logarithm function. Second, we write the sin and cos functions in terms of $\exp(\pm i\alpha)$ and observe that $\int_0^{2\pi} d\alpha\exp(ik\alpha) = 0$ for any integer k except $k=0$. This eliminates many terms, in particular all terms proportional to $\sqrt{1+d^2\cos(2\alpha)}$. This last function is thought of as the Taylor series with respect to $d^2\cos(2\alpha)$, obviously it contains only even powers of $\exp\,(\pm\,i\alpha)$.

Finally, we use the integral

$$\int_0^{2\pi} \frac{d\alpha}{4 + d^4 + 4d^2\, \cos(2\alpha)} = \frac{2\pi}{4 - d^4}\,,$$

and formula (4) for $R_0(q)$. We obtain

$$\left.\frac{\partial \varphi_{\text{cloud}}}{\partial r}\right|_{\substack{\theta=0 \\ r=d}} = \frac{q}{4\pi}\, d\, \left(2 - d^2\right)\,. \tag{21}$$

The nonvanishing component of the force exerted on the charge located at the point $x^1 = b$, $x^2 = 0$ is computed from formula (18)

$$F^1 = -\frac{q^2}{4\pi b}\left(1 - \frac{b^2}{R_0^2(q)}\right)^2 = -\frac{q^2}{4\pi b}\left(1 - \frac{\pi g b^2}{q}\right)^2\,. \tag{22}$$

Here, we have returned to the original coordinate $b = R_0(q)d$. The force is attractive one, as expected from the scalar field. Formula (22) holds for $b < R_0(q)$, outside this range of b the force vanishes[1].

The factor $-q^2/4\pi b$ in formula (22) represents the standard two-dimensional Coulomb force characteristic for the free field ($g = 0$). This force dominates at short distances, $l \ll l_0$, also in the case of self-interacting field. Here, $l = 2b$ is the distance between the charges, and $l_0 = 2R_0(q)$ is the critical distance between them. At distances $l \lessapprox l_0$, the force becomes weaker, and it exactly vanishes for $l \geq l_0$. When the distance l reaches l_0, the screening cloud splits into two nonoverlapping circular clouds that screen each charge separately.

Formula (22) for the force can easily be generalized to the case one particle is located at the point $\vec{x}$ and the other one at $\vec{y}$. It is sufficient to substitute $b = l/2$, where $l = |\vec{x} - \vec{y}\,|$, and to include the unit vector $\vec{n} = (\vec{x} - \vec{y})/|\vec{x} - \vec{y}|$ directed from $\vec{y}$ to $\vec{x}$. Thus, the force exerted on the particle located at $\vec{x}$ is given by formula

$$\vec{F}(\vec{x}\,) = -\frac{q^2}{2\pi l}\left(1 - \frac{l^2}{l_0^2}\right)^2 \vec{n} \tag{23}$$

if $l = |\vec{x} - \vec{y}| < l_0$, otherwise $\vec{F}(\vec{x}\,) = 0$. This force possesses the potential U_{qq} such that $\vec{F}(\vec{x}\,) = -\partial_{\vec{x}} U_{qq}$, namely

$$U_{qq} = \frac{q^2}{2\pi}\left[\ln\frac{l}{l_0} - \left(\frac{l}{l_0}\right)^2 + \frac{1}{4}\left(\frac{l}{l_0}\right)^4 + \frac{3}{4}\right]$$

if $l < l_0$, otherwise $U_{qq} = 0$.

[1] $R_0(q)$ vanishes in the limit $q \to 0$. Therefore, the force also vanishes in this limit, in spite of the fact that the r.h.s. of formula (22) gives $-\pi g^2 b^3/4$. The point is that in that limit formula (22) is not valid for any b — the force vanishes for all $b \geq R_0(q)$.

4. Summary and remarks

1. As the most valuable result of our work we regard the exact formula (23) for the force exerted by one point charge on the other. This force vanishes quadratically when the distance l between the charges approaches (from below) the critical value $l_0 = 2\sqrt{q/\pi g}$. In the one-dimensional case investigated in [4], the force vanishes linearly, $F = q^2(1-a/a_*)/2$, where a is the distance between the charges, and the critical distance a_* is given by the formula $a_* = q/g$ (we keep the same form of the field equation in all dimensions). The situation in three-dimensional case remains to be investigated because in [4] we have been able to compute the force only approximately, under the assumption that the charges are close to each other.

The field φ is the sum of φ_{cloud}, given by the exact integral formula (19), and of the two logarithmic Coulomb terms. In the one-dimensional case the pertinent field is known analytically [4], while in three dimensions, we only have an approximate formula for it. Note that formula (19) provides the quite convenient starting point for numerical computations of the field.

2. The total screening of the charges coupled to the signum-Gordon scalar field may resemble the phenomenon of total screening of external color charges interacting with a classical Yang–Mills field [1, 2]. One should however note that there are several differences: in the Yang–Mills case the total screening is proven for external charges that are spatially extended; the screening field is time-dependent; the complete screening appears only in a certain limit. Moreover, the analytic form of the screening field is not known. It is not clear to us whether the two cases of the total screening are interrelated in some way.

3. In paper [10], it is shown that a classical scalar field in the presence of point-like external charges can be utilized in order to unravel certain essential features of the corresponding quantum theory, like asymptotic freedom or triviality, depending on the sign of coupling constant. The main role is played by a classical perturbative solution that has the form of a formal series in powers of the coupling constant. The field studied there is the real, massless scalar field with the self-interaction of the form $\lambda\varphi^4$. Similar investigations in the cases of a massive scalar field and Yang–Mills field are presented in [11, 12], respectively. We think it would be interesting to employ such a method to the signum-Gordon model, in particular because very little is known about the properties of the quantum version of this model.

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