THE STRUCTURE OF THE REAL LINE IN QUANTUM MECHANICS AND COSMOLOGY*

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(Received October 19, 2015)

We discuss the recently proposed model, where the spacetime in large scales is parametrized by the usual real line $\mathbb{R}$, while at small (quantum mechanical) scales, the space is parametrized by the real numbers $\mathbb{R}_M$ from some formal model M of Zermelo–Fraenkel set theory. We argue that the set-theoretic forcing is an important ingredient of the shift from micro- to macroscale. The set $\mathbb{R}_M$, describing the space at the Planck era, is merely a *meager* subset of $\mathbb{R}$. It is Lebesgue non-measurable and all its measurable subsets have Lebesgue measure 0. According to this, the contributions to the cosmological constant from the zero-point energies of quantum fields vanish. Moreover, the emerged irregularities in the real line can be considered as the source of the primordial quantum fluctuations.

DOI:10.5506/APhysPolB.46.2375
PACS numbers: 02.10.–v, 03.65.–w, 98.80.–k

1. Introduction

Nowadays, the primordial inflation is a widely accepted scenario for the cosmological evolution of the early Universe (*e.g.* [1]). The scenario can be successfully described by a class of simple models based on general relativity (GR), with the addition of the single scalar inflaton field ϕ with the potential $V(\phi)$. Even though these models are very effective, we still do not know the reason why the inflation emerged. The aim of this paper is to show that turning to the fundamental mathematical structures, like the real line, improves our understanding of the origins of cosmic inflation and sheds some light on the cosmological constant (CC) problem.

The structure of the set of real numbers can be described algebraically as the linearly ordered complete field or topologically (and smoothly) as a 1-dimensional Euclidean manifold, which is the simplest manifold of dimension

* Presented by P. Klimasara at the XXXIX International Conference of Theoretical Physics “Matter to the Deepest”, Ustroń, Poland, September 13–18, 2015.

one. Higher-dimensional manifolds, like $\mathbb{R}^n$, are directly modeled by $\mathbb{R}$. It is a truism to say that physicists refer to differentiable manifolds, hence to real numbers, when developing various models of reality. Besides, $\mathbb{R}$ is the basic mathematical object containing the formal representations of the results of classical and quantum measurements. Thus, it is the basic field to which (almost) all physical theories refer to. Let us just mention that space and time coordinates are described by real numbers. The impact of the nontrivial structure of the real field on physics was discussed mostly in the context of quantum mechanics (QM) and quantum gravity [2, 3]. Here, we study the mathematical model for the evolution of the Universe with varying set of reals. The set of real numbers is a formal object in models of ZFC — the Zermelo–Fraenkel set theory (with the axiom of choice). Since ZFC is the first order theory (its axioms are formulated in the first order language), it is not categorical, *i.e.* there exist infinitely many non-isomorphic models of ZFC. Every model M of ZFC determines the object of real numbers $\mathbb{R}_M$, which is the set of all internal in M subsets of $\mathbb{N}_M \simeq \mathbb{N}$. The structure of $\mathbb{R}_M$, the model-dependent real line is, in general, very rich and complicated with the properties relative to a model [4]. The general tool for exploring the real line is the set-theoretic forcing, the procedure invented by Cohen in 1963 when proving the independence of the continuum hypothesis of the axioms of ZFC (and ZF) [5].

In this paper, we consider the model of the evolution of the Universe in which the change of a model of ZFC took place (during the Planck era). Thus, the set of real numbers is a varying and model-dependent object rather than absolute. We apply the forcing on the measure algebra as a tool to formally grasp the changes of the set of reals. From the point of view of cosmology, forcing can be seen as a process that underlies the cosmic inflation and the inflation potential can be related with the change of the inherent density of reals. Moreover, the gravitational contributions of the zero-point energies of quantum fields described in some model M vanish in the, extended by forcing, generic model $M[G]$. It is shown in Sec. 3. We begin with some results about forcing and QM to justify the use of forcing in cosmology.

2. Forcing in QM

This section is the summary of our previous article regarding the forcing emerging from QM. An interested reader is referred to [6] for a more detailed treatment. A forcing can be seen as the deriving property of some Boolean algebra[1]. By $\mathfrak{B}$, we denote some Boolean algebra of projections chosen from the lattice of projections $\mathbb{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. The spectral theorem,

[1] The algebra should be complete and atomless to support a nontrivial forcing.

in general, gives the correspondence between the algebra of self-adjoint (s-a) operators *which are in the Boolean algebra* $\mathfrak{B}$ [6, 7] and the measure algebra defined on the Borel subsets of $X = \mathbb{R}^3$. Actually, there exists the isomorphism between the algebra $\mathfrak{B}$ generating the family of s-a operators $\{A_\alpha\}$ and the measure algebra of the space[2] (X, μ). Such measure algebra is complete and, in general, non-atomic. Thus, there exists a nontrivial *measure* forcing adding random reals to a model M and resulting in the extended model $M[G]$. Roughly speaking, a forcing allows us to formally grasp the quantumness of the observables in the Boolean contexts [6].

The direct consequence of Wesep's analysis of the LHV program in [8] is that the results of the continuous measurements of the position observable, in the semiclassical Boolean contexts, refer to the random forcing. Thus, as shown in [6], the (random) measure forcing has to be present when approaching the classical geometric limit emerging from the quantum scales. Let us analyze some consequences of this fact.

3. Forcing in cosmology: the CC problem

Suppose that, indeed, the shift from the Planck era to the GR-based era of the evolution of the Universe refers to different models of ZFC. Denoting the model in the Planck era by M, we use $M[G]$ for the shifted one. Thus, there corresponds the shift $\mathbb{R}_M \to \mathbb{R}_{M[G]}$ of the real numbers parameterizing the physical content of the epochs. The description of nowadays large scale structure is based on the full real line $\mathbb{R}$, since the diffeomorphism invariance of GR enforces it.

Now, given a particle of mass m, the zero-point contribution to the energy density is formally calculated as

$$\frac{E}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{\boldsymbol{k}^2 + m^2}}{2}\,. \tag{1}$$

This expression is not Lorentz-invariant and is quartically divergent. Imposing any ultraviolet cut-off Λ_{UV} on the momentum integration, one can obtain the finite value. However, the cut-offs induce the loss of validity of a theory for very big momenta. Instead, working in our forcing-based model, let us evaluate this integral without using any cut-offs.

To this end, let us observe that in the case of random forcing, the set $\mathbb{R}_M$ (of real numbers in M) is a meager subset of the full real line: $\mathbb{R}_M \subset \mathbb{R}$ [4]. Moreover, $\mathbb{R}_M$ is Lebesgue non-measurable and has the lower measure zero

[2] The measure algebra of (X, μ) is the quotient of the Borel measurable subsets of X algebra modulo the ideal of null sets (subsets of X that have measure 0).

and the full outer measure: $\mu_*(\mathbb{R}_M) = 0$, $\mu^*(\mathbb{R}_M) = 1$. Let $\mu(S)$ denote the Lebesgue measure of a set S. Then, the following elementary property holds true:

Lemma 1. *Every Lebesgue measurable subset S of the non-measurable set A with the inner measure zero ($\mu_*(A) = 0$) fulfills $\mu(S) = 0$.*

Proof. Since S is measurable, there is $\mu(S) = \mu_*(S)$. Let us assume that $\mu(S) > 0$. Then, we have $\mu_*(S) > 0$. Since $A = S \cup (A \setminus S)$ and the inner measure is additive, there is $\mu(A) > 0$. But A is non-measurable — a contradiction. Hence $\mu(S) = 0$. □

Note also that the Lebesgue integral of an integrable function f over the set of measure zero vanishes even if the set is uncountably infinite (which is the case here)[3].

In the considered cosmological scenario, the Planck era is described by the tools of some model M of ZFC, hence the zero-modes of quantum fields contributing to the vacuum energy too. In particular, physical quantities are described with respect to the real numbers from $\mathbb{R}_M$. Recall that the change of the model $M \stackrel{\text{forcing}}{\longrightarrow} M[G]$ refers to the random forcing, according to Sec. 2. Then, $\mathbb{R}_M$ has lower measure 0 as a meager subset of $\mathbb{R}_{M[G]}$ (and hence of $\mathbb{R}$) and the Lemma 1 can be applied here.

The description of spacetime at the present epoch requires, according to GR, the differentiable manifold built on the full real line $\mathbb{R}$. This is why we evaluate the integral (1) from the nowadays perspective of the GR-scale-based observer. The zero-point energies lie in the model M, so the integration is taken over the non-measurable subset[4] $\mathbb{R}^3_M \subset \mathbb{R}^3$

$$\frac{E}{V} = \int\limits_{\mathbb{R}^3_M} \frac{d^3k}{(2\pi)^3} \frac{\sqrt{\boldsymbol{k}^2 + m^2}}{2} \,. \tag{2}$$

Although such Lebesgue integral does not exist in general, we can evaluate the contributions given by the integration over all measurable subsets

[3] The Lebesgue integral of a simple function $s = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$, where χ_{A_j} is the characteristic function of A_j, reads

$$\int\limits_E s d\mu = \sum_{j=1}^{n} \alpha_j \mu\left(E \cap A_j\right) \,.$$

If $\mu(E) = 0$, then $\mu(E \cap A_j) = 0$ for all j, hence $\int_E s d\mu = 0$. The Lebesgue integral of any non-negative function f is the supremum of integrals of all simple functions s such that $0 \le s \le f$. Since all these integrals are 0, the supremum is 0 too.

[4] In the model M, the real numbers $\mathbb{R}_M$ parametrize both x and k coordinates.

of $\mathbb{R}^3_M$. But, by Lemma 1, all such contributions vanish and the value of the integral (2) calculated this way is zero. Thus, the model of the evolution of the Universe based on set theory has mechanism which allows us to neglect the contributions from the zero-modes of quantum fields. The measure-theoretic argumentation is universal and represents the solution of this part of the CC problem.

Moreover, one can assign the forcing between the models of ZFC to the inflationary epoch so that the real line $\mathbb{R}_M$ becomes rarefied in $\mathbb{R}_{M[G]}$ and $\mathbb{R}$. However, such quantitative approach to the cosmic inflation requires the use of the non-standard (exotic) geometries on $S^3 \times \mathbb{R}$ or $\mathbb{R}^4$ [9].

4. Discussion

We proposed the model in which, during the evolution of the early Universe, the change of a model of set theory took place. Such change is encoded by the forcing on the measure algebra of $\mathbb{R}^3$ and leads to the change of the real line. It is shown that the vanishing of the contributions of the zero-modes of quantum fields to the cosmological constant is the consequence of such forcing-based evolution. This is only a partial success of the proposed model. Namely, the following question remains to be answered (which is the part of the CC problem): why the value of density ρ_Λ is non-zero and so small? As shown in [9], the answer can be based on the 4-dimensional nontrivial *curved* geometries emerging during the shift $M \to M[G]$ at the primordial era. Thus, the curvature is the source for the correct energy density and the shape of the inflation potential. The forcing origins of such geometries will be discussed elsewhere.

REFERENCES

[1] A.D. Linde, *Phys. Lett. B* **108**, 389 (1982).
[2] A. Döring, C.J. Isham, *J. Math. Phys.* **49**, 053515 (2008).
[3] C. Heunen, N.P. Landsman, B. Spitters, *Bohrification* in: *Deep Beauty: Understanding the Quantum World Through Mathematical Innovation* (Ed. H. Halvorson), Cambridge University Press, New York 2011, pp. 271–313.
[4] T. Bartoszyński, H. Judah, *Set Theory: On the Structure of the Real Line*, A.K. Peters, Wellesley, Massachusetts, USA, 1995.
[5] P.J. Cohen, *Proc. Nat. Acad. Sci. USA* **50**, 1143 (1963).
[6] P. Klimasara, J. Król, *Acta Phys. Pol. B* **46**, 1309 (2015).
[7] G. Takeuti, *Two Applications of Logic to Mathematics*, Kano Memorial Lecture 3, Math. Soc. Jpn., Princeton, USA, 1978.
[8] R.A. Van Wesep, *Ann. Phys.* **321**, 2453 (2006).
[9] T. Asselmeyer-Maluga, J. Król, *Adv. High Energy Phys.* **2014**, 867460 (2014).