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SIMPLIFYING SYSTEMS OF DIFFERENTIAL EQUATIONS. THE CASE OF THE SUNRISE GRAPH*

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Differential equations are one of the most powerful and promising tools for evaluating multi-loop and multi-scale Feynman integrals. We report on a new systematic method for simplifying systems of differential equations. The method is based on the analysis of the integration by parts identities in fixed integer numbers of dimensions. The case of the two-loop massive sunrise is discussed in detail.

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1. Introduction and method

The recent discovery of a Higgs boson-like particle at the LHC [1, 2] has, on the one hand, proved the internal consistency of the Standard Model, while on the other, it has not provided any hints towards possible New Physics phenomena. Precision calculations become therefore of crucial importance in order to look for significant deviations between the theoretical predictions and the experimental measurements at the LHC. Precision calculations are based on perturbative expansion in dimensionally regularised [3–5] Feynman integrals. The differential equation method [6, 7], based on the integration by parts identities (IBPs) [8, 9], has proved to be one of the most powerful tools for their analytic and numerical evaluation. The by now standard way to calculate a given set of Feynman integrals is the following. First of all, the IBPs can be used to relate integrals belonging to the same Feynman graph, but with different powers of propagators and scalar products. The IBPs form a large system of linear identities for the integrals. The latter can be inverted allowing one to express most of the integrals in terms of a small subset of basic integrals, dubbed master integrals (MIs). Building upon the Laporta algorithm [10], this reduction procedure has been automatised in many public codes [11–14].

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The IBPs can be further used to prove that the MIs fulfil a system of linear differential equations. If a given graph is reduced to N independent MIs, then one can derive a system of N *coupled* differential equations for the latter. With the increase in the number of loops and external legs, one is usually left with larger and larger systems of coupled equations. It is well known that an appropriate choice of basis of MIs can substantially simplify the system of differential equations. In particular, since we are interested in the MIs expanded as Laurent series in $(d-4)$, a clear simplification can be achieved if a basis of MIs can be found, such that the system of differential equations becomes triangular in the $d \to 4$ limit. Moreover, it is known that thanks to the Tarasov–Lee shifting relations [15, 16], the physical relevant case $d \to 4$ can be recovered from the Laurent expansion of the MIs in any even number of dimensions, $d \to 2\,n$ with $n \in \mathbb{N}$.

We report here on a new, systematical method for achieving such a simplification, focusing, in particular, on the case of the two-loop massive sunrise graph. A detailed and more general discussion can be found in [17]. While the possibility of writing the IBPs is intrinsically based on the fact that the MIs are evaluated for continuous values of the space-time dimensions d, we will show that considering their $d \to 2\,n$ limit can provide new identities useful for decoupling the differential equations in this same limit. It can be shown that these new identities are equivalent to the Schouten pseudo-identities introduced in [18], in those cases where the latter can be derived. The direct study of the IBPs as $d \to 2\,n$ is, nevertheless, simpler and more general, since it can be applied, in principle, to any topology and in any number of space-time dimensions, irrespective of the number of independent momenta at disposal.

Let us consider again a graph reduced to N independent MIs $\mathcal{I}_i(d;x)$. Upon fixing the number of space-time dimensions to an integer (for the physical interesting case, *even*) number, it can happen that the system of IBPs *degenerates*, such that some of the MIs which used to be linearly independent in d-dimensions, can instead become linearly dependent as $d \to 2\,n$. This degeneracy can be traced back in the d-dimensional IBPs to the existence of relations of the form of

$$K(d;x) = \frac{1}{d-2\,n}\left(b_{11}(d;x)\mathcal{I}_1(d;x) + \ldots + b_{1N}(d;x)\mathcal{I}_N(d;x)\right)\,, \qquad (1)$$

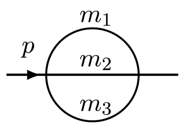
where $K(d;x)$ is an integral of the graph under consideration, x is the set of Mandelstam variables the integrals depend on, and $b_{ij}(d;x)$ are rational functions. As we will show in the next section, if M *independent*[1] relations of the form of (1) can be found, then they can be used to decouple M differential equations from the system in the $d \to 2\,n$ limit.

[1] Here, independent has to be intended in the $d \to 2\,n$ limit.

2. The case of the massive sunrise graph

Let us show how this works for the explicit case of two-loop massive sunrise. The graph is defined as

$$I(d;n_1,n_2,n_3,n_4,n_5) =$$



$$= \int \mathfrak{D}^d k \mathfrak{D}^d l \, \frac{(k\cdot p)^{n_4} (l\cdot p)^{n_5}}{\left(k^2-m_1^2\right)^{n_1} \left(l^2-m_2^2\right)^{n_2} \left((k-l+p)^2-m_3^2\right)^{n_3}} \,. \tag{2}$$

If all three masses have different values, a reduction through IBPs shows that there are four independent MIs, which can be chosen to be

$$\begin{aligned} \mathcal{I}_1(d;s) &= I(d;1,1,1,0,0)\,, & \mathcal{I}_2(d;s) &= I(d;2,1,1,0,0)\,, \\ \mathcal{I}_3(d;s) &= I(d;1,2,1,0,0)\,, & \mathcal{I}_4(d;s) &= I(d;1,1,2,0,0)\,. \end{aligned} \tag{3}$$

In [19], it was shown that these integrals fulfil a coupled system of 4 linear first-order differential equations in d dimensions. The system remains coupled in the $d \to 2\,n$ limits, where $n \in \mathbb{N}$. It was lately shown in [20], using algebraic geometry methods (and as such *a priori* orthogonal to the IBPs), that the scalar integral $\mathcal{I}_1(d;s)$ satisfies a second-order Picard–Fuchs differential equation in $d=2$. In [18], we showed that the same result can be obtained using the Schouten pseudo-identities. We will now discuss how this can be seen simply by studying the IBPs in $d=2$ space-time dimensions.

We generate the IBPs in d dimensions and, before solving them, we fix $d=2$. In this way, we find that two of the four MIs degenerate and become linearly dependent from the other two. Neglecting the sub-topologies, we find the following relations

$$\begin{aligned} m_2^2\, P\left(s,m_1^2,m_2^2,m_3^2\right) \mathcal{I}_3(2;s) \;=\;& \left(m_1^2-m_2^2\right)\left(m_1^2+m_2^2-m_3^2-s\right) \mathcal{I}_1(2;s) \\ & + m_1^2 \Big[\left(m_1^4-3m_2^4+2m_1^2\right)\left(m_2^2-m_3^2-s\right) \\ & + \left(m_3^2-s\right)^2 + 2m_2^2\left(m_3^2+s\right) \Big] \mathcal{I}_2(2;s)\,, \\ m_3^2\, P\left(s,m_1^2,m_2^2,m_3^2\right) \mathcal{I}_4(2;s) \;=\;& \left(m_1^2-m_3^2\right)\left(m_1^2-m_2^2+m_3^2-s\right) \mathcal{I}_1(2;s) \\ & + m_1^2 \Big[m_1^4+m_2^4-3m_3^4+2m_2^2\left(m_3^2-s\right) \\ & + 2m_3^2 s + s^2 - 2m_1^2\left(m_2^2-m_3^2+s\right) \Big] \mathcal{I}_2(2;s)\,. \end{aligned} \tag{4}$$

As discussed, such relations come from the corresponding d-dimensional IBPs with an overall factor $1/(d-2)$. We refer to [17] for details. The limiting value of the two relevant IBPs as $d \to 2$ generates Eqs. (4) and can be used to decouple two of the four differential equations of the two-loop massive sunrise graph by choosing as new basis of master integrals

$$\begin{aligned}
&\mathcal{J}_1(d;s) = \mathcal{I}_1(d;s)\,,\\
&\mathcal{J}_2(d;s) = \mathcal{I}_2(d;s)\,,\\
&\mathcal{J}_3(d;s) = -\left(2m_1^2 - m_2^2 - m_3^2\right)\mathcal{I}_1(d;s) + 2\,m_1^2\left(s - m_1^2\right)\mathcal{I}_2(d;s)\\
&\;+ m_2^2\left(-3m_1^2 + m_2^2 + 3m_3^2 - s\right)\mathcal{I}_3(d;s) + m_3^2\left(-3m_1^2 + 3m_2^2 + m_3^2 - s\right)\mathcal{I}_4(d;s)\,,\\
&\mathcal{J}_4(d;s) = -\left(m_1^2 - 2m_2^2 + m_3^2\right)\mathcal{I}_1(d;s) - 2\,m_2^2\left(s - m_2^2\right)\mathcal{I}_3(d;s)\\
&\;+ m_1^2\left(-m_1^2 + 3m_2^2 - 3m_3^2 + s\right)\mathcal{I}_2(d;s) + m_3^2\left(-3m_1^2 + 3m_2^2 - m_3^2 + s\right)\mathcal{I}_4(d;s)\,.
\end{aligned} \tag{5}$$

It can be easily verified that the last two equations in (5) generate (4) if we put $\mathcal{J}_3 = \mathcal{J}_4 = 0$ and invert them for $\mathcal{I}_3$ and $\mathcal{I}_4$. We refer to [17, 18] for the details on the calculations. We want to stress here that with the choice described above, the differential equations assume a block form in the $d \to 2$ limit, such that instead of having to solve a system of four coupled differential equations, we are left with two coupled equations, plus two (in principle straightforward) integrations by quadrature.

3. Conclusions

Differential equations are one of the most important tools for the calculation of Feynman integrals. When considering complicated multi-loop and/or multi-scale Feynman integrals, one is often left with a large number of MIs and, consequently, a large number of coupled first-order differential equations. Since we are usually interested in the coefficients of the MIs expanded as Laurent series in $(d-4)$, we can try and simplify the system by decoupling some of the equations in the $d \to 4$ limit. We discussed how this simplification can be achieved studying the IBPs in the limit of even integer numbers of dimensions, as shown explicitly in the case of the two-loop massive sunrise graph. A more complete discussion, with applications to a larger number of examples, can be found in [17].

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