BUILDING THE COSMOLOGICAL MODELS VS. DIFFERENT MODELS OF SET THEORY*

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We use the 'local weakening of logic in a space-time' as a mathematical tool suitable also for building cosmological models. One obtains the extensions of the regular space-time solutions of Einstein equations towards solutions with certain space-time singularities. Such space-time models are also natural for addressing the renormalization questions of various quantum field theories. We discuss some examples.

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1. Building different models of set theory and cosmology

Nowadays, a common accepted formalized mathematical foundations of physics, hence also cosmology, lie in Zermelo–Fraenkel (ZF) set theory, possibly with the axiom of choice (AC) — ZFC. Strictly speaking, the main point in the foundations is that one considers a particular global model of ZFC. Such a principle has profound consequences for fundamental theories of physics: given any scale from Planck to cosmological one, a unified set-theoretical apparatus is at hand. We claim that *relaxing* this assumption and admitting some different (possibly weaker) models sheds light on issues connected with singular phenomena in cosmology and QFT. The principle of the model-theoretic variation of mathematical tools in the context of physics appeared already in the past [1, 2]. Here, we allow for the *localization* in

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space-time of the special category called *Basel topos* $\mathcal{B}$, and describe certain results of physical theories on such space-times. $\mathcal{B}$ is a topos of sheaves on a site of C^∞-rings, as constructed in [3]. The main attributes of this approach can be described as follows. First, the internal logic in $\mathcal{B}$ (as in any topos) is intuitionistic, hence neither AC nor excluded middle are allowed. Second, real numbers internal in $\mathcal{B}$ are equipped with nilpotent infinitesimals which allows for synthetic differential geometry, and for the invertible infinitesimals. The latter together with their infinite reciprocals are known from the Robinson's non-standard analysis (NSA). It could be tempting to consider non-standard natural numbers present in $\mathcal{B}$ as a straightforward device to handle divergent phenomena, simply by interpreting limits as such numbers (there were some attempts to apply this line of reasoning *e.g.* in [4]). Instead, we turn here to distribution theory which acquires very special features in $\mathcal{B}$. Namely, one obtains [3].

Theorem. *For any distribution μ on R^n in $\mathcal{B}$, there is a polynomial (non-standard) function $p : R^n \to R$ such that*

$$\mu(f) = \int pf \,.$$

Hence, highly singular distributions, like δ distribution, become smooth functions in $\mathcal{B}$. *This fact alone indicates that $\mathcal{B}$ modifies somehow smoothness structure.* Also, a polynomial representation leads to a globally defined multiplication on the space of distributions, which is of crucial importance in further analysis.

2. Applications to General Relativity and cosmology

We discuss two examples within GR which deal with the Schwarzschild and Reissner–Nordström metrics correspondingly. This kind of results was originally derived using the Colombeau algebra $\mathcal{G}$ of nonlinear generalized functions [5] — yet another rigorous approach to multiplication of distributions. However, it has been shown [6] that the results of Colombeau theory can be established in NSA, hence, *as long as the reasoning is constructive*, also in $\mathcal{B}$.

1. Let us look at the Schwarzschild metric [7]

$$ds^2 = h(r)\, dt^2 - h(r)^{-1}\, dr^2 + r^2 d\Omega\,, \qquad h(r) = -1 + \frac{2m}{r}\,.$$

This metric carries the origin singularity $r = 0$ and coordinate singularity $r = 2m$. In order to obtain distributional results *e.g.* for curvature scalar R, one goes to the Kerr–Schild form of the metric

$$ds^2 = -d\bar{t}^2 + dr^2 + r^2 d\Omega^2 + \frac{2m}{r}\left(d\bar{t}^2 - dr\right)^2 \,,\ \text{where } \bar{t} = t + 2m\log|2m - r|$$

and embeds $\frac{1}{r}$ into the Colombeau algebra $\mathcal{G}$ by convolution with a mollifier $\left(\frac{1}{r}\right)_\epsilon := \frac{1}{r} * \rho_\epsilon$. This leads to the distributional limit of curvature scalar and Einstein tensor

$$(R_\epsilon)_\epsilon \approx 8\pi m\delta \,, \qquad (G^a_{b\ \epsilon})_\epsilon \approx -8\pi m \delta^a_0 \delta^0_b \delta \,.$$

Let us look at this result from the point of view of Basel topos $\mathcal{B}$. Since in $\mathcal{B}$ all distributions are actually smooth functions, the conclusion is that curvature is also smooth (non-singular) quantity.

We state a remark: geodesic incompleteness is closely connected to existence of singularities on a manifold. Since all singularities are absorbed internally in $\mathcal{B}$, does it mean that geodesic completeness is regained? In principle, one could examine *e.g.* the conditions of Hopf–Rinow theorem. Again, we stress that it is inevitable to suspect a change in smoothness structure behind these results.

2. Since [8], there has been an apparent trouble considering ultrarelativistic limit of Reissner–Nordström metric describing the gravitational field of a charged, non-rotating point mass m [9]

$$ds^2 = \left(1 - \frac{2m}{r\left(\bar{r}\right)} + \frac{e^2}{r^2\left(\bar{r}\right)}\right) dt^2 - \left(1 + \frac{m}{\bar{r}} + \frac{m^2 - e^2}{4\bar{r}^2}\right)^2 \left(d\,\bar{r}^2 + d\Omega^2\right)$$

with $r\left(\bar{r}\right) = \bar{r}\left(1 + \frac{m}{\bar{r}} + \frac{m^2 - e^2}{4\bar{r}^2}\right)$. Namely, despite vanishing of electromagnetic field tensor, the stress-energy tensor is δ-like. Again, classically it can not be resolved without referring to the multiplication of distributions.

After a boost (ultrarelativistic) and embedding the components of the field tensor into the Colombeau algebra $\mathcal{G}$, the following relation follows:

$$(F_{ik})_\epsilon \sim \sqrt{\delta} \approx 0$$

which means that F_{ik} is infinitesimally represented by the 0-distribution. However, it can be shown further that we have a representation for the stress-energy tensor in $\mathcal{G}$ such that

$$(T_{00})_\epsilon \approx \frac{3\gamma e^2}{16\left(y^2 + z^2\right)^3} \delta\left(u\right) \,,$$

where u is the null coordinate $u = x' - t'$.

It is perfectly correct that square of infinitesimal distribution gives δ-like distribution, hence the apparent paradox is resolved in this framework. Again, we stress that this result is expected to hold also in $\mathcal{B}$; then, it will be a natural consequence of local modification of space-time by $\mathcal{B}$.

3. Renormalization and Basel topos $\mathcal{B}$

Let us concentrate on UV renormalization from the *causal perturbation theory* (CPT) point of view [10]. It has been proved to be equivalent to the approach via Hopf algebras [11], which already had been confirmed as equivalent to the classical BPHZ forest formula [12].

For the purpose of further analysis, let us remind that quantum fields are operator-valued distributions in general [13]. Thus, the origin of UV divergences lies in the fact that the theory of distributions is purely linear [14]. Namely, a typical Feynman loop amplitude is an integral over powers of propagators, being themselves distributional solutions of EOM. Such integrals are only formal and ill-defined in general.

Let us sketch the CPT approach in the case of massless ϕ^4 theory in 4D [15]: $\Box\phi = 0$, $\Box G = \delta$, hence $G(x) \sim \frac{1}{x^2}$ and $G^2(x) \sim \frac{1}{x^4}$. As an example, we have the following one-loop fish-diagram and the corresponding integral $\int d^4x G^2(x)$ is logarithmic divergent, thus we restrict G^2 to the

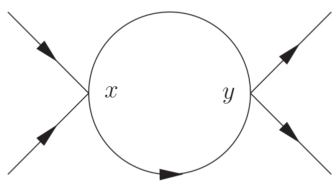

Fig. 1. $\iint d^4x d^4y \phi_0^2(x) G^2(x-y) \phi_0^2(y)$.

subspace of test functions vanishing at 0 to keep the integral finite. Then, by the Hahn–Banach theorem, there exists (non-unique in general) an extension of the restricted distribution

$$G^2_{\mathrm{ren}}(\phi) = \int d^4x \frac{\phi(x) - \phi(0)}{x^4} + c\delta(\phi) \,.$$

The extension is accompanied by additional degrees of freedom — distributions supported 0. These are the counterterms known *e.g.* from BPHZ prescription.

Let us observe that Basel topos $\mathcal{B}$ gives a rather exceptional insight into the subject. Since all distributions, including θ and δ, are regular and smooth, the product G^2 is globally defined on internal distribution space. Now, we have two compelling arguments about the nature of this product. On the one hand, by [3] the global sections functor $\Gamma : \mathcal{B} \to \mathrm{Set}$ induces a bijection between internal and external distributions. Hence, there seems to be no much place for a non-uniqueness of a product giving the renormalization prescriptions. On the other hand, it is known that in the NSA (also in the framework of Colombeau algebras), there is inherent non-uniqueness in the definition of the product [16]: NSA representation does not reduce a singular distribution μ to a function p, but to a class $P = \{p_1, ...\}$. Hence, while in the latter, one would expect that degrees of freedom coming from the class P should overlap with counterterms, the Basel topos approach would realize them perhaps by other means.

4. Discussion

We reviewed some possible applications of local modification of logic in a space-time M, by virtue of using Basel topos $\mathcal{B}$. The applications cover issues connected with black holes and renormalization. The strongest point of this formalism is the nonlinear theory of distributions, internally natural in $\mathcal{B}$. This allows one to derive results requiring multiplication of distributions, inevitable in GR and QFT. There are also strong indications (change in smoothness structure) that locating $\mathcal{B}$ on M can produce exotic smooth structure on $\mathbb{R}^4$. Then, together with [17], one concludes that $\mathcal{B}$ could find an application also in the studies of inflation and early stages of the Universe.

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