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TRAPPING ON DETERMINISTIC MULTIPLEX NETWORKS

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We study the trapping problem associated with a random walk process that takes place in deterministic multiplex networks. To this end, we consider the Average Trapping Time (ATT) and explore the properties of this system by adjusting the coupling strength λ. We get the analytical expression of ATT with the help of the properties of block matrix, and apply it to two types of deterministic multiplex networks. We find that the ATT in our examples presents a minimum with the change of λ and that the emergence of the minimum under some special initial conditions has a potential relationship with the structural difference of the two graphs in the multiplex network. Our results provide a potential way to control the trapping time in multiplex networks.

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1. Introduction

Trapping problem is an old subject [1] and has been studied and cited frequently. The applications of trapping have covered many fields, for example, the light harvesting problems [2], the chemical kinetics problems [3] and the habitat selection in the ecosystem [4]. In recent several decades, there has been an unprecedented increase of the available network data in the natural world [5–8], such as the social networks [9], the biological networks [10], the Internet [11] and so forth. The trapping problem is also widely studied by scientists associated with various dynamical patterns [12] taking place in the networks [13, 14]. Generally, in the study of the trapping problem in networks, the trap (one or more) is located at a pre-given node, and a random walk process is performed on the network until the walker is trapped.

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One of the most important variables in the trapping problem is the Mean First Passage Time (MFPT) [15] which characterizes the mean time a particle takes to first reach the target site (trap) by random walks in the network from the initial site. MFPT is a significant indicator of how fast the information propagates, and has strong relationship with the volume of the network and the distance between the source and target. When considering the properties of the target regardless of the initial site, the Average Trapping Time (ATT) was proposed by averaging MFTP over all the possible initial nodes in the network. Recent works have shown that the ATT is related to the fractal dimension of the network [16, 17], the location of the trap(s), as well as the structural difference of the network, for example, ATT presents diverse behaviors in one-dimensional system [18], regular lattices [19], Small World networks [20], Scale Free networks [21], modular networks [22] and so on.

Recently, a new type of network structure, the multiplex network (also called the interdependent network or the multiplexity) [23, 24], is proposed. The multiplex network consists of two networks with diverse structures between them (also can be explained as one network with different types of edges), and can be used to model the interdependent systems such as the air route network and the railway network (they share some hub nodes), the social network and the online social network (they have duplicated actors). In view of the special structures of the multiplex networks, many phenomena including the percolation [25] and epidemics [26] are much different from the cases in the isolated networks. However, there are few works mentioning the trapping problem in the multiplex networks.

In this paper, we consider the trapping problem in deterministic multiplex networks. We view the coupling strength as the transition probability for the particle to shift between the two networks in the system. The trap(s) is located at one (both) of the networks in the system, and a symmetric random walk process (particle located in a node will hop to its neighbor nodes with equal probability) is performed on the system. The ATT is calculated and the analytical expression is derived with the help of a degenerated master equation. To further study the properties of the trapping in the multiplex network, we select two deterministic multiplex networks as examples — directed linear multiplex graphs and the regular multiplex networks, and all the results show that the ATT has a minimal value with the changing of the coupling strength under some special initial conditions, which provides a potential way to control the efficiency of the multiplex networks by adjusting the coupling strength.

2. Formulation of random walks with trap(s)

We consider the continuous time random walks on a connected and unweighted network. Suppose that the network has N nodes so that the adjacency matrix A is a $N \times N$, 0-1 matrix with element $A_{ij} = 1$, if there is an edge from node i to node j, otherwise 0 (for undirected case, $A_{ij} = A_{ji} = 1$, if there is an edge between node i and node j).

A random walk process is performed on the network, at each time a particle located at node i in the network will hop to its neighbor nodes randomly. The hop probability from node i to node j at any time is given by $P_{ij} = \frac{A_{ij}}{d_i}$, where $d_i = \sum_j A_{ij}$ is the degree of node i, so the transition probability matrix of the process is given by

$$P = D^{-1}A\,, \tag{1}$$

where D is a diagonal matrix with $D_{ii} = d_i$. This dynamical process can be represented by a master equation:

$$\frac{d}{dt}\vec{\rho}\,(t) = Q\vec{\rho}\,(t)\,, \tag{2}$$

where $Q = P^{\mathrm{T}} - I$ (I is identity matrix) is the transition rate matrix and $\vec{\rho}$ is a column vector whose element $\rho_i(t)$ is the probability the particle locates at node i at time t.

In what follows, we consider the case that the trap(s) is located in the network (particle which hops to the trap will be absorbed and never escapes). In this case, the random walk process described above can be represented by a degenerated master equation

$$\frac{d}{dt}\vec{\rho}\,(t) = \tilde{Q}\vec{\rho}\,(t)\,, \tag{3}$$

where $\tilde{Q}$, the degenerated transition rate matrix, is a submatrix of Q by removing the row and column whose index is corresponding to the ID of the trap node, and the trap is also not included in $\vec{\rho}\,(t)$.

Solving Eq. (3), we can get the probability distribution of the particle at each node at time t

$$\vec{\rho}\,(t) = e^{\tilde{Q}t}\vec{\rho}\,(0)\,, \tag{4}$$

where $\vec{\rho}\,(0)$ is the initial distribution of the particle. For convenience, we denote the sum of the elements in $\vec{\rho}\,(t)$ as

$$s(t) = u^T\vec{\rho}\,(t)\,, \tag{5}$$

and u is a column vector with the all elements 1, $s(t)$ can be called the survival probability at time t.

Now, we introduce a random variable, τ, the trapping time for the particle beginning at the initial node and absorbed by the trap in the end. It is easy to see that $Pr(\tau \geq t) = s(t)$, *i.e.*, the particle is still not trapped until time t (with probability $s(t)$) meaning that the trapping time τ for the particle is not less than t. So the Average Trapping Time (ATT), $\langle\tau\rangle$, is the integer of the survival probability $s(t)$, *i.e.*,

$$\langle\tau\rangle = \int_0^\infty s(t)dt = \int_0^\infty u^T\vec{\rho}(t)dt = \int_0^\infty u^T e^{\tilde{Q}t}\vec{\rho}(0)dt = -u^T\tilde{Q}^{-1}\vec{\rho}(0)\,. \qquad (6)$$

Thus the ATT is related to the elements of the inverse degenerated transition rate matrix[1] $\tilde{Q}^{-1}$ and the initial condition $\vec{\rho}(0)$. In the following section, we will generalize this result to the multiplex networks.

3. Trapping on multiplex networks

The multiplex networks, considered here, are two networks with the same size N but different connected patterns, and the nodes in the two networks are one-to-one correspondence. For details, the nodes of the two networks (with different topologies) are numbered $\{1, 2, \ldots, N\}$ respectively, and then the two nodes with the same ID from the two networks are coupled with strength λ, where λ is a probability for the particle to transmit from one network to another. In this system, λ is the only control parameter. Figure 1

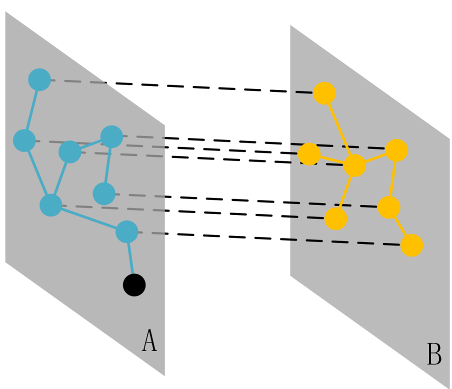

Fig. 1. (Color online) An illustration of the multiplex networks, the nodes (excluding the trap in black color in network A) in network A and B are one-to-one correspondence and coupled with the dashed lines.

[1] Note here that the networks considered in our paper must guarantee that all the nodes (excluding the trap) have at least one path to reach the trap, this is to make the $\tilde{Q}$ invertible.

is an illustration of the multiplex networks (the trap is in black and not numbered). The supra-adjacency matrix of this coupled system can be written as

$$\begin{pmatrix} A & \lambda I \\ \lambda I & B \end{pmatrix}, \tag{7}$$

where A and B are the adjacency matrix (0-1 matrices) of the two networks, and I is an identity matrix.

The random walk process on this multiplex network can be described as follows. At each time step, the particle located at node i in network A (B) will switch to node j in network B (A) with probability λ, or hops to one of its neighbor node j with probability $(1-\lambda)A_{ij}/d_i$. This process will continue until the particle is absorbed by the trap. The degenerated transition rate matrix of this process (as in Eq. (3)) is represented as follows:

$$\tilde{Q} = \begin{pmatrix} (1-\lambda)\tilde{Q}_A - \lambda I & \lambda I \\ \lambda I & (1-\lambda)\tilde{Q}_B - \lambda I \end{pmatrix}, \tag{8}$$

where $\tilde{Q}_A$ and $\tilde{Q}_B$ are the degenerated transition rate matrix of network A and network B respectively. Note that here we regard the coupling strength λ as the bidirectional hop probability for the particle to shift between the two networks.

From the previous section, the average trapping time (ATT) for the coupled system is $\langle\tau\rangle = -u^T\tilde{Q}^{-1}\vec{\rho}(0)$, which is associated with the inverse of the degenerated transition rate matrix $\tilde{Q}$ and the initial condition $\vec{\rho}(0)$. As for the computational complexity of the inverse matrix, especially when the size of the network is very large, we use here the properties of the block matrix, the inverse of $\tilde{Q}$ can be simplified as

$$\tilde{Q}^{-1} = \begin{pmatrix} [\lambda^2 I - L_B L_A]^{-1} & 0 \\ 0 & [\lambda^2 I - L_A L_B]^{-1} \end{pmatrix} \begin{pmatrix} -L_B & \lambda I \\ \lambda I & -L_A \end{pmatrix}, \tag{9}$$

where $L_A = (1-\lambda)\tilde{Q}_A - \lambda I$ and $L_B = (1-\lambda)\tilde{Q}_B - \lambda I$.

In order to explore the properties of ATT in multiplex networks, we perform the random dynamical process in two simple types of deterministic multiplex networks in which the ATT can be calculated analytically in the following two sections.

4. Trapping on directed linear multiplex graphs

The system we study here consists of, as shows Fig. 2, the two directed linear graphs A and B with the opposite direction coupled with a bidirectional strength λ, excluding the trap node, the nodes in the two graphs are one-to-one correspondence, and the coupling strength λ makes the particle transfer between the two graphs. The trap is located at the end of graph A.

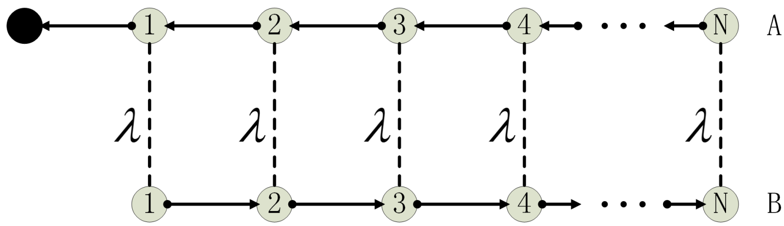

Fig. 2. (Color online) An illustration of the directed linear multiplex graph, nodes with the same ID are coupled with strength λ, the trap is in black and not numbered.

The transition pattern of the particle in this system is very simple, *i.e.*, to its only one neighbor node in the same graph with probability $1-\lambda$ or to its corresponding node in the other graph with probability λ, and $\tilde{Q}_A$ and $\tilde{Q}_B$ in Eq. (8) are given as

$$\tilde{Q}_A = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ 0 & 0 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}, \tag{10}$$

$$\tilde{Q}_B = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & 1 & -1 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{11}$$

In this case, the items $\lambda^2 I - L_B L_A$ and $\lambda^2 I - L_A L_B$ in Eq. (9) are all simple tridiagonal matrices and can be solved under some simple initial conditions.

We consider the three initial conditions:

1. $\vec{\rho}_1(0) = [0, 0, \ldots, 0, 1/2, 0, 0, \ldots, 0, 1/2]^T$ with the N^{th} and $2N^{\text{th}}$ elements $1/2$,
2. $\vec{\rho}_2(0) = \frac{1}{2N}[1, 1, \ldots, 1]^T$ with all elements $1/2N$,
3. $\vec{\rho}_3(0) = [0, 0, \ldots, 0, 1, 0, 0, \ldots, 0]^T$ with the N^{th} element 1.

In condition 2, it is hard for us to get the analytical expression of ATT, while in condition 1, the ATT of the system is given by

$$\langle \tau \rangle = \frac{1}{2\lambda} + \frac{(N^2 - N)\lambda + 2N}{1 - \lambda}, \tag{12}$$

and in condition 3,

$$\langle \tau \rangle = \frac{(N^2 - N)\lambda + 2N}{1 - \lambda}. \tag{13}$$

In Fig. 3, we plot the ATT as a function of the coupling strength λ with different number of nodes N in the initial condition 1 (solid lines), 2 (dashed lines) and 3 (dotted lines) respectively. When $\lambda \longrightarrow 1$, the particle will shift between the two networks and will not be trapped, the figure captures this limiting case. We can see that there is a minimal point in condition 1 and 2 with different N, which indicates an optimal coupling strength for this trapping process. By comprising the lines for the three initial conditions in Fig. 3, we can see that the presence of the minimal point is sensitive to the initial conditions.

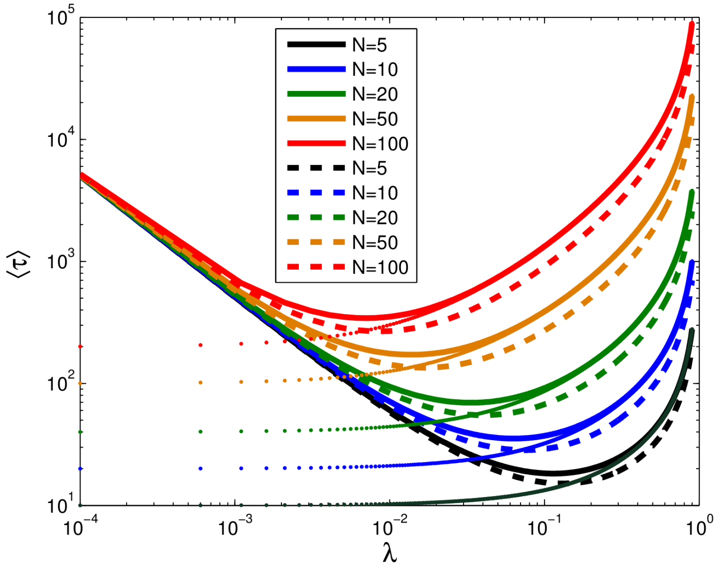

Fig. 3. (Color online) The log–log plot of the ATT as a function of λ in the linear multiplex graph with $N = 5, 10, 20, 50$ and 100. The solid, dashed and dotted lines correspond to the initial condition 1, 2 and 3 respectively.

Furthermore, in condition 2, we can get the minimal point λ^* by $d\langle\tau\rangle/dt = 0$ in Eq. (12), and λ^* is a function of the system size

$$\lambda^* = \frac{1}{\sqrt{2(N^2+N)}+1}\,. \tag{14}$$

λ^* decreases with the increase of the system size N, which means that we can use very small coupling strength to get the optimal trapping efficiency when the system size is considerably large.

5. Trapping on regular multiplex networks

We consider here one type of simple regular deterministic network, the enhanced wheel graph, to perform our random walk process. The enhanced wheel graph (see Fig. 4 for illustration) can be controlled by two parameters (N, k), and we use this type of graph in both layers of our model, taking (N, k_1) and (N, k_2) to denote the parameters of the graphs in the two layers A and B respectively. The two trapping nodes are the central nodes connecting all the other N nodes in A and B respectively, so there are two trapping nodes in this system.

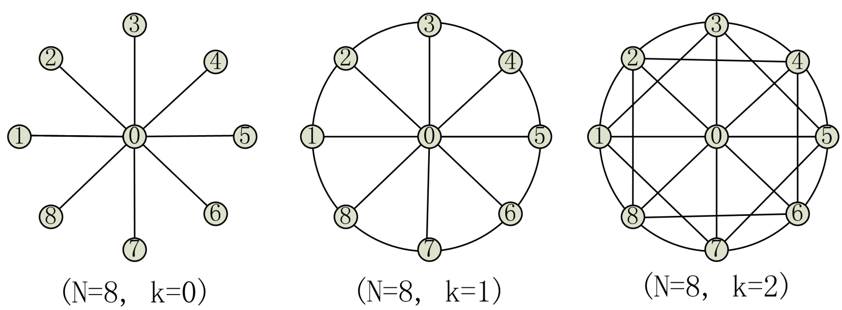

Fig. 4. (Color online) Illustration of the enhanced wheel graph. At first, all the nodes with label $1, 2, \ldots, N$ are listed by a circle and connected to node 0, and then all the nodes except for node 0 add $2k$ edges between its $2k$ nearest neighbors.

This multiplex network is simple and partially symmetric. Each node in the same layer (excluding the trapping node) is equivalent, and any two nodes in different layers are dissimilar because of the distinction of the parameters k_1 and k_2. In addition, the degenerated transition rate matrices, $\tilde{Q}_A$ and $\tilde{Q}_B$ in Eq. (8), are all circulant matrices. The items $[\lambda^2 I - L_B L_A]^{-1}$ and $[\lambda^2 I - L_A L_B]^{-1}$ in Eq. (9) are also circulant matrices, we can easily get the results of $u^T \tilde{Q}^{-1}$ in Eq. (6).

We suppose that the particle initially locates at each node with probability proportional to the node's degree, *i.e.*, the initial condition vector $\vec{\rho}(0) = \frac{1}{N(d_1+d_2)}[d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2]^T$, here $d_i = 2k_i + 1$ is the degree of the nodes in layer A (d_1) and B (d_2) respectively. The ATT of the system under this initial condition is given by

$$\langle\tau\rangle = \frac{\left(-2\,{d_1}^2\,d_2 + {d_1}^2 - 2\,d_1\,{d_2}^2 + {d_2}^2\right)\,\lambda - \left({d_1}^2 + {d_2}^2\right)}{(d_1 + d_2 + 2)\left[(d_1 + d_2 - 1)\,\lambda^2 - (d_1 + d_2 - 2)\,\lambda - 1\right]}\,. \tag{15}$$

We can see that $\langle\tau\rangle$ is independent of the size N. In Fig. 5, we plot the analytical result from Eq. (15). All the lines present a minimum as the change of λ, and the minimal point λ^* diverse with different pairs of (k_1, k_2).

To get the expression of the minimal point (the value of λ indicating the lowest point), we calculate the differential of Eq. (15) by λ and get λ^* as follows:

$$\lambda^* = \frac{-\left(d_1+d_2-1\right)\left({d_1}^2+{d_2}^2\right) - \sqrt{2}\left({d_1}^2-{d_2}^2\right)\sqrt{{d_1}^2 d_2 + d_1 {d_2}^2 - d_1 d_2}}{\left(d_1+d_2-1\right)\left(2{d_1}^2 d_2 - {d_1}^2 + 2 d_1 {d_2}^2 - {d_2}^2\right)}\,,$$

$$(k_2 \geq k_1)\,. \tag{16}$$

Here, we only present the case of $k_2 \geq k_1$ because of the symmetric of the two layers and $d_i = 2k_i + 1$.

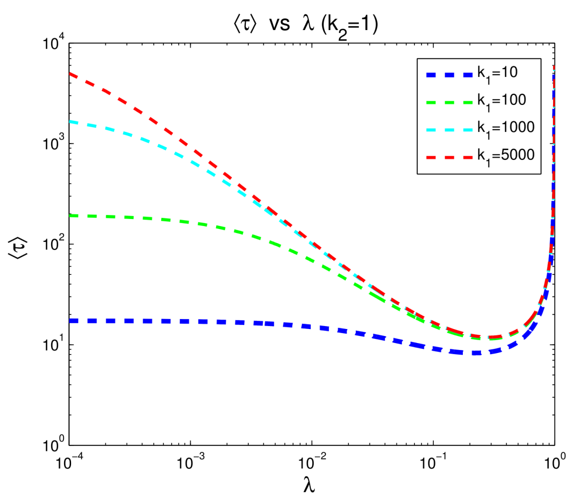

Fig. 5. (Color online) The plot of the result from Eq. (15). The enhanced wheel graph in layer B is fixed with $k_2 = 1$, while the different plots correspond to the different k_1 in layer A. All the graphs in layer A and B have node $N = 20\,000$.

In Fig. 6, we plot the value λ^* for different pair of (k_1, k_2) in the two layers. From the figure it can be seen that the closer of the structures of the two graphs, the smaller the λ^*, and also the more different of the two graph structures, the higher of the minimal point. We can also see a region where the minimum disappears, and this region is becoming broad as the values of (k_1, k_2) becoming large.

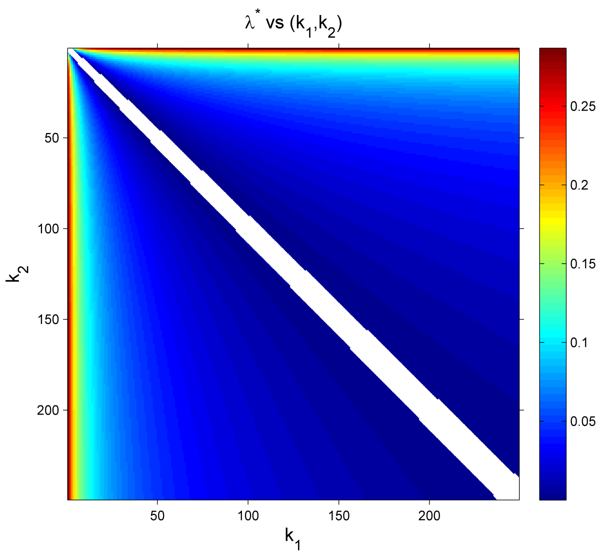

Fig. 6. (Color online) The color representation of the minimal point λ^* which changes with the different pairs of (k_1, k_2) in the two layers. The white color indicates the region where λ^* is absent.

6. Summary

We study the trapping problem in the random walk processes upon a novel system — the multiplex networks with coupling strength λ. The difference between the random walks on the isolated network and the multiplex network is that the particle can randomly shift between two networks with probability λ. The ATT is calculated by the help of a degenerated master equation, and the complexity of the calculation is the same as in the case of the isolated network because of the properties of the block matrix. The analytical expression (Eq. (6)) of ATT is associated with the inverse of the degenerated transition rate matrix and the initial condition.

We further study the trapping problem by applying the random walk processes to two simple deterministic multiplex networks. In the case of the initial condition in our example, both of the $\langle\tau\rangle$ (ATT) in the two systems present a decrease then surge trends as the increase of coupling strength λ, which indicates a point where the minimal trapping time is reached. In the directed linear multiplex graphs, we show that the minimum declines with the increase of the size of the network (Eq. (14)), while the point does not depend on N (Eq. (16)) in the regular multiplex networks because of the symmetry of the system.

This point of minimal trapping time λ^* captures some invisible structural change of the system, which provides a possible access to the pursuit of optimal coupled system or the control of the efficiency of random processes such as searching, localization, navigation *etc.*

REFERENCES

[1] E.W. Montroll, *J. Math. Phys.* **10**, 753 (1969).

[2] A. Bar-Haim, J. Klafter, *J. Phys. Chem.* **B102(10)**, 1662 (1998).

[3] I.M. Sokolov, J. Mai, A. Blumen, *Phys. Rev. Lett.* **79**, 857 (1997).

[4] F. Per, T. Torkild, *Ecology* **84**, 282 (2003).

[5] R. Albert, A.L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).

[6] M.E.J. Newman, *Networks: An Introduction*, Oxford University Press, Oxford 2009.

[7] D.J. Watts, S.H. Strogatz, *Nature* **393**, 440 (1998).

[8] A.L. Barabási, R. Albert, *Science* **286**, 509 (1999).

[9] A. Rapoport, W.J. Horvath, *Behavioral Science* **6**, 279 (1961).

[10] S. Suweis, F. Simini, J.R. Banavar, A. Maritan, *Nature* **500**, 449 (2013).

[11] R. Albert, H. Jeong, A.L. Barabási, *Nature* **401**, 130 (1999).

[12] J.D. Noh, H. Rieger, *Phys. Rev. Lett.* **92**, 118701 (2004).

[13] F. Jasch, A. Blumen, *Phys. Rev.* **E63**, 041108 (2001).

[14] M.F. Shlesinger, *Nature* **443**, 281 (2006).

[15] S. Redner, *A Guide to First-Passage Processes*, Cambridge University Press, Cambridge 2001.

[16] S. Hwang, D.-S. Lee, B. Kahng, *Phys. Rev. Lett.* **109**, 088701 (2012).

[17] B. Meyer, C. Chevalier, R. Voituriez, O. Bénichou, *Phys. Rev.* **E83**, 051116 (2011).

[18] A. Bar-Haim, J. Klafter, *J. Chem. Phys.* **109**, 5187 (1998).

[19] A. García Cantú, E. Abad, *Phys. Rev.* **E77**, 031121 (2008).

[20] F. Jasch, A. Blumen, *Phys. Rev.* **E64**, 066104 (2001).

[21] L.K. Gallos, *Phys. Rev.* **E70**, 046116 (2004).

[22] Z. Zhang, Y. Yang, Y. Lin, *Phys. Rev.* **E85**, 011106 (2012).

[23] V. Nicosia, G. Bianconi, V. Latora, M. Barthelemy, *Phys. Rev. Lett.* **111**, 058701 (2013).

[24] J. Gao, S.V. Buldyrev, H.E. Stanley, S. Havlin, *Nat. Phys.* **8**, 40 (2012).

[25] S.W. Son *et al.*, *Europhys. Lett.* **97**, 16006 (2012).

[26] S. Gómez *et al.*, *Phys. Rev. Lett.* **110**, 028701 (2013).