NEUTRINO-ASSISTED FERMION–BOSON TRANSITIONS

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We study fermion–boson transitions. Our approach is based on the 3×3 subequations of Dirac and Duffin–Kemmer–Petiau equations, which link these equations. We demonstrate that free Dirac equation can be invertibly converted to spin-0 Duffin–Kemmer–Petiau equation in the presence of a neutrino field. We also show that in special external fields, upon assuming again existence of a neutrino (Weyl) spinor, the Dirac equation can be transformed reversibly to spin-0 Duffin–Kemmer–Petiau equation. We argue that such boson–fermions transitions are consistent with the main channel of pion decay.

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1. Introduction

There are many ideas connected with fermion–boson (FB) analogies in the literature. For example, there is FB equivalence, FB duality, FB transmutations, to name a few. There are also intermediate statistics — parastatistics and anyons. There is, finally, supersymmetry. More on these ideas can be found in Refs. [1–4], see also [5–8] and [9]. It seems, however, that a broader and unifying picture is still missing.

Important step in understanding FB analogy was made by Polyakov who discovered possibility of fermion–boson transmutation of elementary excitations of a scalar field interacting with the topological Chern–Simons term in $(2+1)$ dimensions [10]. Recently, the smooth and controlled evolution from a fermionic Bardeen–Cooper–Schrieffer (BCS) superfluid state to a molecular Bose–Einstein condensate (BEC) has been realized in ultracold Fermi gases [11]. On the other hand, we have shown recently that solutions of the Dirac equation can be transformed in the non-interacting case, assuming the existence of a constant spinor, to solutions of the spin-0 Duffin–Kemmer–Petiau (DKP) equation and *vice versa* [9]. Possible analogy between BCS–BEC transition and our findings is motivation of our work.

In the present paper, we generalize results of [9] in two directions. Firstly, we generalize the fermion–boson transformation connecting solutions of free Dirac and spin-0 DKP equations in presence of Weyl spinor. Secondly, we construct analogous transformation in presence of external fields.

The paper is organized as follows. In the next section, we review 3×3 subequations of the Dirac and the spin-0 DKP equations in special external fields and we raise the problem of their covariance. Covariance of these equations was established in [13], while the problem of covariance of their solutions was solved, to some extent, in Ref. [9].

New results are described in Sections 3 and 4. In Section 3, we start with the free Dirac equation and, assuming existence of a Weyl spinor, we derive the spin-0 DKP equation, improving our construction described in [9]. It is suggested that the mechanism of boson to massive fermion and massless neutrino transition is related to pion decay. In Section 4, we show that in external longitudinal fields, upon assuming again existence of a Weyl spinor, the Dirac equation can be transformed to a set of two 3×3 equations in longitudinal fields, similar, but not identical, to equations derived in [13]. Finally, we show that if we switch over to crossed fields we arrive at 3×3 equations which are equivalent to the spin-0 DKP equations. We discuss our results in the last section with special emphasis on pion decay. In what follows, we use notation and conventions described in [12, 13].

2. Subequations of Dirac and DKP equations in external fields

The Dirac equation in external field can be written in spinor notation as [14]

$$\left.\begin{array}{rcl} \pi^{A\dot{B}}\eta_{\dot{B}} & = & m\xi^{A} \\ \pi_{A\dot{B}}\xi^{A} & = & m\eta_{\dot{B}} \end{array}\right\}, \tag{2.1}$$

where $\pi^{A\dot{B}}$ is defined as $\pi^{A\dot{B}} = \left(\sigma^0\pi^0 + \overrightarrow{\sigma}\cdot\overrightarrow{\pi}\right)^{A\dot{B}}$, $\pi^{\mu} = p^{\mu} - qA^{\mu}$, and $\pi_{1\dot{1}} = \pi^{2\dot{2}}$, $\pi_{1\dot{2}} = -\pi^{2\dot{1}}$, $\pi_{2\dot{1}} = -\pi^{1\dot{2}}$, $\pi_{2\dot{2}} = \pi^{1\dot{1}}$. Equation (2.1) can be written in the spinor representation of γ matrices as $\gamma^{\mu}\pi_{\mu}\Psi = m\Psi$, where $\Psi = \left(\xi^1, \xi^2, \eta_{\dot{1}}, \eta_{\dot{2}}\right)^T$.

In longitudinal fields [15], Eq. (2.1) can be splitted into two 3×3 sub-equations [13]

$$\left.\begin{array}{rcl} \pi^{1\dot{1}}\eta_{\dot{1}} & = & m\psi^{1\dot{1}}_{\dot{1}} \\ \pi^{2\dot{1}}\eta_{\dot{1}} & = & m\psi^{2\dot{1}}_{\dot{1}} \\ \pi^{2\dot{2}}\psi^{1\dot{1}}_{\dot{1}} - \pi^{1\dot{2}}\psi^{2\dot{1}}_{\dot{1}} & = & m\eta_{\dot{1}} \end{array}\right\} , \tag{2.2}$$

$$\left.\begin{array}{rcl} \pi^{1\dot{2}}\eta_{\dot{2}} & = & m\psi^{1\dot{2}}_{\dot{2}} \\ \pi^{2\dot{2}}\eta_{\dot{2}} & = & m\psi^{2\dot{2}}_{\dot{2}} \\ -\pi^{2\dot{1}}\psi^{1\dot{2}}_{\dot{2}} + \pi^{1\dot{1}}\psi^{2\dot{2}}_{\dot{2}} & = & m\eta_{\dot{2}} \end{array}\right\} . \tag{2.3}$$

Each of these equations can be written in covariant form [12, 13] yet some components of spinor $\psi^{A\dot{B}}_{\dot{C}}$ are missing. This problem, mentioned in Introduction, will be solved in the next section.

The DKP equation in the interacting case can be written within spinor formalism as

$$\left.\begin{array}{rcl} \pi^{A\dot{B}}\psi & = & m\psi^{A\dot{B}} \\ \pi_{A\dot{B}}\psi^{A\dot{B}} & = & 2m\psi \end{array}\right\} . \tag{2.4}$$

In crossed fields [15], we can write Eq. (2.4) as a set of two equations [13]

$$\left.\begin{array}{rcl} \pi^{1\dot{1}}\psi & = & m\psi^{1\dot{1}} \\ \pi^{2\dot{1}}\psi & = & m\psi^{2\dot{1}} \\ \pi_{1\dot{1}}\psi^{1\dot{1}} + \pi_{2\dot{1}}\psi^{2\dot{1}} & = & m\psi \end{array}\right\} , \tag{2.5}$$

$$\left.\begin{array}{rcl} \pi^{1\dot{2}}\psi & = & m\psi^{1\dot{2}} \\ \pi^{2\dot{2}}\psi & = & m\psi^{2\dot{2}} \\ \pi_{1\dot{2}}\psi^{1\dot{2}} + \pi_{2\dot{2}}\psi^{2\dot{2}} & = & m\psi \end{array}\right\} , \tag{2.6}$$

each of which describes particle with mass m. Equation (2.4) and the set of two equations (2.5), (2.6) are equivalent.

The 3×3 equations (2.5), (2.6) and (2.2), (2.3) are similar. However, they differ in spinor contents and Eqs. (2.2), (2.3) involve longitudinal fields, while Eqs. (2.5), (2.6) correspond to crossed fields.

3. Generalized fermion–boson transition in non-interacting case

The problem of missing components of spinor $\psi^{A\dot{B}}_{\dot{C}}$, mentioned in the previous section, is rather serious because it means that theory is not fully covariant. The problem was solved in Ref. [9] assuming that $\eta_{\dot{B}}(x) = \hat{\alpha}_{\dot{B}}\,\chi(x)$, where $\hat{\alpha}_{\dot{B}}$ was a constant spinor. This ansatz for $\eta_{\dot{B}}(x)$ leads, however, to difficulties of another kind since constant spinors do not appear in nature (although constant Grassman spinors are postulated in some variants of supersymmetrical theories). To solve both problems at once, we make a more general assumption

$$\eta_{\dot{B}}\left(x\right)=\alpha_{\dot{B}}\left(x\right)\chi\left(x\right)\,, \tag{3.1}$$

where $\alpha_{\dot{B}}\left(x\right)$ is a two-component neutrino spinor, *i.e.* it fulfills the Weyl equation, $p^{A\dot{B}}\alpha_{\dot{B}}\left(x\right)=0$. We note that $p^{A\dot{B}}\alpha_{\dot{B}}\chi=\chi\, p^{A\dot{B}}\alpha_{\dot{B}}+\alpha_{\dot{B}}\, p^{A\dot{B}}\chi=\alpha_{\dot{B}}\, p^{A\dot{B}}\chi$ and thus we can rewrite the free Dirac equation as

$$\left.\begin{array}{rcl}\alpha_{\dot{B}}p^{A\dot{B}}\chi & = & m\xi^{A}\\ p_{A\dot{B}}\xi^{A} & = & m\alpha_{\dot{B}}\chi\end{array}\right\}. \tag{3.2}$$

Equation (3.2) can be further written as

$$\left.\begin{array}{rcl}\alpha_{\dot{1}}p^{1\dot{1}}\chi & = & m\psi_{\dot{1}}^{1\dot{1}}\\ \alpha_{\dot{2}}p^{1\dot{2}}\chi & = & m\psi_{\dot{2}}^{1\dot{2}}\\ \alpha_{\dot{1}}p^{2\dot{1}}\chi & = & m\psi_{\dot{1}}^{2\dot{1}}\\ \alpha_{\dot{2}}p^{2\dot{2}}\chi & = & m\psi_{\dot{2}}^{2\dot{2}}\\ \left(p_{1\dot{1}}\psi_{\dot{1}}^{1\dot{1}}+p_{2\dot{1}}\psi_{\dot{1}}^{2\dot{1}}\right)+\left(p_{1\dot{1}}\psi_{\dot{2}}^{1\dot{2}}+p_{2\dot{1}}\psi_{\dot{2}}^{2\dot{2}}\right) & = & m\alpha_{\dot{1}}\chi\\ \left(p_{1\dot{2}}\psi_{\dot{1}}^{1\dot{1}}+p_{2\dot{2}}\psi_{\dot{1}}^{2\dot{1}}\right)+\left(p_{1\dot{2}}\psi_{\dot{2}}^{1\dot{2}}+p_{2\dot{2}}\psi_{\dot{2}}^{2\dot{2}}\right) & = & m\alpha_{\dot{2}}\chi\end{array}\right\}, \tag{3.3}$$

where

$$\left.\begin{array}{rcl}\psi_{\dot{1}}^{1\dot{1}}+\psi_{\dot{2}}^{1\dot{2}} & = & \xi^{1}\\ \psi_{\dot{1}}^{2\dot{1}}+\psi_{\dot{2}}^{2\dot{2}} & = & \xi^{2}\end{array}\right\}. \tag{3.4}$$

To reduce number of spinor components, we demand that

$$\psi_{\dot{B}}^{C\dot{D}}=\alpha_{\dot{B}}\left(x\right)\chi^{C\dot{D}}\left(x\right)\,, \tag{3.5}$$

with the same spinor $\alpha_{\dot{B}}\left(x\right)$, fulfilling the Weyl equation $p^{A\dot{B}}\alpha_{\dot{B}}\left(x\right)=0$. Substituting (3.5) into Eqs. (3.3), we obtain

$$\left.\begin{array}{rcl}\alpha_{\dot{1}}p^{1\dot{1}}\chi & = & m\alpha_{\dot{1}}\chi^{1\dot{1}}\\ \alpha_{\dot{1}}p^{2\dot{1}}\chi & = & m\alpha_{\dot{1}}\chi^{2\dot{1}}\\ \left(p_{1\dot{1}}\alpha_{\dot{1}}\chi^{1\dot{1}}+p_{2\dot{1}}\alpha_{\dot{1}}\chi^{2\dot{1}}\right)+\left[p_{1\dot{1}}\alpha_{\dot{2}}\chi^{1\dot{2}}+p_{2\dot{1}}\alpha_{\dot{2}}\chi^{2\dot{2}}\right] & = & m\alpha_{\dot{1}}\chi\\ \alpha_{\dot{2}}p^{1\dot{2}}\chi & = & m\alpha_{\dot{2}}\chi^{1\dot{2}}\\ \alpha_{\dot{2}}p^{2\dot{2}}\chi & = & m\alpha_{\dot{2}}\chi^{2\dot{2}}\\ \left[p_{1\dot{2}}\alpha_{\dot{1}}\chi^{1\dot{1}}+p_{2\dot{2}}\alpha_{\dot{1}}\chi^{2\dot{1}}\right]+\left(p_{1\dot{2}}\alpha_{\dot{2}}\chi^{1\dot{2}}+p_{2\dot{2}}\alpha_{\dot{2}}\chi^{2\dot{2}}\right) & = & m\alpha_{\dot{2}}\chi\end{array}\right\}. \tag{3.6}$$

Since the Weyl spinor $\alpha_{\dot{B}}(x)$ is arbitrary, equations $p^{A\dot{B}}\chi = m\chi^{A\dot{B}}$, defining components of $\chi^{A\dot{B}}$, follow immediately. We can thus remove spinor components $\alpha_{\dot{1}}$, $\alpha_{\dot{2}}$ from equations defining $\chi^{A\dot{B}}$.

We have shown that for constant spinor, $\alpha_{\dot{B}}(x) = \hat{\alpha}_{\dot{B}}$, the system of equations (3.6) splits into two 3×3 equations [9]. We are going to find conditions enabling similar splitting for the Weyl spinor $\alpha_{\dot{B}}(x)$.

We write the first and the second term in square brackets, [1] and [2], respectively, in the form

$$[1] = \chi^{1\dot{2}} p_{1\dot{1}} \alpha_{\dot{2}} + \chi^{2\dot{2}} p_{2\dot{1}} \alpha_{\dot{2}} + \tfrac{1}{m} \alpha_{\dot{2}} \left(p_{1\dot{1}} p^{1\dot{2}} + p_{2\dot{1}} p^{2\dot{2}} \right) \chi \,, \tag{3.7a}$$

$$[2] = \chi^{1\dot{1}} p_{1\dot{2}} \alpha_{\dot{1}} + \chi^{2\dot{1}} p_{2\dot{2}} \alpha_{\dot{1}} + \tfrac{1}{m} \alpha_{\dot{1}} \left(p_{1\dot{2}} p^{1\dot{1}} + p_{2\dot{2}} p^{2\dot{1}} \right) \chi \,, \tag{3.7b}$$

where equations $p^{A\dot{B}}\chi = m\chi^{A\dot{B}}$ have been used.

Since both terms in (3.7), proportional to $\frac{1}{m}$, vanish identically we can decouple equations (3.6) obtaining

$$\left.\begin{array}{rcl} p^{1\dot{1}}\chi & = & m\chi^{1\dot{1}} \\ p^{2\dot{1}}\chi & = & m\chi^{2\dot{1}} \\ p_{1\dot{1}}\chi^{1\dot{1}} + p_{2\dot{1}}\chi^{2\dot{1}} & = & m\chi \end{array}\right\} , \tag{3.8}$$

$$\left.\begin{array}{rcl} p^{1\dot{2}}\chi & = & m\chi^{1\dot{2}} \\ p^{2\dot{2}}\chi & = & m\chi^{2\dot{2}} \\ p_{1\dot{2}}\chi^{1\dot{2}} + p_{2\dot{2}}\chi^{2\dot{2}} & = & m\chi \end{array}\right\} , \tag{3.9}$$

where unnecessary components $\alpha_{\dot{1}}$, $\alpha_{\dot{2}}$ have been removed, provided that the following equations are fulfilled

$$\left.\begin{array}{l} \left(\chi^{1\dot{1}} p_{1\dot{1}} \alpha_{\dot{1}} + \chi^{2\dot{1}} p_{2\dot{1}} \alpha_{\dot{1}}\right) + \left(\chi^{1\dot{2}} p_{1\dot{1}} \alpha_{\dot{2}} + \chi^{2\dot{2}} p_{2\dot{1}} \alpha_{\dot{2}}\right) = 0 \\ \left(\chi^{1\dot{1}} p_{1\dot{2}} \alpha_{\dot{1}} + \chi^{2\dot{1}} p_{2\dot{2}} \alpha_{\dot{1}}\right) + \left(\chi^{1\dot{2}} p_{1\dot{2}} \alpha_{\dot{2}} + \chi^{2\dot{2}} p_{2\dot{2}} \alpha_{\dot{2}}\right) = 0 \end{array}\right\} . \tag{3.10}$$

Taking into account the form of solutions of the Weyl equation, $\alpha_{\dot{B}}(x) = \hat{\alpha}_{\dot{B}} e^{ik\cdot x}$, where $\hat{\alpha}_{\dot{B}}$ is a constant spinor and $k^\mu k_\mu = 0$, we rewrite Eqs. (3.10) in the form

$$\left.\begin{array}{l} k_{1\dot{1}}\varphi^1 + k_{2\dot{1}}\varphi^2 = 0 \\ k_{1\dot{2}}\varphi^1 + k_{2\dot{2}}\varphi^2 = 0 \end{array}\right\} , \tag{3.11}$$

where

$$\left.\begin{array}{l} \varphi^1 = \hat{\alpha}_{\dot{1}}\chi^{1\dot{1}} + \hat{\alpha}_{\dot{2}}\chi^{1\dot{2}} \\ \varphi^2 = \hat{\alpha}_{\dot{1}}\chi^{2\dot{1}} + \hat{\alpha}_{\dot{2}}\chi^{2\dot{2}} \end{array}\right\} . \tag{3.12}$$

Non-zero solutions φ^1, φ^2 are possible if determinant of the system of equations (3.11) is zero. The determinant, $k_{1\dot{1}}k_{2\dot{2}} - k_{1\dot{2}}k_{2\dot{1}} = k^\mu k_\mu$, vanishes in two physically distinct cases: for $k^\mu = 0$ and for $k^\mu k_\mu = 0$. In the first case, the spinor $\alpha_{\dot{B}}(x)$ is constant, $\alpha_{\dot{B}}(x) = \hat{\alpha}_{\dot{B}}$, and no restrictions are imposed on φ^1, φ^2. The fermion–boson transformation in presence of a constant spinor was investigated in the non-interacting case in [9].

We consider now the second possibility. Since we have assumed that $\alpha_{\dot{B}}(x) = \hat{\alpha}_{\dot{B}}e^{ik\cdot x}$ is a solution of the Weyl equation, the condition $k^\mu k_\mu = 0$ is fulfilled. Moreover, equations (3.8), (3.9) are the set of two 3×3 equations equivalent to the spin-0 DKP equation. Therefore, χ fulfills the Klein–Gordon equation. Thus $\chi(x) = Ce^{il\cdot x}$, where l^μ is a four-vector, $l^\mu l_\mu = m^2$.

Since the Weyl equation as well as the set of equations (3.8), (3.9) are covariant, we can consider special reference frames. In a frame $k^\mu = (1,0,0,-1)$, $l^\mu = (m,0,0,0)$, we have $\hat{\alpha}_{\dot{1}} = 0$, $\chi^{1\dot{2}} = 0$, $\varphi^1 = 0$, $k_{2\dot{2}} = 0$ and it follows that equations (3.11), (3.12) are fulfilled.

We have thus described invertible transition from the free Dirac equation for a spin-$\frac{1}{2}$ fermion, in presence of a massless spin-$\frac{1}{2}$ fermion, described by a dotted Weyl spinor, to the free DKP equation for a spin-0 boson. Indeed, starting from Eqs. (3.8), (3.9) and assuming conditions (3.10), which as has been stated above can be easily fulfilled, we can return to equation (3.2) and, finally, to the Dirac equation (2.1) in non-interacting case. Hence the inverse transformation, from boson to fermions, can be written as

$$B \longrightarrow (F\,\bar{\nu}_F)\ , \tag{3.13}$$

where B and F stand for boson and fermion, respectively, while $\bar{\nu}_F$ denotes antineutrino associated with the fermion F. In this reaction, $(F\,\bar{\nu}_F)$ is a two-fermion composite state. The process (3.13) seems to correspond to the first stage of the main channel of pion decay [16]

$$\pi^- \longrightarrow \left(\mu^-\,\bar{\nu}_\mu\right) \longrightarrow \mu^- + \bar{\nu}_\mu\,, \tag{3.14}$$

where we have postulated formation of the intermediate complex $(\mu^-\,\bar{\nu}_\mu)$.

Indeed, the boson to fermions transformation, constructed in this section, cannot describe direct decay of a pion into muon and neutrino. This follows from the fact that masses of a boson particle and a fermion, into which it transforms in presence of a neutrino field, are equal in our theory. Therefore, the present formalism seems to apply to the first stage of the reaction (3.14) only, with postulated formation of the intermediate complex state $(\mu^-\,\bar{\nu}_\mu)$ of mass $m = m_{\pi^-} = 139.570$ MeV$/c^2$. In the second stage of the process (3.14), the complex state decays into muon of mass $m_{\mu^-} = 105.658$ MeV$/c^2$ and massless neutrino, the energy excess $\left(m_{\pi^-} - m_{\mu^-}\right)c^2$ converted into neutrino (mainly) and muon kinetic energies. We shall comment on the suggested reaction in the last section.

4. Fermion–boson transition induced by a change-over of external fields

In this section, we carry out splitting of the Dirac equation in the interacting case. Let us assume as before the ansatz (3.1) and $p^{A\dot{B}}\alpha_{\dot{B}}(x) = 0$. Computing $\pi^{A\dot{B}}\eta_{\dot{B}}$, we get

$$(p - qA)^{A\dot{B}}\,\alpha_{\dot{B}}\chi = \left(p^{A\dot{B}}\alpha_{\dot{B}}\right)\chi + \alpha_{\dot{B}}\left(p^{A\dot{B}}\chi\right) - qA^{A\dot{B}}\alpha_{\dot{B}}\chi = \alpha_{\dot{B}}\pi^{A\dot{B}}\chi\,, \tag{4.1}$$

and we can rewrite the Dirac equation (2.1) as

$$\left.\begin{array}{rcl} \alpha_{\dot{B}}\pi^{A\dot{B}}\chi & = & m\xi^A \\ \pi_{A\dot{B}}\xi^A & = & m\alpha_{\dot{B}}\chi \end{array}\right\}, \tag{4.2}$$

where we assume that external field is longitudinal, *i.e.* fulfills conditions $\left[\pi^0 \pm \pi^3, \pi^1 \pm i\pi^2\right] = 0$.

We can now repeat all steps described in the preceding section arriving at equivalent of Eq. (3.7)

$$[1] = \chi^{1\dot{2}}p_{1\dot{1}}\alpha_{\dot{2}} + \chi^{2\dot{2}}p_{2\dot{1}}\alpha_{\dot{2}} + \tfrac{1}{m}\alpha_{\dot{2}}\left(\pi_{1\dot{1}}\pi^{1\dot{2}} + \pi_{2\dot{1}}\pi^{2\dot{2}}\right)\chi\,, \tag{4.3a}$$

$$[2] = \chi^{1\dot{1}}p_{1\dot{2}}\alpha_{\dot{1}} + \chi^{2\dot{1}}p_{2\dot{2}}\alpha_{\dot{1}} + \tfrac{1}{m}\alpha_{\dot{1}}\left(\pi_{1\dot{2}}\pi^{1\dot{1}} + \pi_{2\dot{2}}\pi^{2\dot{1}}\right)\chi\,. \tag{4.3b}$$

We note now that terms in rounded brackets vanish identically in longitudinal fields and we get

$$\left.\begin{array}{rcl} \pi^{1\dot{1}}\chi & = & m\chi^{1\dot{1}} \\ \pi^{2\dot{1}}\chi & = & m\chi^{2\dot{1}} \\ \pi_{1\dot{1}}\chi^{1\dot{1}} + \pi_{2\dot{1}}\chi^{2\dot{1}} & = & m\chi \end{array}\right\}, \tag{4.4}$$

$$\left.\begin{array}{rcl} \pi^{1\dot{2}}\chi & = & m\chi^{1\dot{2}} \\ \pi^{2\dot{2}}\chi & = & m\chi^{2\dot{2}} \\ \pi_{1\dot{2}}\chi^{1\dot{2}} + \pi_{2\dot{2}}\chi^{2\dot{2}} & = & m\chi \end{array}\right\}, \tag{4.5}$$

provided that, again, conditions (3.10) are fulfilled but now $\chi^{A\dot{B}}$, χ are solutions of Eqs. (4.4), (4.5) in longitudinal fields.

Since solutions of the Weyl equation are of the form of $\alpha_{\dot{B}}(x) = \hat{\alpha}_{\dot{B}}e^{ik\cdot x}$, where $\hat{\alpha}_{\dot{B}}$ is a constant spinor and $k^\mu k_\mu = 0$, we rewrite Eqs. (3.10) as equations (3.11), (3.12). It follows that determinant of Eqs. (3.11), $k_{1\dot{1}}k_{2\dot{2}} - k_{1\dot{2}}k_{2\dot{1}} = k^\mu k_\mu$, vanishes. Therefore, equations (3.11), (3.12) express one constraint only, which can be written, for example, as

$$\left(k_{1\dot{1}}\hat{\alpha}_{\dot{1}}\pi^{1\dot{1}} + k_{2\dot{1}}\hat{\alpha}_{\dot{2}}\pi^{2\dot{2}}\right)\chi = \left(-k_{1\dot{1}}\hat{\alpha}_{\dot{2}}\pi^{1\dot{2}} - k_{2\dot{1}}\hat{\alpha}_{\dot{1}}\pi^{2\dot{1}}\right)\chi\,. \tag{4.6}$$

Now, we note that Eq. (4.6) can be solved, for longitudinal potentials, by separation of variables, if we put $\chi(x) = f\left(x^0, x^3\right) g\left(x^1, x^2\right)$ since $\pi^{1\dot{1}}$, $\pi^{2\dot{2}}$ act only on x^0, x^3, while $\pi^{1\dot{2}}$, $\pi^{2\dot{1}}$ act only on x^1, x^2.

Let longitudinal fields in (4.4), (4.5) be switched off and then crossed fields, obeying $\left[\pi^0, \pi^3\right] = \left[\pi^1, \pi^2\right] = 0$, are turned on. It follows that equations (4.4), (4.5), in presence of such crossed fields, are the 3×3 equations (2.5), (2.6) obtained from the spin-0 DKP equations (2.4). Therefore, we can pass directly from equations (4.4), (4.5), now in crossed fields, to the DKP equations (2.4).

5. Discussion

In Section 2, we have reviewed the procedure of splitting the Dirac equation into two 3×3 equations in the non-interacting [12] as well as interacting case [13]. These equations can be written in covariant form as Dirac equations with some projection operators but do not contain all components of spinors used in the splitting. This problem was solved in the free case in Ref. [9] where we assumed that $\eta_{\dot{B}}(x) = \hat{\alpha}_{\dot{B}}\chi(x)$, where $\hat{\alpha}_{\dot{B}}$ was a constant spinor. This ansatz for $\eta_{\dot{B}}(x)$ is not fully satisfactory since constant spinors do not appear in nature.

To solve these problems, we have assumed in the present paper ansatzes (3.1), (3.5), involving neutrino field. The free Dirac equation has been converted to the set of 3×3 equations (3.8), (3.9), all spinors in the equations appearing in complete form. There are constraints, (3.11), (3.12), but they can be easily fulfilled. It follows that equations (3.8), (3.9) are equivalent to the spin-0 DKP equations. The inverse transformation, from boson to fermion, in presence of neutrino field, seems to correspond to the first stage of main channel of pion decay (3.14) where formation of the intermediate complex state $(\mu^- \, \bar{\nu}_\mu)$ has been assumed. Therefore, kinematics of the reaction products is missing in the present theory. There is, of course, another problem, since muon neutrino is massless in this formalism, although neutrino oscillations indicate that neutrinos are massive [16].

We have also constructed analogous transition for the Dirac equation in longitudinal fields. The resulting equations (4.4), (4.5) are not equivalent to the DKP equation, not in the case of longitudinal fields. However, switching over to crossed fields in these equations, after removing neutrino components $\alpha_{\dot{1}}$, $\alpha_{\dot{2}}$, we get immediately the DKP equation (2.4). This continuous and invertible fermion–boson transition induced by switch-over of external fields bears some analogy to BCS-BEC transition in ultracold Fermi gases [11] mentioned in Introduction. Further analysis of this problem is needed.

There are several necessary ingredients of these mechanisms: 3×3 equations, Weyl spinor $\alpha_{\dot{A}}(x)$ and, in the interacting case, switch-over of external fields. The present mechanism is an improvement over that of paper [9] since it applies to the interacting case and uses physically meaningful neutrino spinor $a_{\dot{B}}(x)$ rather than a troublesome constant spinor $\hat{a}_{\dot{B}}$. The 3×3 equations may provide a clue to the nature of the transition mechanism.

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