JACOBI FIELDS AND CONJUGATE POINTS ON TIMELIKE GEODESICS IN SPECIAL SPACETIMES

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Several physical problems such as the "twin paradox" in curved spacetimes have a purely geometrical nature and are reduced to studying properties of bundles of timelike geodesics. The paper is a general introduction to systematic investigations of the geodesic structure of physically relevant spacetimes. These are focussed on the search of locally maximal timelike geodesics. The method is based on determining conjugate points on chosen geodesic curves. The method presented here is effective at least in the case of radial and circular geodesics in static spherically symmetric spacetimes. Our approach shows that even in Schwarzschild spacetime (as well as in other static spherically symmetric ones), one can find a new unexpected geometrical feature: each stable circular orbit contains besides the obvious set of conjugate points two other sequences of conjugate points. The obvious limitations of the approach arise from one's inability to solve involved ordinary differential equations and the recent progress in the field allows one to increase the range of metrics and types of geodesic curves tractable by this method.

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1. Introduction

This paper serves as a generic introduction to and a formulation of a systematic research programme for studying the geodesic structure of static spherically symmetric (SSS) spacetimes and then of a wider class of physically relevant ones. After studying a few spacetimes, we have realized that

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some results are universal and need not be separately derived for each spacetime of the class (this observation is not quite obvious since spacetimes with similar isometry groups may considerably differ in their geodesic structure). The programme of investigating the geodesic structure of various spacetimes has originally been motivated by the famous "twin paradox" considered in curved spacetimes [1–5]. It turns out that contrary to the conjecture stated in [5], no general rule concerning of which twin is younger exists and one must study each case separately. The problem is of purely geometrical nature and consists in computing the lengths of various timelike curves having common points; in the Lorentzian geometry only the problem of determining the longest curve is meaningful and directly leads to searching the geodesic structure of the spacetime and this is why it is worth pursuing.

The problem of maximally long timelike curves consists of two separate problems: local and global. In the local problem, one considers a bunch of infinitesimally close timelike curves emanating from the initial point p and intersecting at the endpoint q. A geodesic γ of the bunch is the longest curve in it if the segment pq of γ does not contain a point conjugate to p. In the global problem, one searches for the longest curve in the whole space of all timelike curves with common endpoints. If a geodesic γ is locally the longest, it needs not be globally the longest one and γ is globally maximal on a segment pq if the segment does not contain a future cut point of p. One learns from a monograph on global Lorentzian geometry [6] that all propositions concerning the maximal length curves are "existence theorems" providing no analytic tools to establish if the given (geodesic) curve is globally maximal or to find out the maximal geodesic emanating from the given point. This is a direct consequence of the nonlocal nature of the problem, which cannot be solved by employing a local tool such as a differential equation. Only in spacetimes with special high symmetries, one can directly apply a global theorem to recognize the cut points (or their absence). For this reason, we mainly deal with a more tractable problem of finding out locally maximal curves with the aid of Jacobi vector fields and conjugate points on timelike geodesics. In some cases, one may indicate which segments of special timelike geodesics are globally maximal.

We emphasize that in the search for locally maximal worldliness, one must solve the geodesic deviation equation (GDE) and to this end, one must know an explicit parametric representation of the given geodesic. Complete sets of analytic solutions to the geodesic equation are known in very few spacetimes, *e.g.* for Schwarzschild metric [7, 8] and recently for Schwarzschild–(anti)-de Sitter spacetimes [9, 10] (and references therein). At least in the case of Schwarzschild metric, all timelike geodesics may be expressed in terms of known transcendental functions. In a general SSS spacetime, the radial and circular geodesics are exceptional in that their parametric description is in terms of simple elementary functions. Besides these two special cases (and few other exceptions), the GDE is intractable and thus it is reasonable to first learn about the geodesic structure of SSS spacetimes by investigating these two kinds of geodesics. In this work, we study exclusively radial and circular timelike geodesics.

The paper is organized as follows. In Section 2, the GDE is recast in the form of three scalar equations which are more suitable for our purposes and the general procedure is outlined. These equations are explicitly derived for the radial geodesics in Section 3. Section 4 contains the main result found by the procedure presented in Section 2: the GDEs on circular geodesics (if these exist) are the same for all SSS spacetimes and the solutions found earlier for Schwarzschild metric universally apply. Actually, a preliminary version of the present work had first appeared as `arXiv:1402.3976v1 [gr-qc]` and preceded two published papers [11, 12] which contained detailed results for a few simplest spacetimes found by applying the general method developed here. Therefore, to avoid repetitions, the presentation of the method here is not fully complete and we refer the reader to those two papers for a discussion of some of its aspects. Brief conclusions are given in Section 5.

2. Locally maximal timelike curves: Jacobi fields and conjugate points

A timelike geodesic connecting points p and q is locally maximal if there are no conjugate points to p on its segment pq and these are determined by zeros of any Jacobi vector field on it. A Jacobi field on a timelike geodesic γ with a unit tangent vector field $u^\alpha(s)$ is any vector field $Z^\mu(s)$ being a solution of the GDE on γ,

$$\frac{D^2}{ds^2}\,Z^\mu = R^\mu{}_{\alpha\beta\gamma}\,u^\alpha\,u^\beta\,Z^\gamma\,, \tag{1}$$

which is orthogonal to γ, $Z^\mu\,u_\mu = 0$. Due to the presence of the second absolute derivative D^2/ds^2, the GDE is very complicated and one simplifies it by replacing this derivative by the ordinary one. To this end, one expands Z^μ in a basis consisting of three spacelike orthonormal vector fields $e_a{}^\mu(s)$, $a = 1, 2, 3$ on γ, which are orthogonal to γ and are parallelly transported along the geodesic, *i.e.*

$$e_a{}^\mu\,e_{b\mu} = -\delta_{ab}\,, \qquad e_a{}^\mu\,u_\mu = 0\,, \qquad \frac{D}{ds}e_a{}^\mu = 0\,. \tag{2}$$

(Since we are dealing with timelike curves, it is convenient to apply the metric signature $+\,-\,-\,-$.) Then $Z^\mu = \sum_a Z_a e_a{}^\mu$ and the covariant vector equation (1) is reduced to three scalar ODEs for the Jacobi scalars[1] $Z_a(s)$,

$$\frac{d^2}{ds^2} Z_a = -e_a{}^\mu R_{\mu\alpha\beta\gamma} u^\alpha u^\beta \sum_{b=1}^{3} Z_b e_b{}^\gamma , \quad (3)$$

a general Jacobi field depends on 6 integration constants. Any Killing vector field K^μ generates a first integral of Eq. (1) of the form [13]

$$K_\mu \frac{D}{ds} Z^\mu - Z^\mu \frac{D}{ds} K_\mu = \mathrm{const} . \quad (4)$$

This formula may be recast in terms of the scalars Z_a. To this end, one introduces a tetrad $e_A{}^\mu$, $A = 0, 1, 2, 3$, along γ consisting of the vectors $e_a{}^\mu(s)$ supplemented by $e_0{}^\mu \equiv u^\mu$. The tetrad is orthonormal, $e_A{}^\mu e_{B\mu} = \eta_{AB} = \mathrm{diag}(1, -1, -1, -1)$. Expanding Z^μ and $K^\mu = \sum_{A=0}^{3} K_A e_A{}^\mu$ in the tetrad and inserting them into (4), one gets

$$\sum_{a=1}^{3} \left(Z_a \frac{dK_a}{ds} - \frac{dZ_a}{ds} K_a \right) = \mathrm{const} , \quad (5)$$

where $K_a = -K^\mu e_{a\mu}$. In many cases, the constants (5) generated by independent Killing vectors turn out to be dependent. Besides the simplest spacetimes, the integrals (5) are essential in solving (3).

There are two approaches to finding the Jacobi vector fields. Bażański [14] gave a generic algorithm for solving the GDE in cases where one knows a complete integral of the Hamilton–Jacobi equation for timelike geodesics. When applied to Schwarzschild spacetime [15], it has turned out that this beautiful formalism is of restricted practical use: it does not apply to circular geodesics. When the formalism is applied where it works, it requires to first find the general solution of the GDE and then carefully take appropriate limits in it to the particular type of the geodesic, what makes the procedure rather cumbersome. Furthermore, at least in the Schwarzschild metric, the formalism works in the case of radial geodesics only for curves escaping to the spatial infinity, what excludes finite geodesics, such as those considered in the twin paradox [16]. This is why our approach is closer to that of Fuchs, who found a generic solution to the GDE in SSS spacetimes for some types of timelike geodesics [17]. His formula expresses the Jacobi field in terms of four integrals of expressions made up of Killing vectors and constants of motion they generate. It is our experience that employing this formula is not simpler than solving the GDE for radial geodesics from the very beginning.

[1] The vector index of a Jacobi vector field will always be written as a superscript and the number of the Jacobi scalar — as a subscript.

It should be stressed that also the Fuchs' formula does not apply to the circular geodesics and these must be dealt with separately. We therefore solve the GDE independently for each type of geodesic curves.

To summarize, the procedure is as follows:

— Choose an interesting spacetime with some isometries.

— Choose a geometrically interesting timelike geodesic γ having a simple parametric description $x^\alpha = x^\alpha(\tau)$.

— Choose the triad ${e_a}^\mu(s)$ on γ. The triad is not uniquely determined by Eqs. (2) and should be properly selected as to render equations (3) as simple as possible.

— Solve the GDE (3) applying the first integrals and find a generic solution $Z_a(\tau)$.

— Consider all possible special solutions with $Z_a(0) = 0$ and seek for their zeros, $Z_a(\tau_0) = 0$ for $\tau_0 > 0$.

Then the geodesic γ is uniquely locally maximal on the segment $0 \leq \tau < \tau_0$ and is non-uniquely locally maximal on $0 \leq \tau \leq \tau_0$. For $\tau_1 > \tau_0$, there is a timelike curve from $\gamma(0)$ to $\gamma(\tau_1)$ which is longer than γ. It is clear that the crucial point is to solve equations (3). Besides the simplest cases, it is possible due to the recent progress in techniques of dealing with ODEs.

In the remainder of the paper, we shall apply this general procedure to radial and circular geodesics in SSS spacetimes.

3. Jacobi fields on timelike radial geodesics in static spherically symmetric spacetimes

Before dealing with the Jacobi vectors, we make a comment on globally maximal radial geodesics in SSS spacetimes.

3.1. Globally maximal radial geodesics

The comoving, *i.e.* Gaussian normal geodesic (GNG) coordinates allow one to easily establish that some segments of radial geodesics are globally maximal. The geodesics which are radial in the standard spherical coordinates remain radial in the GNG coordinates, where they coincide with the lines of the proper time. To avoid confusion with the Gullstrand–Painlevé coordinates, which are also made up with the aid of freely falling particles, we stress that in the GNG system the radial geodesics have all spatial coordinates constant, then the metric is

$$ds^2 = d\tau^2 + g_{ij}\left(\tau, x^k\right)\, dx^i\, dx^j\,. \tag{6}$$

Denote by D the GNG chart domain. Let a segment of the future directed radial geodesic γ lying in D be parameterized by the time τ in the interval (τ_1, τ_2), then its length is $s(\gamma) = \tau_2 - \tau_1$. Assume that the domain D is so large that any future directed timelike curve σ joining points $\gamma(\tau_1)$ and $\gamma(\tau_2)$ lies in D. Then, it is obvious that σ is shorter than γ. This fact becomes interesting if D is a sufficiently large part of the whole manifold. In SSS spacetimes, one may effectively construct the transformation from the standard spherical coordinates (wherein the metric is usually given) to the GNG ones directly applying the method developed for the Schwarzschild metric [18] (Lemaître coordinates). Then, one may determine the size of D. It turns out that for the Reissner–Nordström black hole with $M^2 > Q^2$ the domain is the same as that of the standard spherical coordinates, *i.e.* the spacetime outside the outer event horizon, $r_+ < r < \infty$ with $r_+ = M + \sqrt{M^2 - Q^2}$. The radial geodesics are globally maximal for all $r > r_+$.

The Kottler (Schwarzschild–de Sitter) black hole for $\Lambda > 0$ and $9M^2\Lambda < 1$ is static between the black hole event horizon $r = r_m$ and the cosmological de Sitter horizon (the Killing horizon) $r = r_M$. The metric component g_{00} has the maximum for $r = r_e = (3M/\Lambda)^{1/3}$ implying that actually there exist two distinct and non-overlapping GNG charts for r in (r_m, r_e) and in (r_e, r_M).

3.2. Equations for Jacobi fields

The method of GNG coordinates effectively applies only to radial geodesics and as the case of the Kottler black hole shows, in many SSS spacetimes, the domain D is smaller than that of the standard spherical ones. We are, therefore, interested here in locally maximal curves and in this section, we derive the geodesic deviation equation for radial geodesics. We assume the standard form of the SSS metric,

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - F^2(r)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{7}$$

functions ν and λ are real for $r \in (r_m, r_M)$, we assume $r_m \geq 0$. We assume $F(r) = r$ for a generic SSS metric and postpone discussing the case $F(r) = \text{const} = a$ to the next section (the special case of the Bertotti–Robinson spacetime has been studied separately [11]). The timelike Killing vector is $K^\alpha = \kappa\delta_0^\alpha$ and $\kappa = \text{const}$ is a normalization factor (chosen either at $r = 0$ or at spatial infinity). The conserved energy of a particle of mass m on a radial geodesic is E and $k \equiv E/(mc^2) > 0$ is dimensionless.

Let us choose one radial geodesic and denote it by C. The starting point of C is $r = r_0$, $r_m < r_0 < r_M$, and the initial velocity is $\dot{r}(r_0) \equiv u$. The form of C depends on the behaviour of e^ν. We assume that e^ν is monotonic from r_m to r_M (the case of the Kottler spacetime is more involved and requires a separate study).

(i) e^ν is decreasing for $r > r_0$ (*e.g.* dS metric). The gravitation is repulsive: for $u \geq 0$, the curve C flies upwards to the domain boundary $r = r_M$ and will never return; for $u < 0$, the curve goes down, in general, it reaches a minimal height $r = \rho$ at which $\dot{r}(\rho) = 0$, then C turns back and escapes to $r = r_M$.

(ii) e^ν increases for $r > r_0$ (CAdS and R–N), the gravity is attractive. For $u \leq 0$, the curve C falls down towards the lower boundary $r = r_m$; we do not follow it in the time-dependent inner region. For $u > 0$, it goes upwards, reaches the maximal height $r = R$, where $\dot{r}(R) = 0$, then C turns back and falls down to r_m and farther.

In both the cases, we study the more general situation: C consists of two segments, the incoming and the outgoing one. It is convenient to parameterize C with a suitably chosen variable η, $x^\alpha = x^\alpha(\eta)$ via $r = f(\eta)$. The vector tangent to the geodesic C is then

$$u^\alpha = \frac{dx^\alpha}{ds} = \left(\dot{t}, \dot{r}, 0, 0\right) = \left[\frac{k}{\kappa} e^{-\nu}, \varepsilon e^{-\lambda/2} \left(\frac{k^2}{\kappa^2} e^{-\nu} - 1\right)^{1/2}, 0, 0\right], \tag{8}$$

where $\varepsilon = +1$ for the outgoing segment and $\varepsilon = -1$ for the incoming one. The spacelike triad orthogonal to C and satisfying (2) is chosen in the possibly simplest form

$$\begin{aligned} e_1^\alpha &= \left[\varepsilon e^{-\nu/2} \left(\frac{k^2}{\kappa^2}\, e^{-\nu} - 1\right)^{1/2}, \frac{k}{\kappa}\, e^{-(\nu+\lambda)/2}, 0, 0\right], \\ e_2^\alpha &= \left[0, 0, \frac{1}{r}, 0\right], \qquad e_3^\alpha = \left[0, 0, 0, \frac{1}{r}\right] \end{aligned} \tag{9}$$

with $\varepsilon = \pm 1$ as above. The Riemann tensor has six non-vanishing components $R_{\mu\nu\mu\nu}$. The GDEs (3) for the Jacobi scalars are separated,

$$\frac{d^2}{ds^2} Z_1 = \frac{1}{4}\left(\nu'\lambda' - 2\nu'' - \nu'^2\right) e^{-\lambda} Z_1\,, \tag{10}$$

$$\frac{d^2}{ds^2} Z_2 = -\left[\frac{k^2}{2\kappa^2}\frac{1}{r} e^{-(\nu+\lambda)}(\nu' + \lambda') - \frac{\lambda'}{2r}\, e^{-\lambda}\right] Z_2\,, \tag{11}$$

and the equation for Z_3 is identical with that for Z_2. Here $\nu' = d\nu/dr$ *etc.* Since on the LHS of these equations one has derivatives w.r.t. the proper time, it is here that the use of the suitably chosen variable η is necessary. In terms of η one finds more involved equations

$$\frac{d^2 Z_1}{d\eta^2} - \frac{df}{d\eta}\left[\left(\frac{df}{d\eta}\right)^{-2} \frac{d^2 f}{d\eta^2} + \frac{\lambda'}{2} + \frac{k^2}{2\kappa^2}\nu' \left(\frac{k^2}{\kappa^2} - e^\nu\right)^{-1}\right] \frac{dZ_1}{d\eta}$$

$$= \frac{1}{4}\left(\nu'\lambda' - 2\nu'' - \nu'^2\right)\left(\frac{k^2}{\kappa^2}\,e^{-\nu} - 1\right)^{-1}\left(\frac{df}{d\eta}\right)^2 Z_1\,, \tag{12}$$

$$\frac{d^2Z_2}{d\eta^2} - \frac{df}{d\eta}\left[\left(\frac{df}{d\eta}\right)^{-2}\frac{d^2f}{d\eta^2} + \frac{\lambda'}{2} + \frac{k^2}{2\kappa^2}\nu'\left(\frac{k^2}{\kappa^2} - e^{\nu}\right)^{-1}\right]\frac{dZ_2}{d\eta}$$
$$= -e^{\lambda}\left(\frac{k^2}{\kappa^2}e^{-\nu} - 1\right)^{-1}\left(\frac{df}{d\eta}\right)^2\frac{1}{2r}\left[\frac{k^2}{\kappa^2}e^{-(\nu+\lambda)}(\nu' + \lambda') - \lambda' e^{-\lambda}\right] Z_2\,, \tag{13}$$

and the equation for Z_3 is identical with (13); in the equations, one sets $r = f(\eta)$. The first integrals (5) for these equations are generated by $K_t^\alpha = \kappa\delta_0^\alpha$ and the three standard spacelike rotational Killing fields. There are three independent first integrals, which are also separated. K_t^α gives rise to

$$\frac{1}{2}e^{\nu}\nu'\,\frac{df}{d\eta}\,Z_1 + \left(\frac{k^2}{\kappa^2} - e^{\nu}\right)\frac{dZ_1}{d\eta} = C_1\varepsilon\left|\frac{df}{d\eta}\right| e^{(\nu+\lambda)/2}\,, \tag{14}$$

whereas the rotational Killing fields generate two identical expressions,

$$f(\eta)\frac{dZ_a}{d\eta} - \frac{df}{d\eta}\,Z_a = C_a\left|\frac{df}{d\eta}\right| e^{\lambda/2}\left(\frac{k^2}{\kappa^2}e^{-\nu} - 1\right)^{-1/2}\,, \tag{15}$$

here $a = 2, 3$ and C_1, C_2 and C_3 are arbitrary constants. These equations together with their first integrals may be solved only if the functions $\nu(r)$, $\lambda(r)$ and $f(\eta)$ are explicitly given. Currently, these equations are under study in a number of spacetimes. Previously, their solutions were found for the Schwarzschild [16] and R–N metric [12].

4. Jacobi fields on circular geodesics in static spherically symmetric spacetimes

First, we check the very existence of the circular geodesic for some $r = r_0$, $r_m < r_0 < r_M$, since in Minkowski and de Sitter spacetime such a curve does not exist. If it exists, we denote it by B and write $\nu_0 = \nu(r_0)$, $\nu_0' = d\nu(r_0)/dr$ *etc.* First, we consider the metric (7) with $F(r) = a$ (*e.g.* the Bertotti–Robinson spacetime). In this case, the radial component of the geodesic equation implies that $r = r_0$ is a geodesic if and only if $\nu_0' = 0$. If $\nu'(r) \neq 0$ in the whole range as is in the B–R case, then circular geodesics do not exist. Functions $\nu(r)$ admitting isolated points at which $\nu' = 0$ do not correspond to physically interesting spacetimes and we shall not discuss them here. In what follows, we assume $F(r) = r$.

A particle on a circular geodesic B has energy k and angular momentum $J = mcL$ given by

$$k^2 = \frac{2\kappa^2\,e^{\nu_0}}{2 - r_0\nu_0'}\,, \qquad L^2 = \frac{r_0^3\,\nu_0'}{2 - r_0\nu_0'}\,, \tag{16}$$

these quantities do not depend on $\lambda(r)$. One infers from (16) that the necessary and sufficient conditions for circular geodesics to exist are $r_0\nu_0' < 2$ and $\nu_0' > 0$, what implies that $g_{00} = e^{\nu(r)}$ is an increasing function around $r = r_0$; these conditions were found in a different way in [19]. In ultrastatic spherically symmetric spacetimes, where $\nu(r) \equiv 0$, circular geodesics do not exist since they reduce to the case of a particle remaining at rest in the space [12]. We, therefore, assume the generic case $\nu'(r) \neq 0$ (besides isolated points). A circular orbit is stable if an effective potential reaches minimum on it and this amounts to

$$\nu_0'' - \nu_0'^2 + \frac{3\nu_0'}{r_0} > 0\,. \tag{17}$$

4.1. Equations for the Jacobi scalars

We choose the basis triad on B in the form

$$\begin{aligned} e_1^\alpha &= [-T\sin qs, X\cos qs, 0, -Y\sin qs]\,, \\ e_2^\alpha &= \left[0, 0, \frac{1}{r_0}, 0\right], \qquad e_3^\alpha = -\frac{1}{q}\frac{d}{ds}e_1^\alpha\,, \end{aligned} \tag{18}$$

where the constants are

$$T = \left(\frac{r_0\nu_0' e^{-\nu_0}}{2 - r_0\nu_0'}\right)^{1/2}, \qquad X = e^{-\lambda_0/2}, \qquad Y = \frac{1}{r_0}\left(\frac{2}{2 - r_0\nu_0'}\right)^{1/2}$$

$$\text{and} \qquad q = \left(\frac{\nu_0'}{2r_0}\, e^{-\lambda_0}\right)^{1/2}, \tag{19}$$

and the vector tangent to the geodesic B is

$$u^\alpha = \left[\frac{k}{\kappa}\, e^{-\nu_0}, 0, 0, \frac{L}{r_0^2}\right]. \tag{20}$$

These four vectors depend only on the constants determined by the metric functions ν and λ. In the spherical coordinates, the Jacobi scalars Z_a have dimension of length. After a longer computation, the geodesic deviation equations are derived

$$\frac{d^2}{ds^2}Z_1 = q^2\left[\left(b\cos^2 qs - 1\right)Z_1 + bZ_3 \sin qs\, \cos qs\right], \tag{21}$$

$$\frac{d^2}{ds^2}Z_2 = -\frac{2q^2}{2 - r_0\nu_0'}\, e^{\lambda_0}\, Z_2\,, \tag{22}$$

$$\frac{d^2}{ds^2}Z_3 = q^2\left[bZ_1 \sin qs\, \cos qs + \left(b\sin^2 qs - 1\right)Z_3\right], \tag{23}$$

where

$$b=\frac{2}{2-r_0\nu_0'}\left(1-r_0\nu_0'-r_0\frac{\nu_0''}{\nu_0'}\right) . \tag{24}$$

These equations were first derived for the Schwarzschild [16] and then for R–N metric [12] with different constants and it seemed then that their identity were due to the specific form of $\nu(r)$ and $\lambda(r)$ in these metrics. Here, it is shown that these equations are universal for all SSS spacetimes admitting circular geodesics, only the numerical coefficients for the given r_0 depend on the metric functions. The range of b depends on the spacetime; we exclude $b=0$ (CAdS space) and assume $b>0$, *e.g.* for R–N metric $3<b<\infty$. The equations for Z_1 and Z_3 are coupled and their RHSs are similar, but not exactly symmetric.

Again, the first integrals (5) of the equations are generated by the four Killing fields of the SSS spacetime. The vectors $K_t=\partial/\partial t$ and $K_z=\partial/\partial\phi$ generate the same first integral for Eqs. (21) and (23),

$$-\frac{dZ_1}{ds}\sin qs+Z_1q\cos qs+\frac{dZ_3}{ds}\cos qs+Z_3q\sin qs=C_1\,, \tag{25}$$

whereas the other two rotational vectors give rise to two independent first integrals for Eq. (22),

$$\begin{aligned} r_0\,\frac{dZ_2}{ds}\sin\phi-\left(\frac{r_0\nu_0'}{2-r_0\nu_0'}\right)^{1/2}Z_2\cos\phi &= C_2\,,\\ r_0\,\frac{dZ_2}{ds}\cos\phi+\left(\frac{r_0\nu_0'}{2-r_0\nu_0'}\right)^{1/2}Z_2\sin\phi &= C_3\,. \end{aligned} \tag{26}$$

The constants allow one to solve Eq. (22) without any integration,

$$Z_2=C'\sin\frac{Ls}{r_0^2}+C''\cos\frac{Ls}{r_0^2}\,, \tag{27}$$

C' and C'' have dimension of length. The universality of the equations implies universality (modulo the values of the constants) of conjugate points on B. Solutions giving rise to two of the three sequences of conjugate points on B were previously found in [16] and in [12] we presented some properties of nearby timelike geodesics intersecting B at these points.

4.2. Conjugate points generated by the Jacobi scalar Z_2

The deviation vector field generated by Z_2 is $Z^\mu=Z_2(s)e_2^\mu$ and is directed off the 2-surface $\theta=\pi/2$. To determine points on B conjugate to $P_0(s=0,t=t_0,r=r_0,\theta=\pi/2,\phi=\phi_0)$, one takes the vector field vanishing at P_0, $Z^\mu=\frac{C'}{r_0}\,\delta_2^\mu\sin\frac{Ls}{r_0^2}$.

The field has infinite number of zeros at points $Q_n(s_n)$ with

$$s_n = n\pi \frac{r_0^2}{L} = n\pi \left[\frac{r_0}{\nu_0'}(2 - r_0\nu_0')\right]^{1/2}, \qquad n = 1, 2, \ldots \tag{28}$$

The location of these points on the circle $r = r_0$ in the space is found from the expression for the coordinate ϕ on B, $\phi - \phi_0 = Ls/r_0^2$, then the points are equidistant with $\Delta\phi = \pi$. Thus, for n even, the points Q_n coincide in the space with P_0, whereas for n odd, they are points antipodic to P_0 on the circle. This result is geometrically quite obvious: if one rotates in the space the 2-surface $\theta = \pi/2$ by a small angle about the axis joining the spatial projections of P_0 and Q_1, then the nearby circular timelike geodesics emanating from P_0 will successively intersect at Q_n, $n = 1, 2, \ldots$, in the spacetime. This effect was previously found for Schwarzschild [16].

4.3. Jacobi fields spanned on the basis vectors e_1^μ and e_3^μ — an infinite sequence of conjugate points

The coupled equations (21) and (23) have a complete system of basis solutions consisting of four pairs of solutions (Z_{1N}, Z_{3N}), $N = 1, 2, 3, 4$ and the general solution to them is

$$Z_1 = \sum_{N=1}^{4} A_N\, Z_{1N} \qquad \text{and} \qquad Z_3 = \sum_{N=1}^{4} A_N\, Z_{3N} \tag{29}$$

with arbitrary A_N. Since these equations are the same for all SSS spacetimes, the solutions found for the Schwarzschild metric [16] apply. This implies that the value $b = 4$ of the parameter is universally distinguished. From the definition (24), the critical value $b = 4$ corresponds to the equality in (17) and determines the innermost stable circular orbit (ISCO) and the stability criterion (17) requires $b < 4$. For physical reasons, we are interested in conjugate points on stable orbits and expect that there are no conjugate points on unstable orbits. The solutions show that this is the case.

In search for conjugate points, the relevant Jacobi fields must satisfy $Z_1(0) = 0 = Z_3(0)$ and this restricts the coefficients A_N. One separately studies the cases $b > 4$, $b = 4$ and $b < 4$. For the ISCO, $b = 4$, the resulting scalars Z_1 and Z_3 do not have common roots for $s \neq 0$ and the same occurs for the unstable orbits; for $b \geq 4$, there are no conjugate points to $s = 0$. In the most interesting case of stable orbits, the analysis performed in [16] was incomplete. The deviation field vanishing at $s = 0$ depends on arbitrary A_1 and A_4 whereas $A_2 = -\frac{1}{2}\sqrt{4-b}\, A_4$ and $A_3 = -\frac{1}{2}\, A_1$. By substituting the explicit forms of Z_{1N} and Z_{3N} [16] and denoting $y \equiv \sqrt{4-b}qs$, one gets the deviation vector $Z^\mu(s)$

$$\begin{aligned} Z^0 &= T\left[-A_1(1-\cos y)+A_4\left(\tfrac{1}{2}by-2\sin y\right)\right], \\ Z^1 &= X\sqrt{4-b}\left[\tfrac{1}{2}A_1\sin y-A_4(1-\cos y)\right], \qquad Z^2=0\,, \\ Z^3 &= Y\left[-A_1(1-\cos y)+A_4\left(\tfrac{1}{2}by-2\sin y\right)\right]=\frac{Y}{T}\,Z^0\,. \end{aligned} \tag{30}$$

Searching for conjugate points to P_0, one considers three cases depending on values of A_1 and A_4 and in this subsection we study two of these. First, for $A_1=0$ and $A_4\neq 0$, the components Z^0 and Z^1 do not have common roots for $s\neq 0$. In the second case, $A_1\neq 0$ and $A_4=0$, one immediately sees from (30) that $Z^\mu(s)$ is zero at the infinite sequence of points $Q'_n(s'_n)$ on B, where

$$s'_n=\frac{2n\pi}{q\sqrt{4-b}}\,, \qquad n=1,2,\ldots \tag{31}$$

The expression is divergent for $b\to 4$ indicating that ISCOs do not contain conjugate points. The location of Q'_1 is given by its angular distance from P_0

$$\phi'_1-\phi_0=\frac{Ls'_1}{r_0^2}=\frac{2\pi L}{qr_0^2\sqrt{4-b}}\,. \tag{32}$$

Due to arbitrariness of $\lambda(r)$ appearing in q, the distance may be arbitrary and for each SSS spacetime it must be separately computed. For the Schwarzschild metric, $\phi'_1-\phi_0=2\pi[r_0(r_0-6M)^{-1}]^{1/2}>2\pi$. The geometrical interpretation of the sequence $\{Q'_n(s'_n)\}$ geodesics is unclear. For CAdS space $b=0$ and the sequence coincides with that of conjugate points $\{Q_n(s_n)\}$.

Consider a bundle of geodesics $\gamma(\varepsilon)$ which are at ε-distance from the $\mathrm{B}\equiv\gamma(0)$ and emanate from P_0 and their spatial orbits entirely lie in the surface $\theta=\pi/2$, then they are connected to B by the vector $\varepsilon Z^\mu(s)$. It is interesting to see whether these $\gamma(\varepsilon)$ which infinitely many times intersect B at Q'_n have closed orbits. To this end, we notice that all the orbits are contained between the minimal and maximal value of the radius, $r_{\min}=r_0-\frac{1}{2}\varepsilon A_1X\sqrt{4-b}$ and $r_{\max}=r_0+\frac{1}{2}\varepsilon A_1X\sqrt{4-b}$. The successive maxima of r are for $y_n=\sqrt{4-b}\,q\tilde{s}_n=(2n+\frac{1}{2})\pi$ and the arc length of both $\gamma(\varepsilon)$ and B between two successive maxima of r is independent of n

$$Ds\equiv\tilde{s}_{n+1}-\tilde{s}_n=\frac{2\pi}{q\sqrt{4-b}}=s'_1\,. \tag{33}$$

Then also the angular distances between the maxima, $D\phi\equiv\phi(\tilde{s}_{n+1})-\phi(\tilde{s}_n)=\phi(s'_{n+1})-\phi(s'_n)=Ls'_1/r_0^2$, are equal. The orbit of $\gamma(\varepsilon)$ is closed if $D\phi = 2\pi l/m$ for some integers l and m. Then after m periods of change from $r_{\rm max}$ to $r_{\rm min}$ and back to $r_{\rm max}$, the angle ϕ increases by $2\pi l$ and the orbit returns to the same point in the surface $\theta = \pi/2$. Hence the orbit is closed if

$$\frac{\nu_0' e^{\lambda_0}}{3\nu_0' - r_0 \nu_0'^2 + r_0 \nu_0''} = \frac{l^2}{m^2}\,; \tag{34}$$

for every SSS spacetime, it is an algebraic equation for the radius r_0 of B.

4.4. Jacobi fields spanned on e_1^μ and e_3^μ — infinite set of single conjugate points

Finally, we study the third, general, case of search for zeros of the deviation vector, $A_1 \neq 0$ and $A_4 \neq 0$, this case was overlooked in [16]. Since Z^μ is determined up to a constant factor, we put $A_1 = 2$, then

$$Z_1 = 2Z_{11} - \tfrac{1}{2}\sqrt{4-b}A_4 Z_{12} - Z_{13} + A_4 Z_{14} \tag{35}$$

and Z_3 is given by the same combination of Z_{3N}. In search for zeros of Z_1 and Z_3, we apply the solutions given in [16] and replace the two equations by an equivalent simpler system (as above $y = \sqrt{4-b}qs$),

$$\begin{aligned} \sin y + A_4(\cos y - 1) &= 0\,, \\ 2A_4 \sin y - 2\cos y - \tfrac{1}{2}A_4 b y + 2 &= 0\,, \end{aligned} \tag{36}$$

these are equations for A_4 and y; we seek for roots $y \neq 0$. For $y = 2n\pi$, one gets $A_4 = 0$ and returns to the second case and the sequence $\{Q_n'(s_n')\}$. One computes A_4 from the first equation, $A_4 = (1-\cos y)^{-1}\sin y$ for $y \neq 2n\pi$, and inserts it into the other of (36). After some manipulations, one gets

$$\cos y + \frac{b}{8} y \sin y - 1 = 0\,. \tag{37}$$

All positive roots (excluding $2n\pi$) form an infinite sequence $y_n(r_0) = (2n+1)\pi - \delta_n(b(r_0))$, $n = 1, 2, \ldots$, where $\delta_n > 0$ are found numerically. The term $\delta_1(b)$ is of the order of unity for $0 < b < 4$ and decreases for increasing b. The sequence $\{\delta_n(b)\}$ is decreasing and for large n its terms behave as

$$\delta_n \to \frac{16(2n+1)\pi}{(2n+1)^2\pi^2 b - 16}\,. \tag{38}$$

Each root $y_n(r_0)$ determines a separate deviation vector field

$$Z^\mu(n, r_0, s) = Z_1(n, r_0, s)\, e_1^\mu(s) + Z_3(n, r_0, s)\, e_3^\mu(s) \tag{39}$$

connecting the circular $\mathrm{B}(r_0)$ to the nearby geodesic $\gamma(\varepsilon, n, r_0)$ which emanates from $P_0(s = 0)$, lies in $\theta = \pi/2$ and intersects B once at $\bar{Q}_n(\bar{s}_n)$, where

$$\bar{s}_n = \frac{y_n(r_0)}{q(r_0)\sqrt{4 - b(r_0)}}\,. \tag{40}$$

From (31), one sees that $\bar{Q}_n$ lies almost in the centre between Q'_n and Q'_{n+1}, since $\bar{s}_n = s'_n + (\pi - \delta_n)(q\sqrt{4-b})^{-1}$ and $s'_{n+1} = \bar{s}_n + (\pi + \delta_n)(q\sqrt{4-b})^{-1}$. As an example showing the location of $\bar{Q}_1$, we take the Schwarzschild metric:

(i) for $r_0 = 6,26087M$, one has $b = 3,92$, then $\bar{s}_1 = 497,249M$ what corresponds to the angular distance $\bar{\phi}_1 - \phi_0 = 14\pi - 6 \times 10^{-4}$;

(ii) for $r_0 = 78M$, one has $b = 3,04$ then $\bar{s}_1 = 6219,826M$ and $\bar{\phi}_1 - \phi_0 = 2\pi + 2,9246$;

the larger r_0 is, the closer (in terms of the angular distance) to P_0 the conjugate point $\bar{s}_1$ is, but always $\bar{\phi}_1 - \phi_0 > 2\pi$.

5. Conclusions

The main result of the method developed here is that in a general static spherically symmetric spacetime admitting circular timelike geodesics, each stable circular geodesic contains, besides the trivial infinite sequence of conjugate points arising directly from the spherical symmetry, two other infinite sets of conjugate points, whose geometrical interpretation is unclear. At least in the Schwarzschild case, the first conjugate point of each of the two sets appears after making more than one full revolution. This unexpected result shows that the general method for searching for locally maximal timelike curves is effective at least for SSS spacetimes. Furthermore, our current outcomes indicate that the method works well also in some cases beyond this class. Clearly, the practical efficiency of the method crucially depends on one's capability of solving the geodesic deviation equations. This restricts its range to simple metrics and geodesics with simple parametric forms, though the present progress in solving ODEs has increased this range. We expect that in more involved cases, the geodesic structure will be sufficiently revealed by numerical solutions. At present, our published and unpublished results show that the geodesic structure of curved spacetimes is richer than it might be expected.

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