THERMODYNAMICS OF IDEAL FERMI GAS UNDER GENERIC POWER LAW POTENTIAL IN d-DIMENSIONS

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Thermodynamics of the ideal Fermi gas trapped in an external generic power law potential $U = \sum_{i=1}^{d} c_i |\frac{x_i}{a_i}|^{n_i}$ is investigated systematically from the grand thermodynamic potential in d-dimensional space. These properties are explored carefully in the degenerate limit ($\mu \gg K_{\mathrm{B}}T$), where the thermodynamic properties are greatly dominated by the Pauli exclusion principle. Pressure and energy along with the isothermal compressibility are nonzero at $T = 0K$. The nonzero value of compressibility implies that zero point pressure is not a constant but depends on volume.

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1. Introduction

The constrained role of external potential can change the characteristics of quantum gases [1–6]. Increasing attention was given to the subject after it was possible to create the Bose–Einstein condensation (BEC) in magnetically trapped alkali gases [7–9]. A lot of studies have been conducted to understand the behaviour of the ideal Bose gas [10–14] as well as ideal Fermi gas [11, 12]. However, unlike the Bose gas, Fermi gas does not condense due to the Pauli exclusion principle. Hence, the question of a large number of particles occupying a single energy state does not even arise in this case. At sufficiently low temperature, the Fermi gas displays its own brand of interesting behaviour [11, 12] as its fugacity z_{F} can take unrestricted values: $0 < z_{\mathrm{F}} < \infty$ [11], unlike the Bose gas. The latter has a restricted value of fugacity $0 < z_{\mathrm{B}} \leqslant 1$ [11]. The behaviour of thermodynamic quantities of

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the Fermi gas is remarkably governed by the Pauli exclusion principle. For instance, the ground state pressure also known as degeneracy pressure [11] in the ideal Fermi gas is nonzero, unlike the Bose gas or classical gas [11]. Nevertheless, a lot of effort is made towards the understanding of different properties of a Fermi system such as magnetism [15], conductivity [16], transport properties [17], equivalence with the ideal Bose gas [18, 19], dimensionality effects [20], degeneracy [21], polylogarithms [22], q-deformed system [23, 24].

Although the interaction between particles exists in a real system, taking it into account makes the problem difficult to solve analytically. Nevertheless, in order to understand the effect of interactions in quantum gases and retain the essential physics, we approximately present the real system by noninteracting particles in the presence of an external potential [2, 25]. The trapping potential in atomic gases provide the opportunity to manipulate the quantum statistical effects. Some drastic changes have been noted in the Bose system [1, 2] under trapping potential. For instance, BEC is possible in $d = 2$ in the trapped Bose gas, which was not a case in the ideal Bose gas [1, 2]. Therefore, it will be interesting to investigate how the trapping potential does change the properties of the Fermi gas. In their work, Li *et al.* [4] have presented internal energy, heat capacity, ground state energy of the Fermi gas under spherically symmetric potential ($U = br^t$) in arbitrary dimension. However, in the present study, we have investigated properties of the ideal Fermi gas under the generic power law potential $U = \sum_{i=1}^{d} c_i |\frac{x_i}{a_i}|^{n_i}$ in d-dimensions, which will be symmetric under certain choice n_i, x_i and c_i. Thus, in principle, one can reconstruct the results of Li *et al.* [4] choosing this special condition. At first, the density of states has been calculated which enables us to determine the grand potential. Then, from the grand potential of the system, the thermodynamic quantities such as internal energy E, entropy S, pressure P, number of particles N, Helmholtz free energy A, isothermal compressibility κ_T, specific heat at constant volume C_V and constant pressure C_P, and their ratio are derived. In the high temperature limit, the thermodynamic quantities of free quantum gases reduce to the form of classical gas [11]. The same trend is observed in the case of trapped system [1]. Therefore, the low temperature limit of the thermodynamic quantities of quantum gases is particularly important, as the true quantum nature is explicit in that region. In the Bose system, the low temperature limit refers to condensed phase. It was found that the trapping potential changes the general criterion of BEC as well as the condition of jump of specific heat [1, 2]. Thus, the low temperature limit of thermodynamic quantities related to the trapped Fermi system has been examined using the Sommerfeld expansion [11]. It will be very intriguing to investigate energy and pressure of the trapped Fermi gas, while the gas is in the degenerate limit, to check whether they remain nonzero under generic trapping potential at $T = 0K$. Isothermal compressibility (inverse of bulk modulus) is also calculated to check whether the ground state pressure has a volume dependency or it is merely just a constant. A point to note is that in the Hamiltonian, instead of $\frac{p^2}{2m}$ type kinetic part, we have taken ap^s, where p denotes momentum, a is constant and s is an arbitrary kinematic parameter. A different kinematic parameter of quantum systems leads to different characteristics features [1, 27, 28]. Therefore, significant conclusions can be reached in a more generalised way by using arbitrary kinematic parameter. In the present study, it is found that the concept of effective volume plays an important role in the trapped Fermi gas, as seen in the trapped Bose gas [1].

The report is organized in the following way. The density of states and grand potential are calculated in Section 2. Section 3 is devoted to investigating the thermodynamic quantities. Properties of the degenerate Fermi gas are presented in Section 4. Results and discussions are presented in Section 5. The report is concluded in Section 6.

2. Density of states and grand potential of Fermi gas under generic power law potential in d-dimensions

Considering the ideal Fermi gas with kinematic parameter l in a confining external potential in a d-dimensional space with energy spectrum,

$$\epsilon(p, x_i) = bp^l + \sum_{i=1}^{d} c_i \left|\frac{x_i}{a_i}\right|^{n_i} , \tag{1}$$

where p is the momentum, x_i is the i^{th} component of coordinate of a particle and b, l, a_i, c_i, n_i are all positive constants. Note that $x_i < a_i$. Here, c_i, a_i and n_i determine the depth and confinement power of the potential. Using $l = 2$, $b = \frac{1}{2m}$, one can get the energy spectrum of the Hamiltonian used in the literature [2, 10–12]. For the free system, all $n_i \longrightarrow \infty$. As $|\frac{x_i}{a_i}| < 1$, the potential term goes to zero.

Density of states can be obtained from the following formula

$$\rho(\epsilon) \;=\; \int\int \frac{d^d r d^d p}{(2\pi\hbar)^d}\delta(\epsilon - \epsilon(p, r))\,. \tag{2}$$

Therefore, from the above equation, density of states is

$$\rho(\epsilon) = B\frac{\Gamma\left(\frac{d}{l}+1\right)}{\Gamma(\chi)}\epsilon^{\chi-1}\,, \tag{3}$$

where

$$B=\frac{gV_dC_d}{h^da^{d/l}}\prod_{i=1}^{d}\frac{\Gamma\left(\frac{1}{n_i}+1\right)}{c_i^{\frac{1}{n_i}}}\,. \tag{4}$$

Here, $C_d=\frac{\pi^{\frac{d}{2}}}{\Gamma(d/2+1)}$, g is the spin degeneracy factor, $V_d=2^d\prod_{i=1}^{d}a_i$ is the volume of an d-dimensional rectangular whose $i^{\rm th}$ side has length $2a_i$. $\Gamma(l)=\int_0^\infty dx x^{l-1}e^{-x}$ is the gamma function and $\chi=\frac{d}{l}+\sum_{i=1}^{d}\frac{1}{n_i}$.

The grand potential for the Fermi system,

$$q=\sum_{\epsilon}\ln(1+z\exp(-\beta\epsilon))\,, \tag{5}$$

$\beta=\frac{1}{kT}$, k being the Boltzmann constant and $z=\exp(\beta\mu)$ is the fugacity, μ being the chemical potential. Using the Thomas–Fermi semi-classical approximation [25] and re-writing the previous equation, we get

$$q=q_0+\int\limits_0^\infty\rho(\epsilon)\ln(1+z\exp(-\beta\epsilon))\,. \tag{6}$$

So, using the density of states of Eq. (3), we finally get the grand potential

$$q=q_0+B\Gamma\left(\frac{d}{l}+1\right)(kT)^\chi f_{\chi+1}(z)\,, \tag{7}$$

where $q_0=\ln(1+z)$ and $f_l(z)$ is the Fermi function which is defined as

$$f_p(z)=\int\limits_0^\infty dx\frac{x^{p-1}}{z^{-1}e^x+1}=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{z^j}{j^p}\,. \tag{8}$$

3. Thermodynamics of the Fermi gas under generic power law potential in *d*-dimensions

3.1. Number of particles

The number of particles N can be obtained

$$N=z\left(\frac{\partial q}{\partial z}\right)_{\beta,V}=N_0+\frac{gC_n\Gamma\left(\frac{d}{l}+1\right)V_d\prod_{i=1}^{d}\Gamma\left(\frac{1}{n_i}+1\right)}{h^db^{d/l}\prod_{i=1}^{d}c_i^{1/n_i}}(kT)^\chi f_\chi(z)\,. \tag{9}$$

Here, $N_0=\frac{z}{1+z}$ is the ground state occupation number.

Now, we define

$$V'_d = V_d \prod_{i=1}^{d} \left(\frac{kT}{c_i}\right)^{1/n_i} \Gamma\left(\frac{1}{n_i}+1\right), \tag{10}$$

$$\lambda' = \frac{hb^{\frac{1}{l}}}{\pi^{\frac{1}{2}}(kT)^{\frac{1}{l}}} \left[\frac{d/2+1}{d/l+1}\right]^{1/d}. \tag{11}$$

It is noteworthy,

$$\lim_{n_i \to \infty} V'_d = V_d \,, \tag{12}$$

$$\lim_{n_i \to \infty} \chi = \frac{d}{l} \,, \tag{13}$$

$$\lim_{l \to 2, b \to \frac{1}{2m}} \lambda' = \lambda_0 = \frac{h}{(2\pi mkT)^{1/2}} \,. \tag{14}$$

Thus, if we choose $l = 2$ and $b = \frac{1}{2m}$ from Eq. (14), we get $\lambda_0 = \frac{h}{(2\pi mkT)^{1/2}}$, which is the thermal wavelength of nonrelativistic massive fermions as well as massive bosons. However, it should be noted that when $l \neq 2$, λ' depends on dimension. With $d = 3$ and $d = 2$, thermal wavelength of photons (boson) and neutrinos (fermion) are, respectively, $\frac{hc}{2\pi^{1/2}kT}$ and $\frac{hc}{(2\pi)^{1/2}kT}$ which can be obtained from Eq. (11) by choosing $b = c$, where c is the velocity of light. So, one can reproduce the thermal wavelength of both massive and massless particles from the definition of λ' with more general energy spectrum. But one needs to consider the effects of antiparticles to calculate the thermodynamic quantities of ultrarelativistic quantum gas [26].

The number of particles equation is then written as

$$N - N_0 = g \frac{V'_d}{\lambda'^d} f_\chi(z) \,. \tag{15}$$

The number of particles equation for free massive fermions (with $l = 2$, $a = \frac{1}{2m}$, all $n_i \longrightarrow \infty$) in d-dimensional space can be obtained from Eq. (15),

$$N - N_0 = g \frac{V_d}{{\lambda_0}^d} f_{\frac{d}{2}}(z) \,, \tag{16}$$

which gives the exact equation for number of particles at $d = 3$ [11, 12].

3.2. Internal energy

From the Grand Canonical Ensemble, internal energy E is

$$\begin{aligned} E &= -\left(\frac{\partial q}{\partial \beta}\right)_{z,V} \\ &= \frac{gC_n\Gamma\left(\frac{d}{l}+1\right)V_d\prod_{i=1}^{d}\Gamma\left(\frac{1}{n_i}+1\right)}{h^d b^{d/l}\prod_{i=1}^{d}c_i^{1/n_i}}(kT)^{\chi+1}f_{\chi+1}(z) \qquad (17)\\ &= NkT\chi\frac{f_{\chi+1}(z)}{f_\chi(z)}\,. \qquad (18)\end{aligned}$$

In the case of free massive fermions,

$$E = NkT\frac{d}{2}\frac{f_{d/2+1}(z)}{f_{d/2}(z)}\,, \qquad (19)$$

which is in accordance with the exact expression of E for $d = 3$ [11, 12].

Now, as $T \longrightarrow \infty$, from Eq. (19), it can be seen that the internal energy becomes $E = NkT\chi$. For free massive fermions, it is $E = \frac{d}{2}NkT$, which becomes $\frac{3}{2}NkT$, when $d = 3$. Thus E approaches the classical value at high temperature. The same exact trend is seen in the Bose gas too [1].

3.3. Entropy

The entropy S can be obtained from the Grand Canonical Ensemble,

$$\begin{aligned} S &= kT\left(\frac{\partial q}{\partial T}\right)_{z,V} - Nk\ln z + kq \\ &= Nk\left[\frac{v'_d}{\lambda'^d}(\chi+1)f_{\chi+1}(z) - \ln z\right]. \qquad (20)\end{aligned}$$

As before, for free massive fermions, one can find that Eq. (20) approaches

$$S = Nk\left[\frac{v_d}{\lambda^d}\left(\frac{d}{2}+1\right)f_{\frac{d}{2}+1}(z) - \ln z\right]. \qquad (21)$$

Again, at $d = 3$, Eq. (21) reduces to the same expression for entropy as in Refs. [11, 12].

3.4. Helmholtz Free Energy

From the Grand Canonical Ensemble, we get the expression of Helmholtz Free Energy for the Fermi system

$$A = -kTq + NkT\ln z = -NkT\frac{f_{\chi+1}(z)}{f_\chi(z)} + NkT\ln z\,. \tag{22}$$

For free massive fermions, the above expression reduces as below

$$\frac{A}{NkT} = -\frac{f_{\frac{d}{2}+1}(z)}{f_{\frac{d}{2}}(z)} + \ln z\,. \tag{23}$$

Now, for $d = 3$, the above equation produces the exact expression for Helmholtz Free Energy [11, 12].

3.5. Pressure

Re-writing equation (15) stating the number of particles, we get

$$\frac{N-N_0}{V_d\prod_{i=1}^{d}\left(\frac{kT}{c_i}\right)^{1/n_i}\Gamma\left(\frac{1}{n_i}+1\right)} = \frac{N-N_0}{V'_d} = \frac{g}{\lambda'^d}g_\chi(z)\,.$$

Now, we take a very well-known expression for the nonrelativistic d-dimensional ideal free Fermi gas [11]

$$\frac{N-N_0}{V_d} = \frac{g}{\lambda_0^d}f_{d/2}(z)\,.$$

Comparing the above equations, we can say that V'_d is a more generalized extension of V_d, where

$$V'_d = V_d\prod_{i=1}^{d}\left(\frac{kT}{c_i}\right)^{1/n_i}\Gamma\left(\frac{1}{n_i}+1\right)\,.$$

It represents the effect of external potential on the performance of trapped fermions. Calling V'_d the effective volume, the grand potential can be re-written as

$$q = q_0 + g\frac{gV'_d}{\lambda'^d}f_{\chi+1}(z)\,. \tag{24}$$

So, the effective pressure

$$P' = \frac{1}{\beta}\left(\frac{\partial q}{\partial V'_d}\right) = \frac{gkT}{\lambda'^d}f_{\chi+1}(z)\,, \tag{25}$$

which can be re-written as

$$P' = \frac{NkT}{V'_d}\frac{f_{\chi+1}(z)}{f_\chi(z)}\,. \tag{26}$$

The above equation is a very general equation of state for any dimensionality d, any dispersion relation of the form of ($\propto p^s$) having any form of generic power law trap, and, obviously it is expected that it will reproduce the special case of a free system. For the free system, equation (26) becomes

$$P = \frac{1}{\beta}\left(\frac{\partial q}{\partial V_d}\right) = \frac{NkT}{V_d}\frac{f_{d/2+1}(z)}{f_{\frac{d}{2}}(z)} \tag{27}$$

which is in accordance with Refs. [11, 12] at $d = 3$.

Now, comparing Eqs. (18) and (26), we get

$$P'V'_d = \frac{E}{\chi}\,. \tag{28}$$

For d-dimensional free Fermi gas from the previous equation, one can obtain

$$PV_d = \frac{2}{d}E\,. \tag{29}$$

This is an important and familiar relation, $PV = \frac{2}{3}E$ when $d = 3$ [10–13]. This actually shows that equation (28) is a very significant relation for the Fermi gas irrespective of whether they are trapped or free. In the case of trapped fermions, the effective volume and effective pressure play the same role as volume and pressure in current textbooks and literature. Interestingly, the Bose gas also maintains this equation [1].

3.6. Heat capacity

Heat capacity at constant volume C_V can be written as

$$\begin{aligned} C_V &= T\left(\frac{\partial S}{\partial T}\right)_{N,V} \\ &= Nk\left[\chi(\chi+1)\frac{\nu'}{\lambda'^D}f_{\chi+1}(z) - \chi^2\frac{f_\chi(z)}{f_{\chi-1}(z)}\right]. \end{aligned} \tag{30}$$

For free massive fermions, the expression becomes

$$C_V = Nk\left[\frac{d}{2}\left(\frac{d}{2}+1\right)\frac{\nu}{\lambda^D}f_{\frac{d}{2}+1}(z) - \frac{d^2}{2}\frac{f_{\frac{d}{2}}(z)}{f_{\frac{d}{2}-1}(z)}\right]. \tag{31}$$

In the high temperature, the limit of C_V approaches its classical value as it becomes χNk for the trapped system and $\frac{d}{2}Nk$ for the free system, which is $\frac{3}{2}Nk$, when $d = 3$.

Now, heat capacity at constant pressure C_P is

$$\begin{aligned} C_P &= T\left(\frac{\partial S}{\partial T}\right)_{N,P} \\ &= Nk\left[(\chi+1)^2 f^2_{\chi+1}(z) f_{\chi-1}(z)\left(\frac{\nu'}{\lambda'^D}\right)^3 - \chi(\chi+1) f_{\chi+1}(z)\frac{\nu'}{\lambda'^D}\right]. \end{aligned} \tag{32}$$

In the case of free massive Fermi gas, the above equation reduces to

$$C_P = Nk\left[\left(\frac{d}{2}+1\right)^2 f^2_{\frac{d}{2}+1}(z) f_{\frac{d}{2}-1}(z)\left(\frac{\nu'}{\lambda'^D}\right)^3 - \frac{d}{2}\left(\frac{d}{2}+1\right) f_{\frac{d}{2}+1}(z)\frac{\nu'}{\lambda'^D}\right]. \tag{33}$$

It coincides exactly with Refs. [11, 13] for $d = 3$. Again, in the high temperature, the limit C_P becomes $(\chi+1)Nk$ for the trapped system and $(\frac{d}{2}+1)Nk$ for the free system, which is $\frac{5}{2}Nk$, when $d = 3$. So, in the high temperature, the limit C_P approaches its classical value.

Now, the ratio, $\gamma = (\frac{C_P}{C_V})$ for Fermi gas is given by

$$\gamma = \frac{(\chi+1)^2 \frac{f^2_{\chi+1}(z) f_{\chi-1}(z)}{f^3_\chi(z)} - \chi(\chi+1)\frac{f_{\chi+1}(z)}{f_\chi(z)}}{\chi(\chi+1)\frac{\nu'}{\lambda'^D} f_{\chi+1}(z) - \chi^2 \frac{f_\chi(z)}{f_{\chi-1}(z)}}. \tag{34}$$

Now, in high temperature limit, the above equation becomes

$$\gamma = \frac{(\chi+1)^2 - \chi(\chi+1)}{\chi(\chi+1) - \chi^2} = 1 + \frac{1}{\chi}. \tag{35}$$

In the free system, choosing all $n_i \longrightarrow \infty$, we get from the above equation

$$\gamma = 1 + \frac{l}{d}. \tag{36}$$

With $d = 3$ and $l = 2$, γ equals $\frac{5}{3}$, thus obtaining the classical value at high temperature limit.

3.7. Isothermal compressibility

The isothermal compressibility of the Fermi gas can be obtained

$$\kappa_T = -V'_d\left(\frac{\partial V'}{\partial P'}\right)_{N,T} = -V'_d\left(\frac{\partial P'}{\partial z}\right)_{N,T}\left(\frac{\partial z}{\partial V'}\right)_{N,T} = \frac{V'_d}{NkT}\frac{f_{\chi-1}(z)}{f_\chi(z)}, \tag{37}$$

which reproduces the same result for isothermal compressibility of the free massive Fermi gas at $d = 3$ [11]. As $T \longrightarrow \infty$, κ_T takes the classical value for the free system, which is $\frac{1}{P}$.

4. The thermodynamic properties of a degenerate Fermi gas under generic power law potential

At low temperature, we can approximate the Fermi function and write it as quickly convergent Sommerfeld series [11]

$$f_p(z) = \frac{(\ln z)^p}{\Gamma(p+1)} \times \left[1 + p(p-1)\frac{\pi^2}{6}\frac{1}{(\ln z)^2} + p(p-1)(p-2)(p-3)\frac{7\pi^4}{360}\frac{1}{(\ln z)^4} + \ldots\right] . \quad (38)$$

At $T = 0\mathrm{K}$, we can take only the first term of Eq. (38). Substituting this into Eq. (15), we get

$$N - N_0 = N_e = \frac{gC_n\Gamma\left(\frac{d}{l}+1\right)V_d\prod_{i=1}^{d}\Gamma\left(\frac{1}{n_i}+1\right)}{h^d b^{d/l}\prod_{i=1}^{d}c_i^{1/n_i}\Gamma(\chi+1)}E_{\mathrm{F}}^{\chi} , \quad (39)$$

from which one can obtain

$$E_{\mathrm{F}} = \left[\frac{h^d b^{d/l}\prod_{i=1}^{d}c_i^{1/n_i}\Gamma(\chi+1)N_e}{gC_n\Gamma\left(\frac{d}{l}+1\right)V_d\prod_{i=1}^{d}\Gamma\left(\frac{1}{n_i}+1\right)}\right]^{\frac{1}{\chi}} . \quad (40)$$

Following the method of Refs. [4, 11, 12], we approximate the chemical potential and fugacity from Eq. (15)

$$\mu = kT\ln z = E_{\mathrm{F}}\left[1 - (\chi-1)\frac{\pi^2}{6}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right] . \quad (41)$$

Using this approximation, we can calculate the thermodynamic quantities of the previous section

$$\frac{E}{N} = \frac{\chi}{\chi+1}E_{\mathrm{F}}\left[1 + (\chi+1)\frac{\pi^2}{6}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right] , \quad (42)$$

$$\frac{S}{Nk} = \frac{\chi\pi^2}{3E_{\mathrm{F}}}kT , \quad (43)$$

$$P = \frac{E_{\mathrm{F}}N}{(\chi+1)V'}\left[1 + (\chi+1)\frac{\pi^2}{6}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right] , \quad (44)$$

$$\frac{C_V}{Nk} = \frac{\chi\pi^2}{3E_{\mathrm{F}}}kT , \quad (45)$$

$$\kappa_T = \frac{V'\chi}{NE_{\mathrm{F}}}\left\{1 + (1-\chi)\frac{\pi^2}{6}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right\} . \quad (46)$$

In order to determine the pure quantum effect in the degeneracy limit, we need to subtract the classical effect from the above equations. Thus, for any quantity I, the quantum effect in the degeneracy limit is $\Delta I = I - I_{\mathrm{cl}}$. In the case of free massive fermions (choosing $l = 2$), Eqs. (42)–(46) become

$$\frac{E}{N} = \frac{d}{d+2} E_{\mathrm{F}} \left[1 + \left(\frac{d}{2} + 1 \right) \frac{\pi^2}{6} \left(\frac{kT}{E_{\mathrm{F}}} \right)^2 \right], \tag{47}$$

$$\frac{S}{Nk} = \frac{d\pi^2}{6E_{\mathrm{F}}} kT, \tag{48}$$

$$P = \frac{2E_{\mathrm{F}} N}{(d+2)V'} \left[1 + \left(\frac{d}{2} + 1 \right) \frac{\pi^2}{6} \left(\frac{kT}{E_{\mathrm{F}}} \right)^2 \right], \tag{49}$$

$$\frac{C_V}{Nk} = \frac{d\pi^2}{6E_{\mathrm{F}}} kT, \tag{50}$$

$$\kappa_T = \frac{V'd}{2NE_{\mathrm{F}}} \left\{ 1 + \left(1 - \frac{d}{2} \right) \frac{\pi^2}{6} \left(\frac{kT}{E_{\mathrm{F}}} \right)^2 \right\}. \tag{51}$$

At temperature $T = 0$K, entropy $S = 0$ which is in accordance with the third law of thermodynamics. C_V is also zero at $T = 0$K and the internal energy, pressure and isothermal compressibility $T = 0$K,

$$E_0 = \frac{\chi}{\chi + 1} N E_{\mathrm{F}}, \tag{52}$$

$$P_0 = \frac{1}{(\chi + 1)} \frac{N}{V'} E_{\mathrm{F}}, \tag{53}$$

$$\kappa_{T0} = \frac{V\chi}{NE_{\mathrm{F}}}. \tag{54}$$

In the case of the free massive Fermi gas, the above equations reduce to

$$E_0 = \frac{d}{d+2} N E_{\mathrm{F}}, \tag{55}$$

$$P_0 = \frac{2}{(d+2)} \frac{N}{V} E_{\mathrm{F}}, \tag{56}$$

$$\kappa_{T0} = \frac{V}{NE_{\mathrm{F}}} \frac{d}{2}. \tag{57}$$

At $d = 3$, Eqs. (55) and (56) become exactly the same as in Ref. [11].

5. Discussion

Thermodynamics of the ideal Fermi gas in the presence of an external generic power law potential is discussed in this section. It is seen that the effective volume V'_d is a very salient feature of the trapped system. It plays the same role in trapped system as the volume in free system. This enables us to treat the trapped Fermi gas and the Bose gas [1] as a free one. The difference between V'_d and V_d is that the former depends on temperature and power law exponent, while the latter does not. But as all $n_i \longrightarrow \infty$, V'_d approaches V_d. In this process, the more general thermal wavelength λ' is defined with an arbitrary kinematic parameter in any dimension. It was shown how λ' can reproduce the thermal wavelengths for different dimensions cited in the literature. In the case of the trapped Fermi gases, V'_d and λ' enable all the thermodynamic functions of the system to be expressed in a compact form similar to those of the free Fermi gas.

At first, the density of states and grand potential are calculated in Section 2. All the thermodynamic quantities are derived from the grand potential in Section 3. It is seen that all the thermodynamic quantities for the trapped Fermi gas can be expressed in terms of the Fermi function, just as in the case of free Fermi gas. In the former case, the Fermi functions depend on $\chi = \frac{d}{l} + \sum_{i=1}^{d} \frac{1}{n_i}$ and z, while in the latter case Fermi functions depend on $\frac{d}{l}$ and z. As $n_i \longrightarrow \infty$, the mathematical form of thermodynamic quantities of trapped system reduces to that of free system. Nevertheless, it is noteworthy that Eq. (28) is a very remarkable relation for quantum gases as both Bose [1] and Fermi systems maintain it.

In general, the thermodynamic quantities of trapped system differ from free system. We can specifically check this by comparing the free system with the harmonically trapped potential. Let, $d = 3$, $a = \frac{1}{2m}$, $l = 2$, $n_1 = n_2 = n_3 = 2$, $c_i = \frac{m\omega^2}{2}$ and $g = 2$. Results of some of the physical quantities have been listed in Table I. From the table, it is seen that thermodynamic quantities are affected by the trapping potential. The signature of trapping potential is present in the low as well as in the high temperature limit of thermodynamic functions. As we know, chemical potential approaches Fermi energy as $T \longrightarrow 0$ in the case of the free Fermi gas, the same phenomena is also observed in the trapped Fermi gas, although Fermi energy does vary when comparing the trapped system with the free one. Let us turn our attention to the low temperature limit of the Fermi gas. It is seen that both C_V and S has same numerical value in this limit just like the free system and go to zero at $T = 0$K. The latter actually is a manifestation of the third law of thermodynamics. But most significantly internal energy and pressure of the trapped Fermi gas do not go to zero as $T = 0$K like it happens for free Fermi gas. According to equation (46), no matter

TABLE I

Dissimilarity between the free and harmonic-potential-trapped nonrelativistic Fermi gases in $d = 3$.

Physical quantity	Free gas	Trapped gas
Fermi energy	$\frac{\hbar^2}{2m}\left(6\pi^2\left(\frac{N}{V}\right)^{\frac{2}{3}}\right)$	$\hbar\omega(3N)^{1/3}$
Fermi temperature	$\frac{\hbar^2}{2mk}\left(6\pi^2\left(\frac{N}{V}\right)^{\frac{2}{3}}\right)$	$\frac{\hbar\omega}{k}(3N)^{1/3}$
Internal energy	$\frac{3}{2}NkT\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}$	$3NkT\frac{f_4(z)}{f_3(z)}$
Internal energy at lower temperature	$\frac{3}{5}NE_{\mathrm{F}}\left(1+\frac{5}{2}\frac{\pi^2}{6}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right)$	$\frac{3}{4}NE_{\mathrm{F}}\left(1+\frac{2\pi^2}{3}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right)$
Ground state energy	$\frac{3}{5}NE_{\mathrm{F}}$	$\frac{3}{4}NE_{\mathrm{F}}$
Ground state pressure	$\frac{2}{5}\frac{N}{V}E_{\mathrm{F}}$	$\frac{1}{4}\frac{N}{V'}E_{\mathrm{F}}$
Particle number at ground state	$\frac{4\pi V}{3h^3}(2mE_{\mathrm{F}})^{3/2}$	$\frac{1}{3}\left(\frac{E_{\mathrm{F}}}{\hbar\omega}\right)^3$
Internal energy at higher temperature	$\frac{3}{2}NkT$	$3NkT$
Isothermal compressibility	$\frac{V}{NkT}\frac{f_{1/2}(z)}{f_{3/2}(z)}$	$\frac{V'}{NkT}\frac{f_2(z)}{f_3(z)}$
Isothermal compressibility at higher temperature	$\frac{1}{P}$	$\frac{1}{P'}$
Isothermal compressibility at lower temperature	$\frac{3V}{2NE_{\mathrm{F}}}\left[1-\frac{\pi^2}{12}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right]$	$\frac{3V}{NE_{\mathrm{F}}}\left[1-\frac{\pi^2}{3}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right]$
Isothermal compressibility at $T = 0\mathrm{K}$	$\frac{3}{2}\frac{V}{NE_{\mathrm{F}}}$	$\frac{3V}{NE_{\mathrm{F}}}$
Chemical potential at lower temperature	$E_{\mathrm{F}}\left(1-\frac{\pi^2}{12}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right)$	$E_{\mathrm{F}}\left(1-\frac{\pi^2}{3}\left(\frac{kT}{E_{\mathrm{F}}}\right)^2\right)$
Chemical potential at $T = 0\mathrm{K}$	E_{F}	E_{F}
Chemical potential at higher temperature	$-kT\ln\left[\frac{3}{4}\sqrt{\pi}\left(\frac{kT}{E_{\mathrm{F}}}\right)^{3/2}\right]$	$-kT\ln\left[6\left(\frac{kT}{E_{\mathrm{F}}}\right)^3\right]$
$\frac{C_V}{Nk}$	$\left(\frac{15}{4}\right)\frac{f_{\frac{5}{2}}(z)}{f_{\frac{3}{2}}(z)}-\left(\frac{9}{4}\right)\frac{f_{\frac{3}{2}}(z)}{f_{\frac{1}{2}}(z)}$	$12\frac{f_4(z)}{f_3(z)}-9\frac{f_3(z)}{f_2(z)}$
$\frac{C_V}{Nk}$ at lower temperature	$\frac{\pi^2}{2}\frac{kT}{E_{\mathrm{F}}}$	$\pi^2\frac{kT}{E_{\mathrm{F}}}$
$\frac{C_V}{Nk}$ at higher temperature	$\frac{3}{2}$	3

what power law exponent one chooses, the ground state pressure never goes to zero. It suggests that the ground state energy and ground state pressure seen here are clearly a quantum effect arising due to the Pauli exclusion principle. Therefore, fermions spread over a lowest available energy state. More interestingly, isothermal compressibility of the Fermi system is nonzero at $T = 0$K, which indicates that the zero point pressure is not merely a constant but depends on volume.

6. Conclusion

From the grand potential, the thermodynamic properties of the Fermi gas trapped under generic power law potential have been evaluated. The calculated physical quantities reduce to expressions available in the literature, with appropriate choice of power law exponents and dimensionality. The thermodynamic quantities are further studied closely in the degeneracy limit. It is found that pressure, energy and isothermal compressibility are nonzero, with any trapping potential, indicating the governing power of the Pauli exclusion principle. In this manuscript, the discussion was restricted to the case of the ideal system under trapping potential. It will be very interesting to explore the effect of interaction in the degeneracy limit.

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