WILSON LOOPS WITH ARBITRARY CHARGES*

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We discuss how to implement, in lattice gauge theories, external charges which are not commensurate with an elementary gauge coupling. It is shown that an arbitrary, real power of a standard Wilson loop (or Polyakov line) can be defined and consistently computed in lattice formulation of Abelian, two dimensional gauge theories. However, such an observable can excite quantum states with integer fluxes only. Since the non-integer fluxes are not in the spectrum of the theory, they cannot be created, no matter which observable is chosen. Also the continuum limit of above averages does not exist unless the powers in question are, in fact, integer. On the other hand, a new continuum limit exists, which is rather intuitive, and where above observables make perfect sense and lead to the string tension proportional to the square of arbitrary (non-necessary commensurate with gauge coupling) charge.

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1. Introduction

Confinement remains a challenge in spite of the spectacular progress in studying non-perturbative QCD, *e.g.* with lattice methods. Many intuitive models have been proposed over years to elucidate this phenomenon. In particular, the Schwinger model has been a source of valuable inspiration, also in this case. In 1975 Coleman, Jackiw and Susskind have shown [1] that the energy of two external charges, separated by a distance L, grows linearly with L for small mass, m, of dynamical fermions

$$E(L) = me\left(1 - \cos\left(2\pi\frac{q}{e}\right)\right)L\,, \tag{1}$$

where e is a charge of dynamical fermions and q that of external sources. This result was then generalised to non-Abelian systems in the large N (colour) limit with essentially the same string tension [2, 4]. It has also a simple interpretation in terms of screening.

This paper originated in an attempt to confront (1) with lattice calculation and eventually extend it to larger masses of dynamical fermions. However, to do so, one has to introduce on a lattice an external source with an arbitrary charge, not commensurate with the elementary charge e of dynamical fermion, say, a quark. Surprisingly, we have not found any studies of this issue in the literature. Therefore, in this paper, we would like to explore even simpler question: how to represent arbitrary real charges in a lattice version of two dimensional pure U(1) gauge theory (Quantum Maxwell Dynamics, QMD_2).

2. Basics

Partition function of QMD_2 on a $N_x \times N_t$ periodic, lattice is known analytically in terms of modified Bessel functions $I_n(\beta)$, see *e.g.* [5]

$$Z = \int d(\mathrm{links})e^{S(\mathrm{plaquettes})} = \Sigma_n I_n(\beta)^{N_x N_t}\,. \tag{2}$$

In the continuum limit $N_x a = L$, $N_t a = T$, $1/\beta = e^2 a^2$

$$Z \to \#\Sigma_n e^{-E_n T}\,, \qquad E_n = \tfrac{1}{2}e^2 n^2 L\,. \tag{3}$$

Hence, it is saturated by well known states of topological fluxes [3]. Since there are no dynamical degrees of freedom in two dimensions, the theory would have been trivial if not for a non-trivial topology. Only integer multiples of elementary charge are allowed.

External sources are introduced by means of Wilson loops

$$W(\Gamma) = \Pi_{l\in\Gamma} e^{i\theta_l}\,, \tag{4}$$

where l labels U(1) variables (or just links) on a rectangular contour Γ. Their average values

$$Z\langle W(\Gamma)\rangle = \int d(\mathrm{links})W(\Gamma)e^{S} = \Sigma_n I_n(\beta)^{N_t N_x - n_t n_x} I_{n+1}(\beta)^{n_t n_x} \tag{5}$$

have a simple and appealing interpretation in the continuum limit

$$Z\langle W\rangle = \Sigma_n \exp\left(-\frac{e^2}{2}n^2L(T-t)\right)\exp\left(-\frac{e^2}{2}\left(n^2(L-R)+(n+1)^2R\right)t\right) \tag{6}$$

in terms of additional fluxes extending between external charges. One can easily introduce higher charges of external sources (just replace $W[\Gamma] \to (W[\Gamma])^m$), however again, they would have to be integer multiples of e and would result in replacing $n+1$ by $n+m$ in (6).

The obvious candidate for a pair of sources with arbitrary charge q is then an arbitrary real power $(W[\Gamma])^Q$, with $Q = q/e$. At first sight, it may rise some questions of gauge invariance and non-uniqueness. However, a little more careful examination shows that, in fact, there is nothing wrong with this proposal. Since $W[\Gamma]$ is gauge invariant, any function $f(W)$ should also be. Second, the non-uniqueness of $\langle W^Q\rangle$ causes only a technical obstacle, for MC evaluation of (6), which can be readily avoided by consistent followup of a particular branch of the power function. These assertions are confirmed by Fig. 1 where the MC calculations are confronted with the analytical predictions which can be readily obtained as above yielding

$$Z\left\langle W^Q\right\rangle = \Sigma_{m,n}\, I_n^{N_xN_t-n_xn_t}\, I_m^{n_xn_t}\, S(Q-(n-m))^{n_t+n_x}\,. \tag{7}$$

The overlap function

$$S(Q-n+m) = (\sin\pi(Q-n+m)/\pi(Q-n+m))^2$$

results from the integration over two opposite links in the contour Γ. For integer Q, (7) reduces to (5) with already mentioned interpretation. However for arbitrary, real Q things are different as discussed in the next chapter.

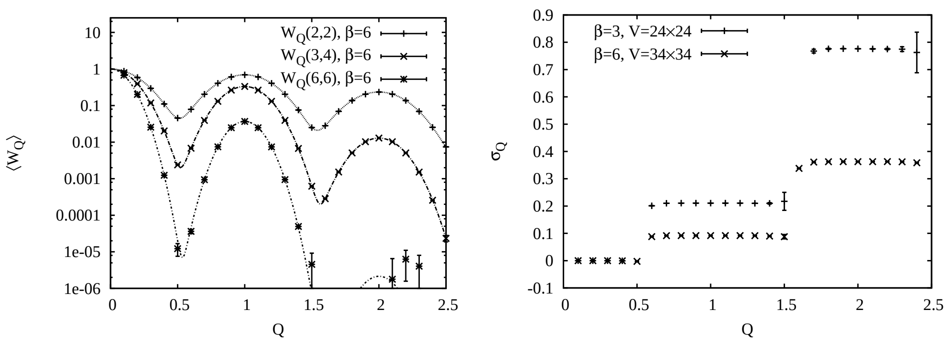

Fig. 1. Numerical data for charged Wilson loops. The Hybrid Monte Carlo algorithm was employed for the simulations, and lattices of size 24×24 at $\beta = 3$ (48 million measurements) and 34×34 at $\beta = 6$ (19.2 million measurements) were used. (a) Monte Carlo data for charged Wilson loops. (b) Extracted string tension σ_Q as a function of charge for $\beta = 3$ and $\beta = 6$.

3. Q-loops

Naively, one might expect that a Q-loop would excite a flux with an arbitrary charge $q = Qe$ with the string tension $\sigma_Q \sim q^2$. This, however, is not the case. Equations (5), (6) tell us that the *only* quantum states in the system are fluxes with multiples of the elementary charge e. Therefore, a Q-loop can excite only these states and the value of Q controls solely the amplitude with which given flux is excited.

Not surprisingly, this is confirmed by lattice calculations. In Fig. 1, the string tension, as extracted from lattice calculation of $\langle W^Q \rangle$, is shown for a range of Qs.

For given real Q, the overlap $S(Q)$ is strongly peaked around the nearest integer, hence only this flux is excited in practice. This explains the steps seen in Fig. 1 (b).

All above was done in the dimensionless lattice units. On the other hand, an attempt to obtain the continuum limit of (7) gives

$$\Sigma_{m,n} \exp\left(-\frac{e^2}{2} n^2 L(T-t)\right) \exp\left(-\frac{e^2}{2}\left(n^2(L-R)+m^2R\right)t\right)$$
$$\times S(Q-(n-m))^{(t+R)/a} ,$$

which does not exist for arbitrary Q. The overlaps S vanish rapidly (with $a \to 0$) for arbitrary Q and are not compatible with integer-valued fluxes m, n unless Q is again integer. This lends credit to the original suspicion that one cannot consistently introduce fractional (or more generally, arbitrary real) charges on a periodic lattice. However, there exists a quite natural limit where (7) makes a perfect sense. It can be termed as a *classical limit* of large Q and will be now discussed in detail.

4. Classical continuum limit of Q-loops

For better illustration, we will calculate the Green's function for the time evolution of a gauge field in the presence of two external charges separated by a distance R. As in [3], the continuous system is defined on a spatial circle with circumference L. Consequently, there is only one degree of freedom. We chose it as a constant (in x) value of the x component of a gauge field: $A_x(t) \equiv A(t)$.

On a lattice, we begin with the matrix element of the transfer matrix for two Polyakov lines[1] separated by n_x lattice units, with charge $q = Qe$

$$G = \left\langle \{\theta'\} \left| \Pi_{PQPQ} \right| \{\theta\} \right\rangle = \int d\{\vartheta\} e^{-iQ\vartheta_0} e^{iQ\vartheta_{n_x}} e^S . \tag{8}$$

[1] That is Wilson loops winding along the periodic time direction. See Ref. [5] for more details.

Similar steps as before (character expansion and integration over vertical links) give for G in the Coulomb gauge on a lattice

$$G\left(\Theta',\Theta\right) = \Sigma_{m,n} I_n^{N_x-n_x} I_m^{n_x} S(Q-(n-m)) e^{in(N_x-n_x)(\Theta-\Theta')} e^{imn_x(\Theta-\Theta')}\,, \tag{9}$$

where Θ (Θ') is a common, in Coulomb gauge, value of all spacial link angles in a lower (upper) time slice.

We are now ready to define the new limit which renders the arbitrary charge q meaningful. This is the *classical limit* of large Q, such that the actual dimensionfull charge $q = Qe$ is finite. It requires the gauge coupling e to tend to zero in appropriate way. Moreover, the large Q and small e limit is taken *before* the continuum limit. The limit can be roughly viewed as consisting of two steps. In the first part β is taken to be large, as usual, however because e is small rather than the lattice constant a (*c.f.* (3)). This gives

$$\begin{aligned} G \;=\; & \Sigma_{m,n} \exp\left(-\frac{1}{2\beta}\left(n^2\left(N_x-n_x\right)+m^2 n_x\right)\right) S(Q-(n-m)) \\ & \times e^{in(N_x-n_x)\Delta\Theta} e^{imn_x\Delta\Theta}\,. \end{aligned}$$

Now, two things happen: (1) at large β and fixed lattice distances, important contributions to the sum come from large fluxes ($m, n \sim b = \sqrt{\beta}$), exponentials become smooth functions of $u = n/b$ and $v = m/b$, and (2) at the same time, Q becomes $\sim b$, so we can write

$$Q = \frac{q}{e} = \frac{b}{g}\,, \qquad g = \frac{1}{qa}\,, \qquad b = \frac{1}{ea} \longrightarrow \infty \tag{10}$$

and obtain

$$\begin{aligned} G \;=\; & \beta \int du dv \exp\left(-\tfrac{1}{2}\left(u^2(N_x-n_x)+v^2 n_x\right)\right) S\left(b\left(g^{-1}-(u-v)\right)\right) \\ & \times e^{ibu(N_x-n_x)(\Theta-\Theta')} e^{ibvn_x(\Theta-\Theta')}\,, \end{aligned}$$

using

$$S\left(b\Delta\right) \overset{b\to\infty}{\longrightarrow} \frac{1}{b}\delta(\Delta)\,, \tag{11}$$

we can do one integral to obtain

$$\begin{aligned} G \;=\; & \sqrt{\beta} \int du \exp\left(-\tfrac{1}{2}\left(u^2(N_x-n_x)+\left(u-g^{-1}\right)n_x\right)\right) \\ & \times e^{iu(N_x-n_x)\left(\tilde{A}-\tilde{A}'\right)} e^{i\left(u-g^{-1}\right)n_x\left(\tilde{A}-\tilde{A}'\right)}\,, \end{aligned}$$

where we have also rewritten the phase factors in terms of the continuum field $\tilde{A} = A/e$, $\Theta_L = eaN_x\tilde{A}$. Now, do the Gaussian integral and take the continuum limit. To this end, rearrange the quadratic terms and the phase factors, $\rho = n_x/N_x$,

$$G = \sqrt{\beta}\left\{\int du \exp\left(-\tfrac{1}{2}(u-\rho/g)^2 N_x\right) e^{i(u-\rho/g)N_x\Delta\tilde{A}}\right\} \times \exp\left(-\tfrac{1}{2}g^{-2}\rho(1-\rho)N_x\right)$$

which gives finally

$$G\left(\tilde{A}', \tilde{A}, \epsilon\right) = \sqrt{\beta}\sqrt{\frac{2\pi a}{L}} \exp\left(-\frac{L}{2}\frac{\left(\tilde{A}-\tilde{A}'\right)^2}{a}\right) \exp\left(-\frac{q^2}{2}\rho(1-\rho)La\right)$$

which is, again, proportional to the kernel for propagation of a free particle over an infinitesimal time lapse $\epsilon = a$, but now the propagation takes place in a constant background potential

$$V = \frac{q^2}{2}\rho(1-\rho)L \tag{12}$$

with arbitrary, real value of a classical charge q.

Notice that the coordinate $\tilde{A}$ is now not periodic. Periodicity $\tilde{A} \to \tilde{A} + \frac{2\pi}{Le}$ was lost while taking the $e \to 0$ limit. For the same reason, the discrete spectrum of topological fluxes has turned into the continuous one of free momenta $u - \rho/g$.

However, some memory of the periodic nature of the microscopic system remains. Namely, the effective string tension

$$\sigma = \frac{q^2}{2}\rho(1-\rho) \tag{13}$$

vanishes at $R = 0, L$. At these configurations, the microscopic string between external sources begins/completes a single winding around the circle increasing the effective flux by one quantum. Since V represents the *difference* between the sum of the two fluxes (inside and outside the pair of sources) and the effective single flux around the circle, it vanishes at $R = 0$ and L. In other words: integer, and only integer, (in terms of winding) fluxes are "screened" (or better: accommodated) by the intrinsic spectrum of periodic QMD_2.

Given existence of the new limit, it is also important to estimate how fast (or slow) it can be reached in practice. This can be answered by confronting the analytical predictions with actual MC simulations. Similar calculations for the trace of (8) give the easily testable prediction for the MC average of two Polyakov lines on a lattice with unit length in time direction

$$\left\langle P_Q^\dagger P_Q \right\rangle \stackrel{\beta\to\infty}{\longrightarrow} \exp\left(-\tfrac{1}{2}q^2a^2\rho(1-\rho)N_x\right). \tag{14}$$

The limit is taken with $Q^2 = q^2a^2\beta$ with qa kept fixed. In the continuum, above prediction reads

$$\left\langle P_Q^\dagger P_Q \right\rangle \stackrel{\text{cont. lim}}{\longrightarrow} \exp\left(-\tfrac{1}{2}q^2\rho(1-\rho)La\right). \tag{15}$$

The above can be generalized for an arbitrary $N_t \times N_x$ lattice and we obtain

$$\left\langle P_Q^\dagger P_Q \right\rangle \stackrel{\beta\to\infty}{\longrightarrow} \exp\left(-\tfrac{1}{2}q^2a^2\rho(1-\rho)N_xN_t\right) \tag{16}$$

and

$$\left\langle P_Q^\dagger P_Q \right\rangle \stackrel{\text{cont. lim}}{\longrightarrow} \exp\left(-\tfrac{1}{2}q^2\rho(1-\rho)LT\right). \tag{17}$$

This is tested in Fig. 2 for $Q = 0.85/\sqrt{2}$ by a simulation on a 6×6 lattice. The actual numerical implementation profited from the equivalence with a spin chain of length 36 [6]. The latter was simulated using the cluster

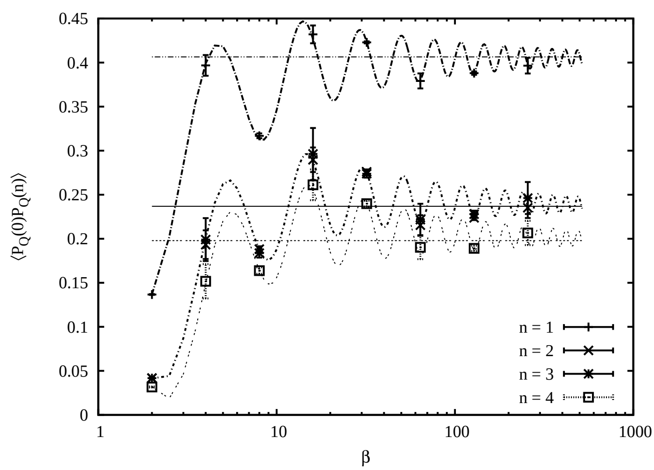

Fig. 2. Comparison of $\langle P_Q^\dagger P_Q \rangle$ measured on a 6×6 lattice (data points) with the analytic prediction of Eq. (7) and with the continuum limit given by Eq. (17) (horizontal lines).

algorithm [7] which practically eliminates the critical slowing down problem and the errors were estimated using the Γ-method, following Ref. [8]. Simulated points perfectly agree with Eq. (7) which is depicted in Fig. 2 as a dashed curves (n_t was set to N_t since we are dealing with two Polyakov loops separated by a distance n_x). The positions of the horizontal lines were calculated using Eq. (17) and correspond to the continuum prediction.

5. Conclusions

Arbitrary, real powers of Wilson loops (or Polyakov lines) are natural candidates if one wishes to study, on a lattice, external charges which are not commensurate with the elementary gauge coupling. However, introduction of such observables raises some subtle questions which show up even in the simplest gauge models. We have discussed them in the case of Quantum Maxwell Dynamics — a pure gauge U(1) theory in two dimensions.

Q-loops, as we call them, can be consistently defined on a lattice: results of MC simulations fully agree with the analytical predictions which are readily available in this simple model.

They do not excite/create quantum states (fluxes) with arbitrary real charge, however. Such states do not exist in the theory. Instead, a Q-loop excites mostly the quantum state of flux with the integer charge which is closest to the "charge carried by a Q-loop". This charge controls the amplitude with which integer (in units of elementary charge) charges are excited.

Above applies to the lattice. The continuum limit of Q-loops does not exist unless a charge of Q-loop is multiple of the elementary gauge coupling.

However, a modification of the continuum limit is possible where Q-loops are well defined and give rise to some interesting physics. The modification is inspired by classical considerations and applies when the arbitrary charge of external sources is much larger than that of the elementary flux. Then, the discrete quantum spectrum becomes continuous and the effective fluxes with arbitrary charge emerge. Existence of this effective, classical behaviour was also confirmed numerically and the ranges of parameters where it is seen were estimated.

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