THE SUNRISE INTEGRAL AROUND TWO AND FOUR SPACE-TIME DIMENSIONS IN TERMS OF ELLIPTIC POLYLOGARITHMS*

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In this paper, we discuss the solution for the sunrise integral around two and four space-time dimensions in terms of a generalised elliptic version of the multiple polylogarithms. In two space-time dimensions, we obtain a sum of three elliptic dilogarithms. The arguments of the elliptic dilogarithms have a nice geometric interpretation. In four space-time dimensions, the sunrise integral can be expressed with the ϵ^0- and ϵ^1-solution around two dimensions, mass derivatives thereof and simpler terms.

1. Introduction

The non-vacuum two-loop sunrise integral is the simplest integral in quantum field theory which cannot be solved in terms of multiple polylogarithms. Because of its fundamental character, the sunrise integral received numerous attention in the past [1–16]. For the equal mass case, an analytic result involving elliptic integrals and elliptic polylogarithms was provided in [10, 16]. The results can be generalised to the arbitrary mass case [13–15], which we will discuss in this paper.

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2. Differential equation for the sunrise integral

In D-dimensional Minkowski space, the two-loop sunrise integral reads

$$S_{\nu_1\nu_2\nu_3}\left(D, p^2, m_1^2, m_2^2, m_3^2, \mu^2\right)$$
$$= \left(\mu^2\right)^{\nu-D} \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{d^D k_2}{i\pi^{\frac{D}{2}}} \frac{1}{\left(-k_1^2+m_1^2\right)^{\nu_1} \left(-k_2^2+m_2^2\right)^{\nu_2} \left(-(p-k_1-k_2)^2+m_2^2\right)^{\nu_3}} \tag{1}$$

with the three internal masses m_1, m_2, m_3. In terms of Feynman parameters, we obtain for the sunrise integral the expression

$$S_{\nu_1\nu_2\nu_3}(D,t) = \frac{\Gamma(\nu-D)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} \left(\mu^2\right)^{\nu-D} \int_\sigma x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1} \frac{\mathcal{U}^{\nu-\frac{3}{2}D}}{\mathcal{F}^{\nu-D}} \omega \tag{2}$$

with $\nu = \nu_1 + \nu_2 + \nu_3$, the momentum squared $t = p^2$, the differential two-form $\omega = x_1 \, dx_2 \wedge dx_3 - x_2 \, dx_1 \wedge dx_3 + x_3 \, dx_1 \wedge dx_2$, the integration region $\sigma = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0, i = 1, 2, 3\}$ and the Feynman graph polynomials $\mathcal{U}$ and $\mathcal{F}$

$$\mathcal{U} = x_1 x_2 + x_2 x_3 + x_3 x_1 \,, \qquad \mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right) \mathcal{U} \,.$$

We are interested in the Laurent expansions of $S_{111}(D,t)$ around two and four space-time dimensions. These expansions start as

$$S_{111}(2 - 2\epsilon, t) = e^{-2\gamma\epsilon} \left[S_{111}^{(0)}(2, t) + \epsilon S_{111}^{(1)}(2, t) + \mathcal{O}\left(\epsilon^2\right) \right] \,,$$
$$S_{111}(4 - 2\epsilon, t) = e^{-2\gamma\epsilon} \left[\frac{1}{\epsilon^2} S_{111}^{(-2)}(4, t) + \frac{1}{\epsilon} S_{111}^{(1)}(4, t) + S_{111}^{(0)}(4, t) + \mathcal{O}(\epsilon) \right] \,. \tag{3}$$

In the following, we are mainly interested in determining $S_{111}^{(0)}(2, t)$ and $S_{111}^{(0)}(4, t)$.

In D dimensions, the sunrise integral $S_{111}(D, t)$ satisfies an inhomogeneous fourth-order differential equation [15]

$$\left[\sum_{i=0}^{4} P_i \frac{d^i}{dt^i} \right] S_{111}(D, t) = \mu^2 \left[c_{12} T_{12} + c_{13} T_{13} + c_{23} T_{23} \right] \,, \tag{4}$$

where the P_i's and c_{ij}'s are polynomials in D, t and the masses, and the T_{ij}'s are products of simpler tadpole integrals. This differential equation can then be expanded for example around $D = 2 - 2\epsilon$.

3. The sunrise integral around two space-time dimensions

In two space-time dimensions, the fourth-order differential equation for the sunrise integral from Eq. (4) can be simplified to a second-order differential equation with a polynomial $p(t)$ depending on t, the masses and logarithms of the masses [11]

$$L_2^{(0)}(2)\, S_{111}^{(0)}(2,t) = \mu^2 p(t)\,. \tag{5}$$

In $D = 2$, the integrand of Eq. (2) simplifies to $\frac{1}{\mathcal{F}}$. The equation $\mathcal{F} = 0$, together with the choice of an origin defines an elliptic curve. A convenient choice for the origin is one of the intersection points of the algebraic variety $\mathcal{F} = 0$ and the integration region σ, where we have three different points $P_1 = [1 : 0 : 0], P_2 = [0 : 1 : 0]$ and $P_3 = [0 : 0 : 1]$. Every chosen origin P_i corresponds to a certain elliptic curve E_i with $i = \{1, 2, 3\}$ which we can transform into the Weierstrass normal form $\hat{E}_i$.

In the next step, we compute the periods Ψ_1, Ψ_2 of the elliptic curve $\hat{E}_i$ which leads to complete elliptic integrals of the first kind

$$\Psi_1 = \frac{4\mu^2}{\sqrt{D}} K(k)\,, \qquad \Psi_2 = \frac{4\mu^2 i}{\sqrt{D}} K\left(k'\right)\,, \tag{6}$$

where the (complementary) modulus $k^{(')}$ depends on the roots e_i of the Weierstrass normal form, while the polynomial D depends on t and the masses. The periods Ψ_1, Ψ_2 are the homogeneous solutions of the second-order differential equation.

After changing the variable from t to the nome $q = e^{2\pi i \tau}$ (*cf.* [16]) which is the natural variable on an elliptic curve with the period ratio $\tau = \frac{\Psi_2}{\Psi_1}$, we can write the special inhomogeneous solution in an overseeable form. The aim is now to express this inhomogeneous solution in terms of the homogeneous solutions and a new (elliptic) generalisation of the classical polylogarithms. The two periods imply a lattice Λ which belongs to the torus representation of the elliptic curve $\mathbb{C}/\Lambda$ to which we can transform by the following elliptic integral

$$[x : y : 1] \to \hat{z} = \frac{1}{\Psi_1} \int\limits_x^\infty \frac{1}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}}\,. \tag{7}$$

This function maps the image of the intersection point $P_j \in E_k$ on the curve $\hat{E}_k$ abbreviated by $Q_{j,k}$ onto the point

$$Q_{j,k} \to \hat{z}_i = \frac{1}{2} \frac{F(u_i, k)}{K(k)} \tag{8}$$

with the incomplete elliptic integral of the first kind $F(z, x)$.

As the last step, we can compute the Jacobi uniformization $\mathbb{C}^*/q^{2\mathbb{Z}}$ of the elliptic curve via the complex exponential $\hat{z} \to w = e^{2\pi i \hat{z}}$ mapping the points $\hat{z}_i$ to

$$\hat{z}_i \to w_i = \exp\left[i\pi \frac{F(u_i, k)}{K(k)}\right]. \tag{9}$$

It turns out that with the following definition of elliptic polylogarithms

$$\mathrm{ELi}_{n;m}(x; y; q) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk},$$

$$\begin{aligned}
\mathrm{E}_{n;m}(x; y; q) = \frac{c_{n+m}}{i} &\left[\left(\tfrac{1}{2}\mathrm{Li}_n(x) + \mathrm{ELi}_{n;m}(x; y; q)\right)\right. \\
&\left. - s_{n+m}\left(\tfrac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}\left(x^{-1}; y^{-1}; q\right)\right)\right]
\end{aligned} \tag{10}$$

(with $c_n = 1$, $s_n = 1$ for n even and $c_n = i$ and $s_n = -1$ for n odd), the full result for the sunrise integral $S_{111}^{(0)}(2, t)$ in two space-time dimensions with arbitrary masses can be expressed as a sum of three elliptic dilogarithms

$$S_{111}^{(0)}(2, t) = \frac{\Psi_1(q)}{\pi} \sum_{i=1}^{3} \mathrm{E}_{2;0}(w_i; -1; -q) \tag{11}$$

with the numbers w_i as arguments [14].

4. The sunrise integral around four space-time dimensions

The sunrise integral around four space-time dimensions can be expressed in the basis [9]

$$\left\{\mu^2 S_{111}(D, t),\ \mu^2 \frac{\partial}{\partial m_i^2} S_{111}(D, t) = -S_{1+\delta_{i,1}\ 1+\delta_{i,2}\ 1+\delta_{i,3}}(D, t)\right\} \tag{12}$$

with the help of Tarasov's dimensional shift relations [17, 18]

$$S_{111}^{(0)}(4, t) = \tilde{L}_3^{(-1)}(2)\, S_{111}^{(1)}(2, t) + \tilde{L}_3^{(0)}(2)\, S_{111}^{(0)}(2, t) + \tilde{R}^{(0)}, \tag{13}$$

where $\tilde{L}_3^{(k)}(2) = C_0^{(k)} + \sum_{i=1}^{3} C_i^{(k)} m_i^2 \frac{\partial}{\partial m_i^2}$ and $\tilde{R}^{(0)}$ is a remainder depending on t and simpler mass terms. The ϵ^1-solution $S_{111}^{(1)}(2, t)$ involves apart from the functions of Eq. (10), one new function $\mathrm{E}_{n_1, n_2; m_1, m_2; 2o}(x_1, x_2; y_1, y_2; q)$ defined in [15]. This function can be viewed as a generalisation of multiple polylogarithms of depth greater than one. If we define the weight of $\mathrm{E}_{n;m}$ by $w = n+m$ and the weight of $\mathrm{E}_{n_1, n_2; m_1, m_2; 2o}$ by $w = n_1+n_2+m_1+m_2+2o$, it

is worth noting that the $\mathcal{O}(\epsilon^1)$-part is not of uniform weight. The $\mathcal{O}(\epsilon^1)$-part of the sunrise integral around two space-time dimensions contains terms of weight three and four.

To understand this issue, one can consider the simpler equal mass case for $S_{111}^{(1)}(2,t)$ where one is concerned with an inhomogeneous second-order differential equation for $S_{111}^{(1)}(2,t)$. With a suitable ansatz for $S_{111}^{(1)}(2,t)$ involving a part proportional to $S_{111}^{(0)}(2,t)$ and a remainder, one obtains a much simpler differential equation. Both parts of the ansatz contain $\log(-q)$-terms, which cancel out in the end, but as a remainder from integrating the $\log(-q)$-term from the second part when solving the differential equation, we obtain the weight four terms.

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