THERMODYNAMICS OF IDEAL BOSE GAS UNDER GENERIC POWER LAW POTENTIAL IN d-DIMENSIONS

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Thermodynamic properties of the ideal Bose gas trapped in an external generic power law potential are investigated systematically from the grand thermodynamic potential in d-dimensional space. In this manuscript, the most general conditions for Bose–Einstein condensate and the discontinuity conditions of heat capacity at the critical temperature in presence of generic power law potential are presented. The dependence of the physical quantities on external potential, particle characteristics and space dimensionality are discussed. The more general results obtained in this paper present an unified illustration of the Bose–Einstein condensation of ideal Bose systems as they reduce to the expressions and conclusions available in the literature with appropriate choice of power law exponent.

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1. Introduction

Many attempts have been made to understand the phenomenon of Bose–Einstein condensation (BEC) after the demonstration by Einstein that there is a possibility of condensation of free bosons [1, 2]. Since then, the bulk behaviour of Bose gas is investigated by many authors [3–13]. An increasing attraction towards this subject is observed, after the achievement to create BEC in magnetically trapped alkali gases [14–16]. As a result, different studies analysing effects of external potentials [17–21], relativistic effect [12, 13, 22], space dimensionality [17, 20, 23], criticality near the transition point [24, 25], Casimir effect [24, 26, 27] of the bosonic system have been performed. Moreover, the recent experimental observation [28] of condensation of massless bosons makes the research area even more challenging.

In a real system, of course, interactions between particles do exist. However, taking them into account makes the problem difficult to solve analytically. Interactions between bosons and their effect on the critical temperature T_c and fraction of condensation $\frac{N_0}{N}$ are understood for the experimental situation, where the contribution of the interaction is low [29]. Thus, it is well approximated that the Bose gas of low density can be treated as an ideal Bose gas. To understand the effect of interactions, we approximately represent the real system by non-interacting particles in the presence of an external potential such as harmonic potential [18], toroidal potential [19], power law potential [17, 21], mean-field potential [26]. The constrains introduced by an external potential influence characteristics of the gases. Thus trapped atomic gases provide the opportunity to manipulate the quantum statistical effects. An excellent study by Salanasich [17] on the Bose gas confined in a trap, described by a symmetric (isotropic) power law potential ($U = Ar^n$) in d-dimensional space, presents a very important relation whether the system will exhibit condensation or not. Salanasich drew a conclusion from the argument of a number of particles that BEC exists if $\frac{d}{2} + \frac{d}{n} > 1$, where n is the exponent of symmetric power law potential. Also, from this relation, one can say that the BEC is possible in the trapped Bose gas under this type of potential in $d = 2$ with appropriate choice of n, which was not the case for the ideal free Bose gas. Thus, it would be interesting to investigate the relation when the potential is not symmetric in general. It should be noted that BEC for free bosons at $d = 3$ is a first order phase transition [7, 8, 10]. But the order of phase transition for the trapped bosons in arbitrary dimensions is yet to be examined.

A lot of effort has been made to understand dimensional dependence of phase transition [3, 10, 12, 13, 17, 20, 23, 30]. Thermodynamic as well as other properties have been calculated in d-dimensions for bosons and fermions [31–33], and interesting conclusions such as equivalence relation between the Bose and Fermi gases in two dimensions have been drawn [34]. In this report, we intend to calculate all the thermodynamic properties of the ideal Bose system in d-dimensions under generic power law potential. To this end, we have first calculated the grand thermodynamic potential. And then, from grand potential, we have determined all the thermodynamic quantities such as internal energy E, entropy S, pressure P, number of particles N, Helmholtz free energy A, isothermal compressibility κ_T, specific heat at constant volume C_V and pressure C_P, their ratio $\gamma = \frac{C_P}{C_V}$. The derived thermodynamic quantities enable us to derive important results regarding the Bose system such as condensate fraction, specific heat jump, latent heat of condensation, critical temperature, equation of state as well as general criteria for existence of BEC for trapped bosons under generic power law potential. Beside this, from the Clausius–Clapeyron equation, it has been shown that, like the ideal free bosons in $d = 3$ [8, 10], BEC for trapped bosons is also a first order phase transition in any dimension. It should be noted that in the Hamiltonian, instead of $\frac{p^2}{2m}$ type kinetic part, we have taken ap^s, where p is momentum and a is constant. Thus, making the important conclusions in more generalized way and taking the results obtained here provide a unified description of behaviour of trapped Bose gases, from which many results in the literature can be derived.

The report is organized in the following way. The density of states and grand potential are calculated in Section 2. Section 3 is devoted to investigating the thermodynamic quantities and checking the order of phase transition. Results and discussion are presented in Section 4. The report is concluded in Section 5.

2. Density of states and grand potential of Bose gas under generic power law potential in d-dimensions

Let us consider the ideal Bose gas in a confining external potential in a d-dimensional space with energy spectrum

$$\epsilon(p, x_i) = bp^l + \sum_{i=1}^{d} c_i \left|\frac{x_i}{a_i}\right|^{n_i}, \tag{1}$$

where p is the momentum and x_i is the i^{th} component of coordinate of a particle and b, l, a_i, c_i, n_i are all positive constants. Note that $x_i < a_i$. Here, c_i, a_i and n_i determine the depth and confinement power of the potential. Using $l = 2$ and $b = \frac{1}{2m}$, one can get the energy spectrum of the Hamiltonian used in the literature [7, 8, 10, 17]. For the free system, all $n_i \longrightarrow \infty$. As $|\frac{x_i}{a_i}| < 1$, the potential term goes to zero when all $n_i \longrightarrow \infty$.

The density of states can be obtained from the following formula:

$$\rho(\epsilon) = \int\int \frac{d^d r d^d p}{(2\pi\hbar)^d} \delta(\epsilon - \epsilon(p, r)). \tag{2}$$

Now, from Eq. (1), the density of states is

$$\rho(\epsilon) = B \frac{\Gamma\left(\frac{d}{l} + 1\right)}{\Gamma\left(\frac{d}{l} + \sum_i \frac{1}{n_i}\right)} \epsilon^{\frac{d}{l} + \sum_i \frac{1}{n_i} - 1}, \tag{3}$$

where

$$B = \frac{g V_d C_d}{h^d a^{d/l}} \prod_{i=1}^{d} \frac{\Gamma\left(\frac{1}{n_i} + 1\right)}{c_i^{\frac{1}{n_i}}}. \tag{4}$$

Here, $C_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$, g is the spin degeneracy factor, $V_d = 2^d \prod_{i=1}^d a_i$ is the volume of a d-dimensional cube, whose $i^{\rm th}$ side has length $2a_i$, and $\Gamma(l) = \int_0^\infty dx x^{l-1} e^{-x}$ is the gamma function.

The grand potential for the Bose system

$$q = -\sum_{\epsilon} \ln(1 - z\exp(-\beta\epsilon)) \,, \tag{5}$$

where $\beta = \frac{1}{kT}$, k is the Boltzmann constant, $z = \exp(\beta\mu)$ is the fugacity and μ the chemical potential. In experiments with trapped gases, thermal energies far exceed the level spacing [15]. So, using the Thomas–Fermi semi-classical approximation [35] and re-writing the previous equation, we get

$$q = q_0 - \int_0^\infty \rho(\epsilon) \ln(1 - z\exp(-\beta\epsilon)) \,. \tag{6}$$

Using the density of states of Eq. (3), we finally get the grand potential

$$q = q_0 + B\Gamma\left(\frac{d}{l} + 1\right)(kT)^\chi g_{\chi+1}(z) \,, \tag{7}$$

where $q_0 = -\ln(1-z)$, $\chi = \frac{d}{l} + \sum_{i=1}^d \frac{1}{n_i}$ and $g_l(z)$ is the Bose function, which is defined as

$$g_l(z) = \int_0^\infty dx \frac{x^{l-1}}{z^{-1}e^x - 1} = \sum_{j=1}^\infty \frac{z^j}{j^l} \,. \tag{8}$$

It should be noted that when $l > 1$, as $z \longrightarrow 1$ (in condensed phase)

$$g_l(1) = \sum_{r=1}^\infty \frac{1}{r^l} = \zeta(l) \,.$$

3. Thermodynamics of Bose gas under generic power law potential in *d*-dimensions

3.1. Number of particles

The number of particles N can be obtained as

$$N = z\left(\frac{\partial q}{\partial z}\right)_{\beta,V} = N_0 + \frac{gC_n\Gamma\left(\frac{d}{l}+1\right)V_d \prod_{i=1}^d \Gamma\left(\frac{1}{n_i}+1\right)}{h^d b^{d/l} \prod_{i=1}^d c_i^{1/n_i}}(kT)^\chi g_\chi(z) \,. \tag{9}$$

Here, $N_0 = \frac{z}{1-z}$ is the ground state occupation number.

Now, we define

$$V_d' = V_d \prod_{i=1}^{d} \left(\frac{kT}{c_i}\right)^{1/n_i} \Gamma\left(\frac{1}{n_i}+1\right), \tag{10}$$

$$\lambda' = \frac{h b^{\frac{1}{l}}}{\pi^{\frac{1}{2}} (kT)^{\frac{1}{l}}} \left[\frac{d/2+1}{d/l+1}\right]^{1/d}. \tag{11}$$

It is noteworthy,

$$\lim_{n_i \to \infty} V_d' = V_d, \tag{12}$$

$$\lim_{n_i \to \infty} \chi = \frac{d}{l}, \tag{13}$$

$$\lim_{l \to 2, b \to \frac{1}{2m}} \lambda' = \lambda = \frac{h}{(2\pi m kT)^{1/2}}. \tag{14}$$

So, if we choose $l = 2$ and $b = \frac{1}{2m}$ from Eq. (14), we get $\lambda_0 = \frac{h}{(2\pi m kT)^{1/2}}$, which is the thermal wavelength of non-relativistic free massive bosons. However, it should be noted that when $l \neq 2$, λ' depends on the dimension. With $d = 3$ and $d = 2$, thermal wavelengths of photons are respectively $\frac{hc}{2\pi^{1/2} kT}$ and $\frac{hc}{(2\pi)^{1/2} kT}$, which can be obtained from Eq. (11) by choosing $b = c$, where c is the velocity of light. So, from the definition of λ' with more general energy spectrum, one can reproduce the thermal wavelength of both massive and massless bosons.

The number of particles equation is then written as

$$N - N_0 = g \frac{V_d'}{\lambda'^d} g_\chi(z). \tag{15}$$

With $l=2$, $a=\frac{1}{2m}$ and all $n_i \longrightarrow \infty$, the number of particles equation for free massive bosons in d-dimensional space can be obtained from Eqs. (12)–(15)

$$N - N_0 = g \frac{V_d}{{\lambda_0}^d} g_{\frac{d}{2}}(z) \tag{16}$$

which is exactly like in Ziff [10], that gives the exact equation for number of particles at $d = 3$ [7, 8].

Now, turning our concentration to find the critical temperature, we know that $T \longrightarrow T_c$, $\mu \longrightarrow 0$ or $z \longrightarrow 1$. Thus, from equation (9), the critical density is

$$N_c = \frac{g C_n \Gamma\left(\frac{d}{l}+1\right) \prod_{i=1}^{d} \Gamma\left(\frac{1}{n_i}+1\right)}{h^d b^{d/l} \prod_{i=1}^{d} c_i^{1/n_i}} (kT)^\chi \zeta(\chi) \tag{17}$$

and the critical temperature is

$$T_{\rm c}=\frac{1}{k}\left[\frac{N_{\rm c}h^d b^{d/l}\prod_{i=1}^{d}c_i^{1/n_i}}{gC_n\Gamma\left(\frac{d}{l}+1\right)V_d\prod_{i=1}^{d}\Gamma\left(\frac{1}{n_i}+1\right)\zeta(\chi)}\right]^{\frac{1}{\chi}}. \tag{18}$$

From equations (17) and (18), we can obtain the fraction of condensation

$$\frac{N_0}{N}=1-\left(\frac{T}{T_{\rm c}}\right)^{\chi}. \tag{19}$$

For free massive bosons, the two above equations become like below, as in Refs. [10, 26]:

$$T_{\rm c} = \frac{h^2}{2\pi mk}\left(\frac{N_{\rm c}/V_d}{\zeta(d/2)}\right)^{\frac{2}{d}}, \tag{20}$$

$$\frac{N_0}{N} = 1-\left(\frac{T}{T_{\rm c}}\right)^{d/2}. \tag{21}$$

From Eq. (20), it is seen that if $d\longrightarrow\infty$, $T_{\rm c}=\frac{h^2}{2\pi mk}$ and we can say that if d approaches infinity, the critical temperature picks a non-zero value, instead of being zero [23]. Also, Eqs. (20) and (21) exactly reproduce the critical temperature and condensate fraction at $d=3$ [7–10].

Now, from equation (15), one can find the most general criterion for BEC to take place. It is seen that it depends on χ. Equation (15) dictates BEC to take place when

$$\chi=\frac{d}{l}+\sum_i^d\frac{1}{n_i}>1 \tag{22}$$

is satisfied. This equation shows that the conditions for BEC to take place depend both on dimension d as well as on kinematic characteristics l and exponent of power law n_i. When the potential is symmetric, *i.e.* $n_1=n_2=\cdots=n_i=\cdots=n_d=n$, Eq. (22) becomes

$$\frac{d}{l}+\frac{d}{n}>1\,. \tag{23}$$

Now, putting $l=2$ for massive bosons in the above equation, we achieve the criterion for BEC to take place in the case of symmetric potential which is the same as in Ref. [17]

$$\frac{d}{2}+\frac{d}{n}>1\,. \tag{24}$$

In the case of free massive bosons, all $n_i \longrightarrow \infty$, so Eq. (24) becomes

$$\frac{d}{2} > 1\,. \tag{25}$$

So, for free massive bosons, we get the usual criterion that d should be greater than 2 for BEC to take place, consistent with the result obtained in Refs. [10, 36].

3.2. Internal energy

From the Grand Canonical Ensemble, internal energy E is

$$\begin{aligned} E &= -\left(\frac{\partial q}{\partial \beta}\right)_{z,V} \\ &= \frac{gC_n\Gamma\left(\frac{d}{l}+1\right)V_d\prod_{i=1}^{d}\Gamma\left(\frac{1}{n_i}+1\right)}{h^d b^{d/l}\prod_{i=1}^{d}c_i^{1/n_i}}(kT)^{\chi+1}g_{\chi+1}(z)\,. \end{aligned} \tag{26}$$

So, the internal energy below and above the critical temperature is

$$E = \begin{cases} NkT\chi\frac{g_{\chi+1}(z)}{g_\chi(z)}\,, & T > T_\mathrm{c}\,, \\ NkT\chi\frac{\zeta(\chi+1)}{\zeta(\chi)}\left(\frac{T}{T_\mathrm{c}}\right)^{\chi}\,, & T \le T_\mathrm{c}\,. \end{cases} \tag{27}$$

In the case of free massive bosons ($l = 2$, $b = \frac{1}{2m}$ and $n_i \longrightarrow \infty$), one can see that Eq. (27) reduces to

$$E = \begin{cases} NkT\frac{d}{2}\frac{g_{d/2+1}(z)}{g_{d/2}(z)}\,, & T > T_\mathrm{c}\,, \\ NkT\frac{d}{2}\frac{\zeta(d/2+1)}{\zeta(d/2)}\left(\frac{T}{T_\mathrm{c}}\right)^{d/2}\,, & T \le T_\mathrm{c}\,, \end{cases} \tag{28}$$

which is in accordance with Ziff [10], also produces the exact expression of E for $d = 3$ [7, 8].

Now as $T \gg T_\mathrm{c}$, from Eq. (27) it is seen that the internal energy becomes $E = NkT\chi$. For free massive bosons, it is $E = \frac{d}{2}NkT$, which becomes $\frac{3}{2}NkT$ when $d = 3$, thus E approaches the classical value at high temperature.

3.3. Entropy

The entropy S can be obtained from the Grand Canonical Ensemble

$$S = kT\left(\frac{\partial q}{\partial T}\right)_{z,V} - Nk\ln z + kq\,.$$

Again, the entropy below and above the critical temperature is

$$S=\begin{cases} Nk\left[\frac{v'_d}{\lambda'^d}(\chi+1)g_{\chi+1}(z)-\ln z\right]\,, & T>T_{\rm c}\,,\\ (N-N_0)k\frac{v'_d}{\lambda'^d}(\chi+1)\zeta(\chi+1)\,, & T\leq T_{\rm c}\,. \end{cases} \tag{29}$$

From the above, it can be seen that $S=0$ as $T\longrightarrow 0$, in accordance with the third law of thermodynamics. As before, for free massive bosons ($l=2$, $b=\frac{1}{2m}$ and $n_i\longrightarrow\infty$), with the help of Eqs. (12)–(14), one can find that Eq. (29) reduces to

$$S=\begin{cases} Nk\left[\frac{v_d}{\lambda^d}\left(\frac{d}{2}+1\right)g_{\frac{d}{2}+1}(z)-\ln z\right]\,, & T>T_{\rm c}\,,\\ (N-N_0)k\frac{v_d}{\lambda^d}\left(\frac{d}{2}+1\right)\zeta\left(\frac{d}{2}+1\right)\,, & T\leq T_{\rm c}\,, \end{cases} \tag{30}$$

which is exactly like in Ziff [10]. Again, at $d=3$ Eq. (30) reduces to the same expression for entropy as in Refs. [7, 8].

3.4. Helmholtz free energy

From the Grand Canonical Ensemble, we get the expression of the Helmholtz free energy

$$A=-kTq+NkT\ln z\,. \tag{31}$$

Now, from the expression of grand potential, we obtain the equation of the Helmholtz free energy below and above $T_{\rm c}$,

$$\frac{A}{NkT}=\begin{cases} -\frac{g_{\chi+1}(z)}{g_\chi(z)}+\ln z\,, & T>T_{\rm c}\,,\\ -\frac{\nu}{\lambda^d}\zeta(\chi+1)\,, & T\leq T_{\rm c}\,. \end{cases} \tag{32}$$

In the case of free massive bosons, the above expression reduces as below

$$\frac{A}{NkT}=\begin{cases} -\frac{g_{\frac{d}{2}+1}(z)}{g_{\frac{d}{2}}(z)}+\ln z\,, & T>T_{\rm c}\,,\\ -\frac{\nu}{\lambda^d}\zeta\left(\frac{d}{2}+1\right)\,, & T\leq T_{\rm c}\,, \end{cases} \tag{33}$$

exactly like in Ziff [10]. Now, for $d=3$, the above equation produces the exact expression for the Helmholtz free energy [7, 8].

3.5. Pressure

Re-writing equation (9) stating the number of particles, we get

$$\frac{N-N_0}{V_d\prod_{i=1}^d\left(\frac{kT}{c_i}\right)^{1/n_i}\Gamma\left(\frac{1}{n_i}+1\right)}=\frac{N-N_0}{V'_d}=\frac{g}{\lambda'^d}g_\chi(z)\,.$$

Now, we take a very well-known expression for the non-relativistic d-dimensional ideal free Bose gas [10]

$$\frac{N-N_0}{V_d} = \frac{g}{\lambda_0^d} g_{d/2}(z)\,.$$

Comparing the above equations, we can say that V'_d is a more generalized extension of V_d. It represents the effect of external potential on the performance of trapped bosons. Calling V'_d the effective volume, the grand potential can be re-written as

$$q = q_0 + g\frac{V'_d}{\lambda'^d} g_{\chi+1}(z)\,. \tag{34}$$

So, the effective pressure is

$$P' = \frac{1}{\beta}\left(\frac{\partial q}{\partial V'_d}\right) = \begin{cases} \frac{gkT}{\lambda'^d} g_{\chi+1}(z)\,, & T > T_{\mathrm{c}}\,, \\ \frac{gkT}{\lambda'^d}\zeta(\chi+1)\,, & T \le T_{\mathrm{c}}\,. \end{cases} \tag{35}$$

It can be also re-written as

$$P' = \frac{1}{\beta}\left(\frac{\partial q}{\partial V'_d}\right) = \begin{cases} \frac{NkT}{V'_d}\frac{g_{\chi+1}(z)}{g_{\chi}(z)}\,, & T > T_{\mathrm{c}}\,, \\ \frac{(N-N_0)}{V'_d} kT \frac{\zeta(\chi+1)}{\zeta(\chi)}\,, & T \le T_{\mathrm{c}}\,. \end{cases} \tag{36}$$

The above equation is a very general equation of state for any dimensionality d, any dispersion relation of the form of ($\propto p^s$) having any form of generic power law trap and, obviously, it is expected that it will reproduce the special case of free system. For the free system, equation (36) becomes

$$P = \frac{1}{\beta}\left(\frac{\partial q}{\partial V_d}\right) = \begin{cases} \frac{NkT}{V_d}\frac{g_{d/2+1}(z)}{g_{\frac{d}{2}}(z)}\,, & T > T_{\mathrm{c}}\,, \\ \frac{(N-N_0)}{V_d} kT \frac{\zeta\left(\frac{d}{2}+1\right)}{\zeta\left(\frac{d}{2}\right)}\,, & T \le T_{\mathrm{c}}\,, \end{cases} \tag{37}$$

which is in accordance with Ziff.

Now, comparing Eqs. (35) and (27), we get

$$P'V'_d = \frac{E}{\chi}\,. \tag{38}$$

For d-dimensional free Bose gas, one can obtain from the previous equation

$$PV_d = \frac{2}{d}E\,. \tag{39}$$

This is an important and familiar relation, $PV = \frac{2}{3}E$ when $d = 3$ [7–10]. This actually shows that equation (38) is a very significant relation for the Bose gas irrespective of whether the bosons are trapped or free. In the case of trapped bosons, the effective volume and pressure play the same role as volume and pressure in the current textbooks and literature.

3.6. Heat capacity at constant volume

Heat capacity at constant volume C_V below and above T_c can be written as

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_{N,V} = \begin{cases} Nk\left[\chi(\chi+1)\frac{g_{\chi+1}(z)}{g_\chi(z)} - \chi^2\frac{g_\chi(z)}{g_{\chi-1}(z)}\right], & T > T_\mathrm{c}\,, \\ Nk\chi(\chi+1)\frac{\zeta(\chi+1)}{\zeta(\chi)}\left(\frac{T}{T_\mathrm{c}}\right)^\chi, & T \leq T_\mathrm{c}\,. \end{cases} \tag{40}$$

From this, we can investigate whether the specific heat will show a jump or not. From Eq. (40), we can obtain the difference between the heat capacities at constant volume, at T_c as

$$\Delta C_V\mid_{T=T_\mathrm{c}} = C_V\mid_{T_\mathrm{c}^-} - C_V\mid_{T_\mathrm{c}^+} = Nk\chi^2\frac{g_\chi(1)}{g_{\chi-1}(1)}\,. \tag{41}$$

The above equation shows that the jump of heat capacity depends on χ. C_V will be discontinuous at $T = T_\mathrm{c}$ if

$$\chi = \frac{d}{l} + \sum_{i=1}^{d}\frac{1}{n_i} > 2 \tag{42}$$

is satisfied and C_V will be continuous at $T = T_\mathrm{c}$ if χ satisfies

$$1 < \chi \leq 2\,. \tag{43}$$

So, in the case of free massive bosons (choosing $l = 2$ and all $n_i \longrightarrow \infty$), C_V will be discontinuous for $d > 4$, in agreement with Ziff [10]. In the high temperature limit, C_V approaches its classical value as it becomes χNk for trapped system and $\frac{d}{2}Nk$ for free system, which is $\frac{3}{2}Nk$, when $d = 3$.

3.7. Isothermal compressibility

The isothermal compressibility of trapped Bose gas can be obtained as

$$\kappa_T = -V\left(\frac{\partial V'}{\partial P'}\right)_{N,T}\,. \tag{44}$$

Now, using the simple chain rule of partial derivative, the above equation becomes

$$\left(\frac{\partial V'}{\partial P'}\right)_{N,T} = \left(\frac{\partial V'}{\partial z}\right)_{N,T}\left(\frac{\partial z}{\partial P'}\right)_{N,T}\,. \tag{45}$$

Using equations (15) and (35), we get

$$\left(\frac{\partial P}{\partial V}\right)_{N,T} = \left(\frac{-1}{NkT}\right)\frac{g_{\chi-1}(z)}{g_\chi(z)} \tag{46}$$

which concludes

$$\kappa_T = \begin{cases} \frac{V}{NkT}\frac{g_{\chi-1}(z)}{g_\chi(z)}\,, & T > T_\mathrm{c}\,, \\ \frac{V}{NkT}\frac{\zeta(\chi-1)}{\zeta(\chi)}\,, & T \le T_\mathrm{c}\,. \end{cases} \tag{47}$$

It reproduces the same result for isothermal compressibility of the free massive Bose gas at $d = 3$ [7]. In the high temperature limit, κ_T takes the classical value for free system which is $\frac{1}{P}$.

3.8. Heat capacity at constant pressure

Now, the heat capacity at constant pressure C_P is

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_{N,P}$$
$$= \begin{cases} Nk\left[(\chi+1)^2\frac{g^2_{\chi+1}(z)g_{\chi-1}(z)}{g^3_\chi(z)} - \chi(\chi+1)\frac{g_{\chi+1}(z)}{g_\chi(z)}\right], & T > T_\mathrm{c}\,, \\ Nk\left[(\chi+1)^2\zeta(\chi+1)\zeta(\chi-1)\left(\frac{\nu'}{\lambda'^D}\right)^3 - \chi(\chi+1)\zeta(\chi+1)\frac{\nu'}{\lambda'^D}\right], & T \le T_\mathrm{c}\,. \end{cases} \tag{48}$$

In the case of free massive Bose gas in $d = 3$, it coincides exactly with Refs. [7, 9], which diverges as $\frac{1}{T-T_\mathrm{c}}$ when the temperature approaches T_c from above. In the condensed state for $d = 3$, $C_P \propto T^{5/2}$. So, keeping P constant, it suggests that $dT = 0$. The temperature will not change, no matter how much heat will enter the system, meaning the volume will grow. Again, in the high temperature limit, C_P becomes $(\chi+1)Nk$ for the trapped system and $(\frac{d}{2}+1)Nk$ for free system, which is $\frac{5}{2}Nk$, when $d = 3$. So, in the high temperature limit, C_P approaches its classical value.

Now, the ratio $\gamma = \left(\frac{C_P}{C_V}\right)$ for $T > T_\mathrm{c}$ is given by

$$\gamma = \frac{(\chi+1)^2\frac{g^2_{\chi+1}(z)g_{\chi-1}(z)}{g^3_\chi(z)} - \chi(\chi+1)\frac{g_{\chi+1}(z)}{g_\chi(z)}}{\chi(\chi+1)\frac{g_{\chi+1}(z)}{g_\chi(z)} - \chi^2\frac{g_\chi(z)}{g_{\chi-1}(z)}}\,. \tag{49}$$

For $T \gg T_c$, the above equation becomes

$$\gamma = \frac{(\chi+1)^2 - \chi(\chi+1)}{\chi(\chi+1) - \chi^2} = 1 + \frac{1}{\chi}\,. \tag{50}$$

In the case of free system, choosing all $n_i \longrightarrow \infty$, we get from the above equation

$$\gamma = 1 + \frac{l}{d}\,. \tag{51}$$

Thus, γ equals $\frac{5}{3}$, when $d = 3$ and $l = 2$, which is the classical value at high temperature limit.

3.9. Clausius–Clapeyron equation

In any first order phase transition, pressure is governed by the Clausius–Clapeyron equation, in the transition line [8, 10]. Like BEC for free Bose gas at $d = 3$, BEC for trapped Bose gas will also be a first order phase transition if they obey the Clapeyron equation. The Clapeyron equation derived from Maxwell's relations is

$$\frac{dP}{dT} = \frac{\Delta S}{\Delta V} = \frac{L}{T\Delta V}\,, \tag{52}$$

where L, ΔS and ΔV are the latent heat, change in entropy and change in volume respectively. The effective pressure in phase transition line is

$$P_0(T) = \frac{kT}{\lambda'^d} g_{\chi+1}(1)\,. \tag{53}$$

Differentiating with respect to T leads to

$$\frac{dP_0}{dT} = \frac{1}{TV_g}\left[(\chi+1)kT\frac{g_{\chi+1}(1)}{g_\chi(1)}\right]\,. \tag{54}$$

Now, when two phases coexist, the non-condensed phase has specific volume V_g whereas the condensed phase has specific volume 0, which concludes $\Delta V = V_g$. This concludes the above equation to be Clausius–Clapeyron equation where the latent heat of transition per particle in the case of trapped bosons is

$$L = \frac{g_{\chi+1}(1)}{g_\chi(1)}(\chi+1)kT\,. \tag{55}$$

Therefore, BEC for trapped bosons is also a first order phase transition in arbitrary dimensions. For free massive bosons in three-dimensional space, the latent heat per particle becomes

$$L = \frac{g_{5/2}(1)}{g_{3/2}(1)}\left(\frac{5}{2}\right)kT\,, \tag{56}$$

which agrees exactly with Ref. [8].

4. Discussion

Properties of the ideal Bose gas in the presence of an external generic power law potential have been discussed in this section. The study done by density of states approach in d-dimensional space allows us to investigate the two phase system that is both in the condensed and non-condensed phase across the phase transition point. The most general conditions for BEC as well as the continuity conditions of the specific heat for ideal Bose gas trapped in generic power law potential can be obtained with any kinematic characteristics in d-dimensional space.

At first, the density of states and grand potential of the Bose gas under generic power law potential in d-dimensions has been calculated from which we can easily derive the expression for number of particles. It gives us the first insight into the condensed phase. Using number of particles equation, the general criterion for BEC has been obtained from which one can reach the same conclusion for symmetric potential [17], harmonic potential [18] as well as more simplified system of free massive bosons [7, 8, 10]. The fraction of condensate as well as the critical temperature was achieved in this process, predicting the exact equation for free system [10]. Now, we turn our attention to thermodynamic quantities such as internal energy E, entropy S, free energy A all of which are evaluated from the grand potential. The obtained expressions coincide with available literature when the potential is symmetric ($n_1 = n_2 = \cdots = n_i = \cdots = n_d$) [17], harmonic ($n_1 = n_2 = \cdots = n_i = \cdots = n_d = 2$) [18] or the system is free (all $n_i \longrightarrow \infty$) [10]. Beside this, from the pressure equation, we have derived the equation of state which determines a relation among the thermodynamic generalized coordinate y and the thermodynamic generalized force Y, and temperature T of the system. Point to note in the case of trapped bosons, pressure P is replaced by effective pressure P' and volume V is replaced by effective volume V'. It is also seen that the high temperature limit of Bose gas is just a classical system obeying the Maxwell–Boltzmann distribution. Therefore, the idea of effective volume and pressure is not only valid for trapped Bose gas but also for trapped classical system and trapped Fermi gas. For free systems, both effective pressure and effective volume reduce to pressure and volume, respectively. Thus, we can conclude the effective volume and pressure to be of a more general notion for trapped systems, which enables us to treat the trapped atomic gases as free ones. The other derived quantities for trapped system such as specific heat at constant volume C_V, specific heat at constant pressure C_P, isothermal compressibility κ_T can be reproduced in their familiar forms found in literature [7–10] in the case of d-dimensional system [9, 10] as well as in $d = 3$ [7, 8].

Now, let us turn our focus towards general criterion of BEC Eq. (22) and the jump of C_V at $T = T_c$ Eq. (42). In general, it has been observed that there is no first order [7, 8, 10] as well as second order [37] phase transition for $d \leqslant 2$ for free system. Equation (25) derived from the general criterion Eq. (22) also suggests the same for the free ideal Bose gas. Turning our attention to the trapped system, we consider a one-dimensional system with $n_1 = 2$ and $l = 2$. Equation (22) then dictates, there is no BEC for such system as $\chi < 1$. In the case of two-dimensional Bose gas ($d = 2$), with $n_1 = n_2 = 2$ (*i.e.* harmonic potential), we can find it fulfills Eq. (22), so BEC exists in such system. That means we have seen the case when BEC can exist for $d \leqslant 2$, with an appropriate choice of power law exponent, which was not the case for the ideal free Bose gas. In the case of jump of C_V in the free Bose system, it is seen that there is a jump of C_V at $T = T_c$, when $d > 4$ [10]. For instance, taking $d = 3$, $l = 2$ and $n_i \longrightarrow \infty$, we can see that there is no discontinuity at $d = 3$ for the free system as $\chi < 2$. But again, one can obtain a jump in C_V at $T = T_c$ for the trapped system when $d \leqslant 4$ with an appropriate choice of n_i. For example, taking $d = 3$ and $l = 2$ with $n_1 = n_2 = n_3 = 2$, one can obtain from Eq. (42) that $\chi > 2$. So, in this case, we can obtain a jump at $d = 3$ in C_V at $T = T_c$ for trapped system, which is, however, not a property for free Bose gas. Neveretheless, Eqs. (40) and (41) show that although $C_{T_c^+}$ is not a monotonical function of χ, both $C_{T_c^-}$ and ΔC_{T_c} increase monotonically as the parameter χ increases. This indicates that, in general, in the vicinity of T_c, the larger the value of χ, the larger the energy needed to raise the Bose system to a higher temperature state, particularly when the system is in condensed phase. On the other hand, when the temperature is very low, $C_{V(T<T_c)}$ decreases as χ increases and the larger the value of χ, the more quickly $C_{V(T<T_c)}$ tends to zero. Also, larger value of χ indicates larger fraction of bosons to be in the ground state, such that lower number of bosons is excited when temperature raises.

The Clausius–Clapeyron equation applies to any first order phase transition. From the expression of effective pressure, it is shown that trapped bosons in arbitrary number of dimensions satisfy the Clausius–Clapeyron equation, from which we can say that BEC is a first order phase transition with or without the existence of external potential. The latent heat of transition per particle in the case of trapped system can also be obtained from this equation, which reduces to exact expression of latent heat in the case of free massive bosons at $d = 3$ [8].

5. Conclusion

From complete thermodynamics of the ideal Bose gas trapped in generic power law potential, we have derived the general criterion for BEC. Also, the calculated physical quantities reduce to expressions available in the literature with appropriate choice of power law exponent. In this manuscript, we have restricted our discussion to the case of ideal system. It will be very interesting to see the effect of interaction on the general criterion of BEC.

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