THE FLAVOUR PROBLEM AND THE FAMILY SYMMETRY BEYOND THE STANDARD MODEL*

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In the framework of a two Higgs doublet model, we try to explain lepton masses and mixing matrix elements assuming that neutrinos are Dirac particles. Discrete family symmetry groups, which are subgroups of U(3) up to the order of 1025 are considered. Like in the Standard Model with one Higgs doublet, we found that discrete family symmetries do not give satisfactory answer to these basic questions in the flavour problem.

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1. Introduction

The Standard Model is so far the best theory describing particles and their interactions. However, it does not explain many issues. Only in the lepton sector we have 10 arbitrary parameters (6 masses + 3 mixing angles + 1 CP phase) in the case of Dirac neutrinos and even 12 parameters (6 masses + 3 mixing angles + 3 CP phases) if we assume neutrinos to be of Majorana nature. Moreover, the existence of 3 families of quarks and leptons, the nature of neutrinos, the mechanism of neutrino mass generation and values of CP-violating phases are still unresolved mysteries within the framework of this theory. The enumerated problems are the part of the so-called “flavour problem” [1].

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Until 2012, it was a common consensus that we were on the right path to find a solution: the TriBiMaximal (TBM) mixing [2] fully explained parameters determining the PMNS mixing matrix. However, thanks to more precise measurements [3–6], it has been discovered that the reactor angle cannot be assumed to be zero. This fact resulted in the need to find another pattern describing the mixing. After 2012, many ideas aimed at solving this problem. Most of them were based on a simple extension of the Standard Model by supplementing it with a discrete symmetry group [7–9]. It was noticed that in order to get a non-trivial mixing, the family symmetry must be broken into two residual symmetries generating separately the forms of mass matrices in the charged lepton and the neutrino sector. In our opinion, this idea is not so convincing: we impose full symmetry, which subsequently must be broken into its subgroups. Anyway, to our knowledge, that reasoning did not lead to any reasonable results: the form of the mixing matrix was not clarified, any prediction for the masses has not been obtained.

All these aspects motivated us to give a different approach a try. Our idea consists in extending not only symmetry of the Standard Model, but also the scalar sector. This concept has been already briefly described in [10].

2. The symmetry group

As it has been already written in the introduction, our model focuses on examination of the consequences of adding to $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_\mathrm{L}\times\mathrm{U}(1)_Y$ gauge symmetry some extra group $\mathrm{G_F}$ with some modification of the Higgs part of the Lagrangian. In our approach, we assume that $\mathrm{G_F}$ (flavour symmetry) is a finite and non-Abelian group. The reason for this choice is quite obvious: TBM pattern was based on A_4 group. The long-lasting success of this model gave us some indications to search for solution in this "area".

Apart from that, we expect from $\mathrm{G_F}$ to be the subgroup of U(3). Another condition for the symmetry comes from the Yukawa Lagrangian

$$\mathcal{L}_\mathrm{Y} = -\sum_{\alpha,\beta=e,\mu,\tau}\sum_{j=1,2}\left[\left(h_j^l\right)_{\alpha\beta}\bar{L}_{\alpha\mathrm{L}}\phi_j l_{\beta\mathrm{R}} + (h_j^\nu)_{\alpha\beta}\bar{L}_{\alpha\mathrm{L}}\tilde{\phi}_j\nu_{\beta\mathrm{R}}\right] + \mathrm{H.c.}\,, \quad (1)$$

where h^l, h^ν are Yukawa couplings, L_L denotes left-handed lepton doublet, $\nu_\mathrm{R}, l_\mathrm{R}$ stand for right-handed singlets of $\mathrm{SU}(2)_\mathrm{L}\times\mathrm{U}(1)_Y$ (of neutrino and charged lepton field respectively). Note that instead of one Higgs field we put here a sum over 2 doublets ($\tilde{\phi} = i\sigma_2\phi^*$). This is the extension of the scalar sector, which we have mentioned before.

From the picture presented above, the last requirement for the group naturally arises. We want the following fields' transformation laws to be valid

$$\begin{aligned} L'_{\alpha\mathrm{L}} &= \left(A^{\mathrm{L}}\right)_{\alpha,\chi} L_{\chi\mathrm{L}}\,, \qquad l'_{\beta\mathrm{R}} = \left(A^{l}\right)_{\beta,\gamma} l_{\gamma\mathrm{R}}\,, \\ \nu'_{\beta\mathrm{R}} &= \left(A^{\nu}\right)_{\beta,\delta} \nu_{\delta\mathrm{R}}\,, \qquad \phi'_i = \left(A^{\phi}\right)_{i,k} \phi_k\,, \end{aligned} \tag{2}$$

where A^{L}, A^{l},A^{ν},A^{ϕ} are irreducible representations of the flavour group $\mathrm{G_F}$. Thus, it becomes quite clear, that 2- and 3-dimensional irreducible representations of $\mathrm{G_F}$ are needed. A^{L}, A^{l} and A^{ν} should be 3-dimensional matrices, while A_{ϕ} must be a 2-dimensional matrix as there are two Higgs doublets in our model.

At this point, attributes of the wanted symmetry can be easily established. Summarising, the desired group should be finite and non-Abelian, it should be a subgroup of U(3) and it should have 2- and 3-dimensional irreducible representations. At this stage, to simplify our search, we take into account only finite groups, which cannot be written as a direct products with cyclic groups. In order to find the groups fulfilling all these requirements, it is convenient to make use of the program GAP for discrete algebra computation [11]. Besides, since we are interested in finding the groups of small orders, the application of the Small Group Library [12, 13] together with the REPSN [14] package, which provides representations, is necessary. Among all groups implemented in this library, up to the order of 1025, we have found that 62 groups fulfil our search criteria. Among them 17 are subgroups of SU(3).

All considered subgroups of U(3) can be identified with the classification of finite subgroups by Blichfeldt, Miller and Dickson [15] and Ludl [16]. The results are presented in Table I.

TABLE I

Classification of considered subgroups of U(3).

$[[i,j]]$	The group description	The group classification	SU(3)?
[[24,12]]	S_4	$\Delta(24)=\Delta(6\times 2^2)$	✓
[[48,30]]	$A_4 \rtimes C_4$	$S_4(2)$	
[[54,8]]	$((C_3\times C_3)\rtimes C_3)\rtimes C_2$	$\Delta(54)=\Delta(6\times 3^2)$	✓
[[96,64]]	$((C_4\times C_4)\rtimes C_3)\rtimes C_2$	$\Delta(96)=\Delta(6\times 4^2)$	✓
[[96,65]]	$A_4 \rtimes C_8$	$S_4(3)$	
[[108,11]]	$((C_3\times C_3)\rtimes C_3)\rtimes C_4$	$\Delta(6\times 3^2,2)$	
[[150,5]]	$((C_5\times C_5)\rtimes C_3)\rtimes C_2$	$\Delta(150)=\Delta(6\times 5^2)$	✓
[[162,10]]	$(((C_3\times C_3)\rtimes C_3\times C_3)\rtimes C_3)\rtimes C_2$		
[[162,12]]	$((C_9\times C_3)\rtimes C_3)\rtimes C_2$		
[[162,14]]	$((C_9\times C_3)\rtimes C_3)\rtimes C_2$	$D(9,1,1;2,1,1)$	✓
[[162,44]]	$((C_9\times C_3)\rtimes C_3)\rtimes C_2$	$\Delta'(6\times 3^2,2,1)$	
[[192,182]]	$((C_4\times C_4)\rtimes C_3)\rtimes C_4$	$\Delta(6\times 4^2,2)$	

Continuation of TABLE I.

$[[i,j]]$	The group description	The group classification	SU(3)?
[[192,186]]	$A_4 \rtimes C_{16}$	$S_4(4)$	
[[216,17]]	$((C_3 \times C_3) \rtimes C_3) \rtimes C_8$	$\Delta(6 \times 3^2, 4)$	
[[216,88]]	$((C_3 \times C_3) \rtimes C_3) \rtimes Q_8$	$\Sigma(72\phi)$	✓
[[216,95]]	$((C_6 \times C_6) \rtimes C_3) \rtimes C_2$	$\Delta(216) = \Delta(6 \times 6^2)$	✓
[[294,7]]	$((C_7 \times C_7) \rtimes C_3) \rtimes C_2$	$\Delta(294) = \Delta(6 \times 7^2)$	✓
[[300,13]]	$((C_5 \times C_5) \rtimes C_3) \rtimes C_4$	$\Delta(6 \times 5^2, 2)$	
[[324,13]]	$(((C_3 \times C_3) \rtimes C_3) \rtimes C_4) \rtimes C_2$		
[[324,15]]	$((C_9 \times C_3) \rtimes C_3) \rtimes C_4$		
[[324,17]]	$(((C_3 \times C_3) \rtimes C_3) \rtimes C_4) \rtimes C_2$		
[[324,102]]	$((C_9 \times C_3) \rtimes C_3) \rtimes C_4$	$\Delta'(6 \times 3^2, 2, 2)$	
[[384,568]]	$((C_8 \times C_8) \rtimes C_3) \rtimes C_2$	$\Delta(384) = \Delta(6 \times 8^2)$	✓
[[384,571]]	$((C_4 \times C_4) \rtimes C_3) \rtimes C_8$	$\Delta(6 \times 4^2, 3)$	
[[384,581]]	$A_4 \rtimes C_{32}$	$S_4(5)$	
[[432,33]]	$((C_3 \times C_3) \rtimes C_3) \rtimes C_{16}$	$\Delta(6 \times 3^2, 4)$	
[[432,239]]	$(((C_3 \times C_3) \rtimes C_3) \rtimes C_4) \rtimes C_4$		
[[432,260]]	$((C_6 \times C_6) \rtimes C_3) \rtimes C_4$	$\Delta(6 \times 6^2, 2)$	
[[486,26]]	$((C_{27} \times C_3) \rtimes C_3) \rtimes C_2$		
[[486,28]]	$((C_{27} \times C_3) \rtimes C_3) \rtimes C_2$		
[[486,61]]	$((C_9 \times C_9) \rtimes C_3) \rtimes C_2$	$\Delta(486) = \Delta(6 \times 9^2)$	✓
[[486,125]]	$((C_9 \times C_3) \times C_3) \rtimes C_3) \rtimes C_2$		
[[486,164]]	$((C_{27} \times C_3) \rtimes C_3) \rtimes C_2$	$\Delta'(6 \times 3^2, 3, 1)$	
[[588,16]]	$((C_7 \times C_7) \rtimes C_3) \rtimes C_4$	$\Delta(6 \times 7^2, 2)$	
[[600,45]]	$((C_5 \times C_5) \rtimes C_3) \rtimes C_8$	$\Delta(6 \times 5^2, 4)$	
[600,179]]	$((C_{10} \times C_{10}) \rtimes C_3) \rtimes C_2$	$\Delta(600) = \Delta(6 \times 10^2)$	✓
[[648,19]]	$(((C_3 \times C_3) \rtimes C_3) \times C_8) \rtimes C_3$		
[[648,21]]	$((C_9 \times C_3) \rtimes C_3) \rtimes C_8$		
[[648,23]]	$(((C_3 \times C_3) \rtimes C_3) \times C_8) \rtimes C_3$		
[[648,244]]	$((C_9 \times C_3) \rtimes C_3) \rtimes C_8$	$\Delta'(6 \times 3^2, 2, 3)$	
[[648,259]]	$((C_{18} \times C_6) \rtimes C_3) \rtimes C_2$	$D(3,1,2;9,3,2)$	✓
[[648,260]]	$((C_{18} \times C_6) \rtimes C_3) \rtimes C_2$		
[[648,266]]	$((C_6 \times C_6 \times C_3) \rtimes C_3) \rtimes C_2$		
[[648,531]]	$C_3.(((C_3 \times C_3) \rtimes Q_3) \rtimes C_3)$		
[[648,532]]	$(((C_3 \times C_3) \rtimes C_3) \times Q_8) \rtimes C_3$	$\Sigma(216\phi)$	✓
[[648,533]]	$(((C_3 \times C_3) \rtimes C_3) \times Q_8) \rtimes C_3$		
[[648,551]]	$((C_9 \times C_3) \rtimes C_3) \rtimes Q_8$		
[[648,563]]	$((C_{18} \times C_6) \rtimes C_3) \rtimes C_2$		
[[726,5]]	$((C_{11} \times C_{11}) \rtimes C_3) \rtimes C_2$	$\Delta(726) = \Delta(6 \times 11^2)$	✓
[[768,1085333]]	$((C_4 \times C_4) \rtimes C_3) \rtimes C_{16}$	$\Delta(6 \times 4^2, 8)$	
[[768,1085335]]	$((C_8 \times C_8) \rtimes C_3) \rtimes C_4$	$\Delta(6 \times 8^2, 2)$	
[[768,1085351]]	$A_4 \rtimes C_{64}$	$S_4(6)$	
[[864,69]]	$((C_3 \times C_3) \rtimes C_3) \rtimes C_{32}$	$\Delta(6 \times 3^2, 16)$	
[[864,675]]	$(((C_3 \times C_3) \rtimes C_3) \rtimes C_4) \rtimes C_8$		
[[864,701]]	$((C_{12} \times C_{12}) \rtimes C_3) \rtimes C_2$	$\Delta(864) = \Delta(6 \times 12^2)$	✓
[[864,703]]	$((C_6 \times C_6) \rtimes C_3) \rtimes C_8$	$\Delta(6 \times 6^2, 4)$	
[[972,29]]	$((C_{27} \times C_3) \rtimes C_3) \rtimes C_4$		
[[972,31]]	$((C_{27} \times C_3) \rtimes C_3) \rtimes C_4$		
[[972,64]]	$((C_9 \times C_9) \rtimes C_3) \rtimes C_4$	$\Delta(6 \times 9^2, 2)$	
[[972,309]]	$(((C_9 \times C_3) \rtimes C_3) \rtimes C_4) \rtimes C_3$		
[[972,348]]	$((C_{27} \times C_3) \rtimes C_3) \rtimes C_4$	$\Delta'(6 \times 3^2, 3, 2)$	
[[1014,7]]	$((C_{13} \times C_{13}) \rtimes C_3) \rtimes C_2$	$\Delta(1014) = \Delta(6 \times 13^2)$	✓

3. The invariance equation and its interpretation

Here, we introduce the main concept, which is the requirement imposed on the Yukawa Lagrangian (Eq. (1)) to be invariant under transformations of Eq. (2). In other words, we demand the following relations to be fulfilled:

$$\begin{aligned}
\sum_{i=1}^{2}\left(A^{\phi}\right)_{ik}\left(A^{\mathrm{L}\dagger}\right)_{\alpha\gamma}\left(h_i^l\right)_{\gamma\delta}\left(A^l\right)_{\delta\beta} &= \left[h_k^l\right]_{\alpha\beta}\,,\\
\sum_{i=1}^{2}\left(A^{\phi}\right)^{*}_{ik}\left(A^{\mathrm{L}\dagger}\right)_{\alpha\gamma}\left(h_i^\nu\right)_{\gamma\delta}\left(A^l\right)_{\delta\beta} &= \left[h_k^\nu\right]_{\alpha\beta}\,.
\end{aligned}\tag{3}$$

We can easily simplify these relations by swapping the indices:

$$\begin{aligned}
\sum_{i=1}^{2}(A^{\phi})^T_{ki}\left(A^{\mathrm{L}\dagger}\right)_{\alpha\gamma}\left(A^l\right)^T_{\beta\delta}\left(h_i^l\right)_{\gamma\delta} &= \left[h_k^l\right]_{\alpha\beta}\,,\\
\sum_{i=1}^{2}\left(A^{\phi}\right)^{\dagger}_{ki}\left(A^{\mathrm{L}\dagger}\right)_{\alpha\gamma}(A^\nu)^T_{\beta\delta}\,(h_i^\nu)_{\gamma\delta} &= \left[h_k^\nu\right]_{\alpha\beta}\,.
\end{aligned}\tag{4}$$

It is possible to present expressions derived in Eq. (4) as two eigenproblems

$$\boldsymbol{N_1}\Gamma^l = \Gamma^l\,, \qquad \boldsymbol{N_2}\Gamma^\nu = \Gamma^\nu\,, \tag{5}$$

where

$$\Gamma^l = \begin{pmatrix} h_1^l \\ h_2^l \end{pmatrix}, \qquad \Gamma^\nu = \begin{pmatrix} h_1^\nu \\ h_2^\nu \end{pmatrix}, \tag{6}$$

and

$$\boldsymbol{N_1} = \left(A^{\phi}\right)^T \otimes \left(A^{\mathrm{L}}\right)^{\dagger} \otimes \left(A^l\right)^T, \qquad \boldsymbol{N_2} = \left(A^{\phi}\right)^{\dagger} \otimes \left(A^{\mathrm{L}}\right)^{\dagger} \otimes \left(A^\nu\right)^T\,. \tag{7}$$

Note that Γ^l, Γ^ν are the vectors composed of appropriate elements of matrices h_k^l and h_k^ν respectively. In both cases, N_1 and N_2 are 18-dimensional.

The way of solving equations Eq. (5) is described in detail in [17]. In general, the algorithm can be summarized as follows: construction of $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ for all generators of the considered flavour group $\mathrm{G_F}$, looking for the eigensubspace for all generators, determining the common eigensubspace, establishing the base vector of the common eigensubspace.

To conclude, the base vector of the common eigensubspace constitute the solution. However, the first step introduced in the presented algorithm requires some comment. In calculations, it is necessary to take into account only the generators' representations. This non-trivial fact results from the following proposition (see *e.g.* [17]): if the invariance equations Eq. (4) are valid for the representations of some generators g_1 and g_2 of the flavour group $\mathrm{G_F}$, then they are automatically satisfied by the representations of their product $g_3 = g_1 g_2$.

It turns out that the solution of the invariance equation has got simple mathematical interpretation. It can be proven (see *e.g.* [17]) that Yukawa couplings h^ν and h^l play the role of Clebsch–Gordan coefficients for the following decompositions:

$$A^{\mathrm{L}} \otimes \left(A^l\right)^* = \oplus_\lambda A_\lambda \,, \qquad A^{\mathrm{L}} \otimes \left(A^\nu\right)^* = \oplus_\lambda A_\lambda \,, \tag{8}$$

where in the first case, we look for the 2-dimensional representation ($A^\phi = A_\lambda$ for some λ), while in the second case, we demand the existence of $(A^\phi)^* = A_\lambda$ for some λ. This condition is necessary to get some solution for the invariance equation.

4. The mass matrices and the mixing matrix in the lepton sector

Having the Yukawa couplings: $h^l_{1,2}$ and $h^\nu_{1,2}$ (see the previous section) and vacuum expectation values of the Higgs fields v_i, one can get the mass matrices

$$M^l_{\alpha,\beta} = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} v_i \left(h^l_i\right)_{\alpha,\beta} \,, \qquad M^\nu_{\alpha,\beta} = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} v_i \left(h^\nu_i\right)_{\alpha,\beta} \,. \tag{9}$$

To proceed with the calculations, one should diagonalize these matrices. It is commonly known in the literature that in order to derive the mass eigenvalues, it is necessary to use bi-unitary transformation

$$V^{l\dagger}_{\mathrm{L}} M^l V^l_{\mathrm{R}} = M^l_{\mathrm{diag}} \,, \qquad V^{\nu\dagger}_{\mathrm{L}} M^\nu V^\nu_{\mathrm{R}} = M^\nu_{\mathrm{diag}} \,. \tag{10}$$

In practice, one diagonalizes $M^l M^{l\dagger}$ and $M^\nu M^{\nu\dagger}$ instead of M^ν or M^l since in this case, one needs only one unitary matrix to perform the diagonalization

$$V^{l\dagger}_{\mathrm{L}} \left(M^l M^{l\dagger}\right) V^l_{\mathrm{L}} = \left(M^l M^{l\dagger}\right)_{\mathrm{diag}} \,, \qquad V^{\nu\dagger}_{\mathrm{L}} \left(M^\nu M^{\nu\dagger}\right) V^\nu_{\mathrm{L}} = \left(M^\nu M^{\nu\dagger}\right)_{\mathrm{diag}} \,.$$

From the obtained matrices V^l_{L} and V^ν_{L}, one can easily create the mixing matrix

$$U_{\mathrm{PMNS}} = V^{l\dagger}_{\mathrm{L}} V^\nu_{\mathrm{L}} \,. \tag{11}$$

Now, we can try to find such a free model parameters, which give correct values of the lepton masses and mixing matrix elements [18].

Our preliminary results indicate that in the set of the examined groups there is none, which is able to reproduce the mixing matrix elements and the lepton masses.

5. Conclusions

Clearly, the results obtained so far are not in agreement with experiment. If it turned out that results match the experimental data, it would be necessary to have a second look at the Higgs potential. We would need to know whether the Higgs potential meets all experimental requirements, gives two different vacuum expectation values, and all additional Higgs particles satisfy existing experimental bounds: *e.g.* the flavour changing neutral current is small.

In the nearest future, we plan to study the model with another extensions of the scalar sector (for example, with three Higgs doublets) and similar models for Majorana neutrinos. Apart from that, we are going to examine the model with left–right symmetry where additional sterile neutrinos are present.

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