OPPENHEIMER AND BONDI LIMITS: THE CASE OF THE SINGULAR CONFORMAL CURVATURE*

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Spherically symmetric, gravitating configurations built of the regularly distributed gas in equilibrium still admit a singular Weyl curvature. In contrast to the Black Holes, the singularity located in the center of the gas cloud is naked — forms the repulsive point-like gravity source. Configurations of this kind can exceed both the Oppenheimer limit and the Bondi limit.

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1. Introduction: the role of conformal curvature

The Ricci curvature is coupled to matter via the Einstein equations. It is uniquely defined when the distribution of matter is specified. On the contrary, the Weyl tensor is a "free parameter" of the theory and can be chosen arbitrary — independently of the matter distribution. The freedom to arbitrary setting the conformal curvature is seldom disclosed in astrophysical investigations, where the symmetry of the energy-momentum tensor is usually replicated to metric tensor. This procedure does not result from the general relativity, neither from observations. Instead, it constitutes an additional constraint, and eventually results in the loss of generality.

Although the source of the Weyl tensor is not specified, this quantity has an important geometrical and physical meaning [1]. The Weyl tensor itself is the source of the geodesic deviation. Conformal curvature acts in the way similar to the tidal forces [1, 2], as well as it contributes to the ADM mass of spherical bodies [3].

2. Nakedly-singular gaseous spheres in equilibrium

In this paper, we discuss static, spherically symmetric configurations with the regularly distributed gas described by the hydrodynamic energy-momentum tensor. Following the classical Oppenheimer–Volkoff approach [4], we assume the same symmetry for both the energy-momentum tensor and the metric tensor

$$ds^2 = -A(r)c^2dt^2 + \left(1 - 2\frac{G}{c^2}\frac{M(r) + M_g}{r}\right)^{-1} dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right).$$
(1)

However, we relax the assumption of the metric tensor regularity [4, 5]. The problem differs from the classical TOV problem in non-vanishing constant of integration M_g which is explicitly present in the metrics (1). The Einstein equations $G_{\mu\nu} = \kappa T_{\mu\nu}$ reduce to non-autonomous dynamical system

$$M'(r) = 4\pi r^2 \rho(r),$$
(2)

$$p'(r) = -G\left(c^2\rho(r) + p(r)\right)\frac{M(r) + M_g + 4\pi r^3 p(r)c^{-2}}{r\left(r - 2G(M(r) + M_g)c^{-2}\right)}.$$
(3)

By computing the Carminati–McLenaghan invariants [6, 7]

$$\mathcal{R}_1 = 3^{1/3}\mathcal{R}_2{}^{2/3},$$
(4)

$$\mathcal{R}_3 = \frac{7\mathcal{R}_2{}^{4/3}}{4\ 3^{1/3}},$$
(5)

$$\mathcal{W}_1 = 6^{1/3}\mathcal{W}_2{}^{2/3},$$
(6)

$$\mathcal{M}_2 = \mathcal{M}_3 = \left(\tfrac{2}{3}\right)^{1/3} \mathcal{R}_2{}^{2/3} \mathcal{W}_2{}^{2/3} , \tag{7}$$

$$\mathcal{M}_4 = -6^{-2/3} \mathcal{R}_2 \mathcal{W}_2{}^{2/3} , \tag{8}$$

$$\mathcal{M}_5 = 3^{-2/3} \mathcal{R}_2{}^{2/3} \mathcal{W}_2 , \tag{9}$$

where

$$\mathcal{R}_2 = 3 \left(\frac{1}{4} \kappa \left(c^2 \rho + p \right) \right)^3 , \tag{10}$$

$$\mathcal{W}_2 = -6 \left(\frac{c^2 \kappa}{8 \pi r^3} \left(M(r) + M_g - \frac{4}{3} \pi r^3 \rho \right) \right)^3 , \tag{11}$$

we establish that M_g is responsible for the conformal curvature singularity (M_g enters $\mathcal{W}_1$ and $\mathcal{W}_2$ invariants), and does not affect the the regularity of the matter distribution (is absent in $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_2$). In order to avoid a collapse to Black Hole, the constant M_g must be negative. The singularity is naked [8–13] and the gaseous cloud encloses a point-like repulsive gravity source at the center.

3. The nonrelativistic neutron polytropic approximation

To build a model, we consider the Fermi gas in nonrelativistic regime with the equation of state approximated by the polytrope [14, 15]

$$\rho(p) = \frac{1}{c^2} \left(\frac{p}{k} \right)^{\frac{1}{\Gamma}} , \qquad \Gamma = 5/3 , \qquad k = \frac{\hbar^2}{15 \pi^2 m_N} \left(\frac{3\pi^2}{c^2 m_N} \right)^{\Gamma} . \tag{12}$$

We assume that maximal pressure $p_{\max} = p(r_{\max})$ does not exceed 10^{34} Pa, as for the regular neutron star. Python language library scipy is used with the vode integrator in order to numerically integrate the system (2)–(3) with the equation of state (12). We adopt the backward differentiation formulas BDF-method (http://www.netlib.org/ode/vode). The central repulsive singularity sweeps the gas out. The region of maximal density forms at some distance $r_{\max}$ from the center. A typical profile is shown in Fig. 1.

The integration was repeated for 59 different values of $r_{\max}$ from the range $r_{\max} \in (1 \text{ km}, 10^{2.5} \text{ km})$, and for 10 different $p_{\max} \in (5.3 \times 10^{33} \text{ Pa}, 4.6 \times 10^{34} \text{ Pa})$ which gives altogether the matrix of 590 profiles. For each profile, the ADM mass, the configuration radius r_s, the repulsive gravity counterpart M_g, and the gravitational surface redshift z were found. Details of the integration procedure are explained in [16]. In the set of profiles under investigation, the configuration of the maximal redshift has parameters:

$$M_{\text{ADM}} = 31 \, M_\odot , \qquad r_s = 92 \text{ km} , \qquad z = 11 . \tag{13}$$

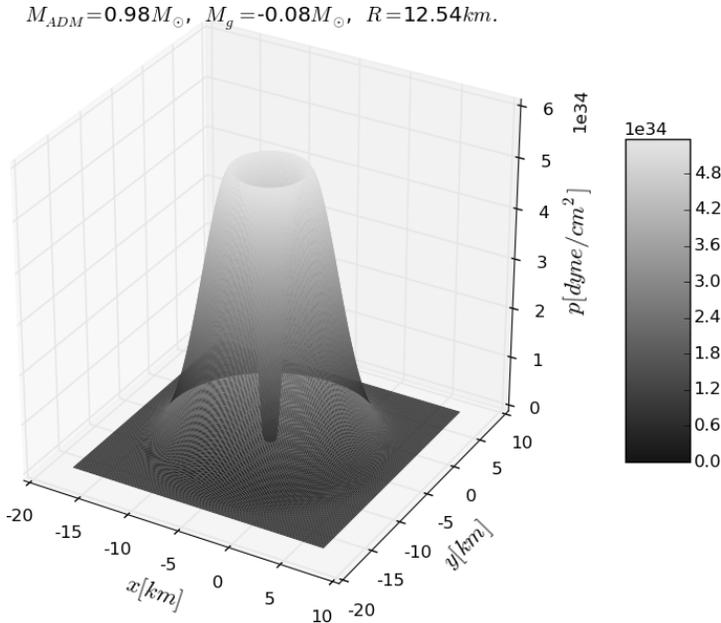M_{ADM}=0.98$M_\odot$, M_g=-0.08$M_\odot$, R=12.54km.

Fig. 1. The pressure profile in $2+1$ dimensional visualization of a typical nakedly-singular, spherical, polytropic cloud.

The maximal value for z (13) comes from the limits for polytrope approximation for the Fermi gas of neutrons and may change with the change of the equation of state. When the regularity conditions for metrics and the cosmic censorship are relaxed, the Oppenheimer limit [4] and the Bondi [17] limit are exceeded. The maximal gravitational redshift we obtained belongs to the range which is presently reserved for cosmology.

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