CALCULATION AND APPLICATION OF OFF-SHELL AMPLITUDES*

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(Received October 16, 2015)

Hadron scattering processes involving low values for the fraction of the momentum of the hadrons transfered to the partonic process require factorization prescriptions other than collinear factorization. These admit pdfs with resummed logarithms of this momentum fraction. One of their characteristics is that they provide an explicit transverse momentum to the parton, rendering it off-shell. Consequently, the exact definition and calculation of the matrix element needs special attention. The program AMP4HEF numerically evaluates multi-parton amplitudes and matrix elements with up to two of them off-shell.

DOI:10.5506/APhysPolB.46.2105
PACS numbers: 12.38.Bx, 13.85.–t

1. Introduction

Factorization formulas are at the basis of cross section calculations of hadron collisions at high energy scattering experiments like the Large Hadron Collider. Heuristically, they separate the partonic cross section, calculable perturbatively within quantum chromodynamics (QCD), from parton density functions (pdfs) describing the colliding hadrons. The latter typically cannot be calculated to the end, and need experimental input to be modeled. Their dependence on some relevant energy scale, however, can be calculated perturbatively within QCD.

For the calculation of processes within central rapidity regions, *collinear factorization* is mostly applied, in which a fraction of the momentum of each hadron, usually denoted by the letter "x", is entering the partonic process as the momentum of each of the initial-state partons. The original idea of the

* Presented at the XXXIX International Conference of Theoretical Physics "Matter to the Deepest", Ustroń, Poland, September 13–18, 2015.

parton model is that it is actually these partons that scatter while carrying those momentum fractions. The possibility of a systematic perturbative approach within collinear factorization is well-established, and it is known to be universal with respect to the content of the partonic cross section.

For events in non-central rapidity regions, one of the partons is carrying far less energy than the other, that is the value of x is much smaller. In this situation, other factorization formulas apply that allow for pdfs that take into account the resummation of logarithms of the small value of x. An example is the hybrid factorization of [1], which applies the so-called high-energy factorization (HEF) [2, 3] with respect to one of the scattering hadrons in dijet production. One of the features that distinguishes this formula from collinear factorization is that one of the pdfs explicitly depends on momentum in the transverse plane, that serves as the transverse momentum of the parton entering the partonic process, rendering this momentum off-shell. Consequently, the definition and calculation of the scattering amplitude needs special attention, since simple application of standard Feynman rules with off-shell kinematics does not lead to gauge invariant quantities. A few approaches that lead to the same tree-level results have been developed to deal with this situation [4–8].

The applicability of different factorization formulas is dictated by the relative values of the relevant scales in the considered process. In forward dijet production, these are the typical transverse momentum P_{T} of a hard jet, the transverse momentum k_{T} of the small-x gluons, and the saturation scale Q_{s}, separating the linear regime from the non-linear regime of parton saturation [9] for the evolution of the low-x pdf. The aforementioned HEF is valid when $Q_{\mathrm{s}} \ll k_{\mathrm{T}} \sim P_{\mathrm{T}}$, and the k_{T}-dependent pdf is also called the *unintegrated pdf*. When $Q_{\mathrm{s}} \sim k_{\mathrm{T}} \ll P_{\mathrm{T}}$, the so-called *transverse momentum dependent* (TMD) factorization [10] is valid, that is exactly in the region where non-linear evolution effects are present. In this case, on-shell matrix elements are used in combination with several unintegrated gluon distributions. The fact that this factorization formula involves a sum over several pdfs that are associated with different color structures for the matrix elements is an essential difference with, for example, collinear factorization. In [11], a factorization formula was derived for forward dijet production that is valid in both regimes, so for $Q_{\mathrm{s}} \lesssim k_{\mathrm{T}} \lesssim P_{\mathrm{T}}$. This formula requires both features for the matrix elements, namely off-shellness and explicit color structures.

2. Off-shell amplitudes from on-shell recursion

The foregoing highlights the need for control over matrix elements with off-shell initial-state partons to the level of separately gauge invariant color structures. In [11], the so-called color-ordered helicity amplitudes were applied. In case all partons are on-shell, these are known to be computable efficiently using on-shell recursion [12, 13]. Recently, it has been shown that also in the case of off-shell partons, the "on-shell" type of recursion can be applied [14, 15]. These amplitudes still come in all helicities for the on-shell partons. For the off-shell partons, the amplitudes take another argument besides the momentum, namely the longitudinal momentum or *direction*, that is (a fraction of) the momentum of the hadron from which the off-shell parton is considered to originate. In general, momenta k_i and directions p_i as arguments of an amplitude satisfy

$$\begin{aligned} k_1^\mu + k_2^\mu + \cdots + k_n^\mu &= 0 \qquad \text{momentum conservation}\,, && (1)\\ p_1^2 = p_2^2 = \cdots = p_n^2 &= 0 \qquad \text{light-likeness}\,, && (2)\\ p_1\!\cdot\! k_1 = p_2\!\cdot\! k_2 = \cdots = p_n\!\cdot\! k_n &= 0 \qquad \text{eikonal condition}\,, && (3)\end{aligned}$$

where we follow the convention that on-shell momenta are identical to the associated directions. With the help of an auxiliary light-like four-vector q^μ, the momentum k^μ can be decomposed in terms of its light-like direction p^μ, satisfying $p\cdot k = 0$, and a transversal part, following

$$k^\mu = x(q)p^\mu - \frac{\kappa}{2}\frac{\langle p|\gamma^\mu|q]}{[pq]} - \frac{\kappa^*}{2}\frac{\langle q|\gamma^\mu|p]}{\langle qp\rangle}\,, \tag{4}$$

with

$$x(q) = \frac{q\cdot k}{q\cdot p}\,, \qquad \kappa = \frac{\langle q|\not{k}|p]}{\langle qp\rangle}\,, \qquad \kappa^* = \frac{\langle p|\not{k}|q]}{[pq]}\,. \tag{5}$$

The coefficients κ and κ^* can easily be shown to be independent of the auxiliary momentum, and play, together with the spinors of the light-like directions and on-shell momenta, the role of fundamental building blocks for the recursively found expression for the helicity amplitudes. As an example, we present a helicity amplitude with three gluons, one of them off-shell, and a quark–anti-quark pair:

$$\begin{aligned}\mathcal{A}\left(q^-, g_1^+, g_2^-, g_3^*, \bar{q}^+\right) &= \frac{[31]^4\langle 3q\rangle^3}{[23][12]\langle q\bar{q}\rangle\langle 3|\not{k}_3+\not{p}_2|1]\langle 3|\not{p}_1+\not{p}_2|3]\langle q|\not{p}_1+\not{p}_2|3]}\\ &+\frac{[3\bar{q}]^2\langle 2q\rangle^3\langle 2|\not{k}_3+\not{p}_{\bar{q}}|3]}{\kappa_3\langle 12\rangle\langle 1q\rangle\left(\not{k}_3+\not{p}_{\bar{q}}\right)^2\left(-\langle 2|\not{k}_3+\not{p}_{\bar{q}}|3]\langle q|\not{k}_3|\bar{q}] - [3\bar{q}]\langle 2q\rangle\left(\not{k}_3+\not{p}_{\bar{q}}\right)^2\right)}\\ &+\frac{[1\bar{q}]^3\langle 32\rangle^4}{\kappa_3^*[q\bar{q}]\langle 3|\not{k}_3+\not{p}_2|1]\langle 2|\not{k}_3|\bar{q}]\langle 3|\left(\not{k}_3+\not{p}_2\right)\not{k}_3|2\rangle}\,. \end{aligned}\tag{6}$$

The number 3 in the spinor products refers to the direction p_3 of the off-shell gluon. If the amplitude is multiplied by $|k_3| = \sqrt{|k_3^2|}$ and the limit $k_3 \to p_3$

is applied, then the first term in (6) vanishes, while the other two give

$$\frac{|k_3|}{\kappa_3}\mathcal{A}\left(q^-,g_1^+,g_2^-,g_3^+,\bar{q}^+\right)+\frac{|k_3|}{\kappa_3^*}\mathcal{A}\left(q^-,g_1^+,g_2^-,g_3^-,\bar{q}^+\right)\,. \tag{7}$$

So the amplitude with the off-shell gluon contains the amplitudes for both helicities if the gluon were on-shell. This coherent sum eventually turns into an incoherent sum of squared amplitudes via the remnant angular integral for $\vec{k}_{\mathrm{T}\,3}$. Due to the phase factors in front of the amplitudes in (7), interference terms after squaring are eliminated by this angular integral.

3. AMP4HEF

Recently, the on-shell recursion for off-shell amplitudes has been applied in a numerical program AMP4HEF to evaluate multi-gluon amplitudes [16]. It includes expressions obtained with the recursion for up to 5 gluons, and numerical recursion for more gluons. Furthermore, it evaluates both amplitudes and matrix elements, *i.e.* squared amplitudes summed over all colors and helicities. This program has now been extended to include amplitudes and matrix elements with one quark–anti-quark pair. More specifically, it includes the following processes:

$$\begin{aligned}
&\emptyset \;\to\; g\,g+4g\,, &&\emptyset\to\bar{q}\,q+3g\,,\\
&\emptyset \;\to\; g^*\,g+4g\,, &&\emptyset\to\bar{q}^*\,q+3g\,,\\
&\emptyset \;\to\; g^*\,g^*+4g\,, &&\emptyset\to\bar{q}\,q^*+3g\,,\\
&\emptyset \;\to\; g^*+\bar{q}\,q+2g\,, &&
\end{aligned} \tag{8}$$

plus those with fewer on-shell gluons. AMP4HEF can be obtained from: `http://bitbucket.org/hameren/amp4hef/downloads`. It is written in Fortran, but there is an interface for and an example of use in, C++ included. Many object-oriented features of the Fortran 2003 standard are used, but to the user only a few subroutines which take arrays of intrinsic types as arguments are provided. To evaluate matrix elements, one only needs to:

```
use amp4hef
```

so one has access to

```
subroutine put_process( id ,Ntotal ,Noffshell ,process )
integer,intent(out) :: id
integer,intent(in ) :: Noffshell,Nonshell,process(*)
```

to set a process and obtain an integer identifier referring to that process,

```
subroutine put_momenta( id ,momenta ,directions )
integer         ,intent(in) :: id
real(kind(1d0)),intent(in) :: momenta(0:3,*) ,directions(0:3,*)
```

to set the values of the momenta k_i and, for off-shell partons, the directions p_i, and

```
subroutine matrix_element( id ,ampSquared )
integer         ,intent(in ) :: id
real(kind(1d0)),intent(out) :: ampSquared
```

to evaluate the matrix element summed over colors and helicities. A more detailed description of the routines and their input/output is provided with the program.

4. Conclusions

Factorization prescriptions relevant for scattering processes involving low values of the fraction of the hadron momenta transfered to the partonic process require matrix elements with off-shell initial-state partons. The program AMP4HEF can evaluate such matrix elements, as well as individual color-ordered amplitudes, numerically, for processes involving several external partons.

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