

# ROOTS OF UNITY AND LEPTON MIXING PATTERNS FROM FINITE FLAVOUR SYMMETRIES\*

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The classification of lepton mixing matrices from finite residual symmetries is reviewed, with emphasis on the role of vanishing sums of roots of unity for the solution of this problem.

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## 1. Introduction

The  $3 \times 3$  mixing matrix or PMNS matrix  $U$  in the lepton sector has two large and one small mixing angle. It could be that this peculiar feature, which is in stark contrast to the CKM matrix in the quark sector, can be explained through an underlying flavour symmetry. Many attempts in this direction have been made, but no convincing scenario has emerged up to now. Some years ago, the idea was put forward that the structure of  $U$  is connected with residual symmetries in the charged-lepton and neutrino mass matrices [1]. In this approach, which is completely independent of any realization of this idea in a model, the diagonalization of the mass matrices is effectively replaced by the diagonalization of the symmetry transformation matrices of the residual symmetries. Using the notation  $|U|^2 \equiv (|U_{ij}|^2)$ , it turns out that this approach can either determine  $|U|^2$  completely or fix one of its rows or one of its columns.

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In [2], we have demonstrated that a complete classification of all possible  $|U|^2$ , up to independent permutations on  $|U|^2$  from the left and right, can be performed under the following assumptions:

- There are three lepton flavours.
- Neutrinos have Majorana nature.
- The flavour group  $G$  is finite.

As a result, we have found that there are 17 sporadic mixing patterns and one infinite series associated with a genuine three-flavour mixing matrix  $U$ . All these mixing patterns had been found before assuming specific groups [3, 4]. Thus, our analysis demonstrates that there are no possible other mixing patterns, no matter which finite flavour group  $G$  one begins with.

We stress that the finiteness of  $G$  is an *ad hoc* assumption for the mathematical treatment of the problem. It is absolutely crucial for the arguments used in [2].

## 2. Residual symmetries

In order to fix the notation, we denote the mass terms of charged leptons and neutrinos by

$$\mathcal{L}_{\text{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2} \nu_L^T C^{-1} M_\nu \nu_L + \text{H.c.}, \quad (1)$$

where the indices L and R indicate the chiralities of the fermion fields and  $C$  is the charge-conjugation matrix. Due to the assumed Majorana nature of the neutrinos, we have  $M_\nu^T = M_\nu$ . Diagonalization of the mass matrices proceeds via

$$U_\ell^\dagger M_\ell M_\ell^\dagger U_\ell = \text{diag}(m_e^2, m_\mu^2, m_\tau^2), \quad U_\nu^T M_\nu U_\nu = \text{diag}(m_1, m_2, m_3), \quad (2)$$

leading to the mixing matrix  $U = U_\ell^\dagger U_\nu$ .

The idea of residual symmetries [1] rests on the fact that the  $\ell_L$  and  $\nu_L$  belong to the same gauge doublet, therefore, in a weak basis they must belong to the same multiplet of the flavour group  $G$ , which is broken to the subgroup  $G_\ell$  in the charged-lepton sector and to  $G_\nu$  in the neutrino sector. Invariance of the mass matrices under the residual groups is formulated as

$$T \in G_\ell \Rightarrow T^\dagger M_\ell M_\ell^\dagger T = M_\ell M_\ell^\dagger, \quad S \in G_\nu \Rightarrow S^T M_\nu S = M_\nu. \quad (3)$$

Since the charged-lepton and neutrino mass spectra are non-degenerate, both  $G_\ell$  and  $G_\nu$  must be Abelian and, therefore,

$$G_\ell \subseteq \text{U}(1) \times \text{U}(1) \times \text{U}(1), \quad G_\nu \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \quad (4)$$

Consequently, all  $T \in G_\ell$  together with  $M_\ell M_\ell^\dagger$  are simultaneously diagonalizable, and the same is true for all  $S \in G_\nu$  and  $M_\nu^\dagger M_\nu$ .

In essence, the diagonalization of  $M_\ell M_\ell^\dagger$  is replaced by the diagonalization of the  $T \in G_\ell$  and the diagonalization of  $M_\nu^\dagger M_\nu$  is replaced by the diagonalization of the  $S \in G_\nu$ .

Some remarks are in order. If a single  $T \in G_\ell$  has non-degenerate eigenvalues, then  $U_\ell$  is uniquely determined and  $G_\ell \cong \mathbb{Z}_N$  with a suitable  $N$ . In this context, it is sufficient that there is one such  $T$  in  $G_\ell$ . If all  $T \in G_\ell$  are degenerate, one can show [2] that one can confine oneself to two generators  $T_1, T_2$  of  $G_\ell$  and  $G_\ell \cong K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , where  $K$  is Klein's four group. With regard to  $G_\nu$ , one can limit oneself to  $G_\nu \cong K$  by requiring that all  $S \in G_\nu$  have  $\det S = 1$ .

It is important to realize what the approach of residual symmetries achieves and what not. Since  $U_\ell$  is determined by the diagonalization of the  $T$  only up to a diagonal matrix of phase factors from the right and the analogous statement holds for  $U_\nu$ , residual symmetries cannot fix Majorana phases. Moreover, since we can switch from a representation of  $G$  to its complex conjugate representation, the sign of the CKM-type phase in  $U$  is not determined either. These two statements can be subsumed by saying that with residual symmetries only  $|U|^2$  can be determined. Since  $|U|^2$  originates in the diagonalization of representation matrices of  $G$ , the resulting entries of  $|U|^2$  are pure numbers determined by group theory, independent of the parameters of any underlying theory. In addition, this approach does not make any connection to lepton masses. Therefore,  $|U|^2$  can only be determined up to independent permutations from the left and right.

In order to go from groups to mixing matrices  $|U|^2$ , one first has to choose a group  $G$  which has the subgroup  $G_\nu = K$ . Then, one has to search for all subgroups  $G_\ell$  of  $G$  which completely fix  $U_\ell$ . Thereafter, one has to compute  $|U|^2$  for all these subgroups. Many authors have chosen this approach — see [2–4] and references therein.

However, a general analysis has to be group independent. It turns out that the key to the general analysis is the determination of all possible forms of  $|T| \equiv (|T_{ij}|)$  with the help of a theorem on vanishing sums of roots of unity [5], which has already been used in such a context before in [6].

### 3. General analysis

In order to determine the possible forms of  $|T|$ , it is useful to choose a basis where  $G_\nu = \{\mathbb{1}, S_1, S_2, S_3\}$  with

$$S_1 = \text{diag}(1, -1, -1), \quad S_2 = \text{diag}(-1, 1, -1), \quad S_3 = S_1 S_2. \quad (5)$$

In this basis,  $U_\nu = \mathbb{1}$ ,  $U = U_\ell^\dagger$  and  $UTU^\dagger = \hat{T}$  is diagonal.

In the following,  $P_1, P_2, P$  are  $3 \times 3$  permutation matrices. There is a series of steps [2] that leads to possible mixing patterns:

1. determination of the five basic forms of  $|T|$  up to permutations  $P_1|T|P_2$ ,
2. determination of the internal (CKM-type) phase of  $T$ ,
3. finding all inequivalent forms of  $|T|$  through  $|T| \rightarrow |T|P$ ,
4. exclusion of two forms of  $|T|$  which do not lead to finite groups,
5. determination of external (Majorana-type) phases of  $T$ ,
6. computation of possible patterns of  $|U|^2$  up to permutations  $P_1|U|^2P_2$  from the possible matrices  $T$ .

Here, we will only discuss the first step. For the other steps, we refer the reader to [2]. In relation to step 3, we note that two matrices  $T$ ,  $T'$  are equivalent, *i.e.* they lead to the same  $|U|^2$  modulo permutations, if they are related by  $T' = V^\dagger T V$  such that  $V$  is a permutation matrix times a diagonal matrix of phase factors.

In order to perform step 1, we consider the matrices  $Y^{(ij)} \equiv T^\dagger S_i T S_j$  of  $G$ . With

$$S_j^{-1} Y^{(ij)} S_j = \left( Y^{(ij)} \right)^\dagger, \quad \det Y^{(ij)} = 1 \quad (6)$$

we see that the eigenvalues of  $Y^{(ij)}$  must be 1,  $\lambda^{(ij)}$ ,  $(\lambda^{(ij)})^*$ . Due to the finiteness of  $G$ , all  $\lambda^{(ij)}$  have to be roots of unity. Because of  $\sum_{k=1}^3 S_k = -\mathbb{1}$ ,

$$\sum_{k=1}^3 \text{Tr } Y^{(kj)} = \sum_{k=1}^3 \text{Tr } Y^{(ik)} = 1. \quad (7)$$

Written in terms of the eigenvalues, these equations give

$$\sum_{k=1}^3 \left( \lambda^{(kj)} + \lambda^{(kj)*} \right) + 2 = \sum_{k=1}^3 \left( \lambda^{(ik)} + \lambda^{(ik)*} \right) + 2 = 0 \quad \forall \quad i, j = 1, 2, 3. \quad (8)$$

It is not difficult to show that there is a relation between  $|T_{ij}|$  and the eigenvalues  $\lambda^{(ij)}$  given by

$$|T_{ij}|^2 = \frac{1}{2} \left( 1 + \text{Re } \lambda^{(ij)} \right). \quad (9)$$

Therefore, the generic equation one has to solve is

$$\sum_{k=1}^3 (\lambda_k + \lambda_k^*) + 2 = 0 \quad (10)$$

with roots of unity  $\lambda_k$ . Using a theorem of Conway and Jones [5], one can prove that equation (10) has, up to reordering and complex conjugation, only the three solutions

$$(\lambda_1, \lambda_2, \lambda_3) = \begin{cases} (i, \omega, \omega) , \\ (\omega, \beta, \beta^2) , \\ (-1, \lambda, -\lambda) \end{cases} \quad (11)$$

with  $\omega = e^{2\pi i/3}$ ,  $\beta = e^{2\pi i/5}$  and  $\lambda = e^{i\vartheta}$  being an arbitrary root of unity.

Any solution  $(\lambda_1, \lambda_2, \lambda_3)$  of equation (10) can correspond via equation (9) to a row or a column of  $|T|$ . In order to combine the solutions of equation (10) to matrices  $|T|$ , one must bear in mind that  $T$  is unitary, which rules out quite a few combinations. Up to independent permutations from the left and right, there are only five forms of  $|T|$ . Two of these forms do not lead to a finite flavour group and a third one gives only one sporadic genuine three-flavour mixing pattern. Thus, the two most relevant forms are

$$|T| = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (12)$$

and

$$|T| = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} \\ \frac{\sqrt{5}+1}{4} & \frac{1}{2} & \frac{\sqrt{5}-1}{4} \\ \frac{\sqrt{5}-1}{4} & \frac{\sqrt{5}+1}{4} & \frac{1}{2} \end{pmatrix}. \quad (13)$$

For the further steps in the derivation of the mixing patterns, see [2].

## 4. Results

Confining ourselves to genuine three-flavour mixing patterns, we have found that, under the assumptions displayed in Section 1, residual symmetries lead to 17 sporadic patterns of  $|U|^2$  and one series. Using data on lepton mixing, it turns out that all sporadic cases are ruled out. The mixing pattern of the infinite series is given by

$$|U|^2 = \frac{1}{3} \begin{pmatrix} 1 + \text{Re } \sigma & 1 & 1 - \text{Re } \sigma \\ 1 + \text{Re } (\omega\sigma) & 1 & 1 - \text{Re } (\omega\sigma) \\ 1 + \text{Re } (\omega^2\sigma) & 1 & 1 - \text{Re } (\omega^2\sigma) \end{pmatrix}. \quad (14)$$

This  $|U|^2$  depends on the parameter  $\sigma = e^{2i\pi p/n}$ , where  $p/n$  is a rational number, *i.e.*  $\sigma$  is a root of unity. Clearly, permutation of the rows in

equation (14) leads to equivalent mixing patterns due to the freedom in  $\sigma$ . Permutation of the columns, however, leads to three distinct cases in the usual ordering of charged leptons and neutrino masses, depending in which column 1/3 is located. Only the choice displayed in equation (14) for which

$$\cos^2 \theta_{13} \sin^2 \theta_{12} = 1/3 \quad (15)$$

holds is compatible with the data. Because of the specific form of  $|U|^2$  in equation (14), the actual parameter which is restricted by the data on the mixing angles is  $\text{Re } \sigma^6$ . Using the fit results of [7], the 3sigma range of  $\sin^2 \theta_{13}$  translates into  $-0.69 \lesssim \text{Re } \sigma^6 \lesssim -0.37$ . Thus, there is indeed a range of  $\sigma$  such that the  $|U|^2$  of equation (14) is compatible with the data.

In summary, among the mixing patterns obtained by the approach of residual symmetries, lepton mixing data single out a unique one-parameter mixing pattern. The rational number occurring in the exponent of the parameter  $\sigma$  is related to the groups which realize the relevant residual symmetries. Such groups are subgroups of SU(3) of type D [8]. Apart from the correlations between the mixing angles which can be read off from the figure in [2], a further prediction of equation (14) is a trivial CKM-type phase. Therefore, a future measurement of CP violation in neutrino oscillations could be a crucial test of this mixing matrix.

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