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TEST AFFINELY-RIGID BODIES IN RIEMANNIAN SPACES AND THEIR QUANTIZATION

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Discussed are some classical and quantization problems of test affinely-rigid bodies moving in Riemannian spaces. We investigate the systems with potential energies for which the variables can be separated. The special case of constant curvature two-dimensional spaces is discussed. Some explicit solutions are found using the Sommerfeld polynomial method.

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1. Introduction

The paper is a continuation of [1] where the model of test metrically-rigid bodies was analyzed on the classical and quantum levels. Here, we consider the concept of an infinitesimal affinely-rigid body, thus in addition to gyroscopic degrees of freedom, the deformative ones are taken into account. Strictly speaking, we investigate in some details two kinds of two-dimensional problems, namely motion of structured material points on the sphere and pseudosphere (Lobachevsky space). Next, we formulate the two-dimensional situation on the quantum level, which is also of some physical interest. Obviously, it may have some direct physical applications when we deal with the dynamics of graphenes, fullerenes and nanotubes [2–4]. Our results may be physically applicable in mechanics of media with microstructure. We mean micromorphic media which are continua of infinitesimal affinely-rigid bodies. Namely, surfaces of such bodies will behave as two-dimensional continua with the effective microstructure induced by the usual three-dimensional microstructure. There are also other possibilities like continua with the layered molecular structure or surface defects. The classical curved space results might be applicable in geophysical or ecological problems. Realize, *e.g.*, catastrophes like those of tankers and their consequences like the resulting motion of two-dimensional pollution “spots” on the oceanic surface, or the sliding motion of continental plates [5].

We follow the standard procedure of quantization in Riemannian manifolds, *i.e.*, we use the L^2-Hilbert space of wave functions in the sense of the usual Riemannian measure (volume element). The classical kinetic energy is replaced by the corresponding quantum expression based on the Laplace–Beltrami operator. The separation of variables is performed and then the corresponding one-dimensional Schrödinger equations are solved using the Sommerfeld polynomial method [6, 7].

2. Classical description

Let (M, g) be an n-dimensional Riemann space, where M is a manifold and g is a metric tensor defined on it. It is clear that in the general case, there is no concept of extended affinely- or metrically-rigid bodies in Riemannian manifolds, because it is rather typical that their isometry and affine groups are trivial. However, we can consider infinitesimal objects of this kind, so small that one can consider them as injected into the tangent spaces. Strictly speaking, such objects are structured material points, *i.e.*, material points with attached linear bases describing internal degrees of freedom. Let us recall that the degrees of freedom were represented by the spatial coordinates x^i $(i = 1, \ldots, n)$ and the components $e^i{}_A$ of the attached co-moving basis e_A $(A = 1, \ldots, n)$ [1]. In the metrically-rigid case, e_A are assumed orthonormal,

$$g_{ij} e^i{}_A e^j{}_B = \delta_{AB} \,, \tag{1}$$

if they are general, our system is an infinitesimal affinely-rigid body (a homogeneous deformable gyroscope). The equations of motion in the metrically-rigid case are noneffective in practical calculations, because the quantities $e^i{}_A$ are not independent generalized coordinates. The way out is to fix some orthonormal field of frames E_A, usually somehow distinguished by the geometry of (M, g). Then, we take the following expansion:

$$e_A(t) = E_B(x(t)) R^B{}_A(t) \,,$$

where $R(t)$ is a time-dependent orthogonal matrix, *i.e.*

$$\delta_{CD} R^C{}_A R^D{}_B = \delta_{AB} \,.$$

There are some standard methods for parameterizing $R(t)$ like Euler angles, rotation vector, *etc.* In the affinely-rigid case, when no constraints on e_A are imposed, there are no essential reasons to use the prescribed reference frame

E_A any longer. But we are interested in problems in which finite rotations interact with extra imposed infinitesimal deformations. It is more convenient and easier to perform calculations if we still use prescribed anholonomic orthonormal frames E_A. In a general case of affine motion, *i.e.*, the one without constraints (1), the expression for the total kinetic energy has the form [1]

$$T = T_{\rm tr} + T_{\rm int} = \frac{m}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} + \frac{1}{2} g_{ij} \frac{De^i{}_A}{Dt} \frac{De^j{}_B}{Dt} J^{AB} \,. \tag{2}$$

Obviously, it remains also true when (1) is imposed. In this formula, the descriptors "tr" and "int" refer obviously to the translational and internal parts, m denotes the mass, and $J^{AB} = J^{BA}$ are co-moving components of the tensor of inertia. If we take the expansion

$$e_A(t) = E_B(x(t)) \phi^B{}_A(t) \,,$$

where $\phi(t) \in \mathrm{GL}(n, \mathbb{R})$ is a general nonsingular matrix then, we obtain for the internal part of kinetic energy the following expression

$$T_{\rm int} = \tfrac{1}{2} \delta_{MN} \phi^M{}_K \phi^N{}_L \widehat{\Omega}^K{}_A \widehat{\Omega}^L{}_B J^{AB} \,. \tag{3}$$

The affine velocity $\widehat{\Omega}$ in the co-moving representation is defined by

$$\frac{De_B}{Dt} := e_A \widehat{\Omega}^A{}_B \,,$$

then

$$\widehat{\Omega}^A{}_B = \left(\phi^{-1}\right)^A{}_F \Gamma^F{}_{DC} \phi^D{}_B \phi^C{}_G v^G + \left(\phi^{-1}\right)^A{}_C \frac{d\phi^C{}_B}{dt} \,, \tag{4}$$

where $\Gamma^F{}_{DC}$ are the anholonomic components of the Levi-Civita affine connection with respect to E_A and the symbols

$$v^G = e^G{}_i \frac{dx^i}{dt}$$

are the co-moving components of the translational velocity.

In the general case of affine motion, the formula (4) may be written in an abbreviated form

$$\widehat{\Omega} = \widehat{\Omega}_{\rm dr} + \widehat{\Omega}_{\rm rl} \,,$$

where

$$\widehat{\Omega}_{\rm dr}{}^A{}_B = \phi^{-1A}{}_F \Gamma^F{}_{DC} \phi^D{}_B \phi^C{}_G v^G \,, \tag{5}$$

$$\widehat{\Omega}_{\rm rl}{}^A{}_B = \phi^{-1A}{}_C \frac{d\phi^C{}_B}{dt} \,. \tag{6}$$

The labels "dr" and "rl" refer respectively to "drift" (or "drive") and "relative". The reason is that ϕ refers to affine rotations with respect to the just passed prescribed reference frame E; the first term describes the time rate of affine rotations contained in the field E itself. When gyroscopic constraints are imposed, all these $\widehat{\Omega}$-objects become skew-symmetric angular velocities. To stress this, sometimes, but not always, we shall then use the symbols $\widehat{\omega}$, $\widehat{\omega}_{\rm dr}$ and $\widehat{\omega}_{\rm rl}$.

One of analytical advantages following from the prescribed reference frame E is the possibility of using the polar and two-polar decompositions [8, 9]

$$\phi = UA = BU = LDR^{-1}\,,$$

where U, L, R are orthogonal, A, B are symmetric, D is diagonal, and obviously,

$$B = UAU^{-1}\,.$$

As usual, U, L, R denote fictitious gyroscopic degrees of freedom extracted from $\phi \in {\rm GL}(n,\mathbb{R})$. The corresponding "co-moving" angular velocities are given by the expressions

$$\widehat{\omega}_{\rm rl} = U^{-1}\frac{dU}{dt}\,, \qquad \widehat{\chi}_{\rm rl} = L^{-1}\frac{dL}{dt}\,, \qquad \widehat{\vartheta}_{\rm rl} = R^{-1}\frac{dR}{dt}\,.$$

Obviously, the "spatial" representation may be used

$$\omega_{\rm rl} = \frac{dU}{dt}U^{-1}, \qquad \chi_{\rm rl} = \frac{dL}{dt}L^{-1}, \qquad \vartheta_{\rm rl} = \frac{dR}{dt}R^{-1}.$$

However, in calculations appearing in practical problems, the "co-moving" objects are more convenient. Of course, in the two-dimensional world, when $n = 2$, these representations coincide.

After some calculations, one can show that the kinetic energy of internal motion $T_{\rm int}$ may be expressed in the following way in terms of the polar decomposition:

$$T_{\rm int} = -\frac{1}{2}{\rm Tr}\left(AJA\widehat{\omega}^2\right) + {\rm Tr}\left(AJ\frac{dA}{dt}\widehat{\omega}\right) + \frac{1}{2}{\rm Tr}\left(J\left(\frac{dA}{dt}\right)^2\right)\,, \qquad (7)$$

where

$$\widehat{\omega} = \widehat{\omega}_{\rm dr} + \widehat{\omega}_{\rm rl} = \widehat{\omega}_{\rm dr} + U^{-1}\frac{dU}{dt}\,, \qquad (8)$$

and obviously $\widehat{\omega}_{\rm dr}$ is the restriction of $\widehat{\Omega}_{\rm dr}$ (5) to the U-rigid motion

$$\widehat{\omega}_{\rm dr}{}^{A}{}_{B} = \left(U^{-1}\right)^{A}{}_{F}\Gamma^{F}{}_{DC}U^{D}{}_{B}U^{C}{}_{E}v^{E}\,. \qquad (9)$$

The two-polar decomposition becomes analytically useful in doubly-isotropic dynamical problems, *i.e.*, the isotropic ones both in the physical and micromaterial spaces. This double isotropy imposes certain restrictions both on the kinetic and potential energies. What concerns the very kinetic energy, the inertial tensor must be isotropic $J = I \cdot \mathrm{id}_n$ (id_n denotes the $n \times n$ identity matrix, I is a scalar constant). Then one can show that (7) becomes

$$T_{\mathrm{int}} = -\frac{I}{2}\mathrm{Tr}\left(D^2\widehat{\chi}^{\,2}\right) - \frac{I}{2}\mathrm{Tr}\left(D^2\widehat{\vartheta}^{\,2}\right) + I\mathrm{Tr}\left(D\widehat{\chi}D\widehat{\vartheta}\right) + \frac{I}{2}\mathrm{Tr}\left(\left(\frac{dD}{dt}\right)^2\right), \tag{10}$$

where now

$$\widehat{\vartheta} = R^{-1}\frac{dR}{dt}, \tag{11}$$

$$\widehat{\chi} = \widehat{\chi}_{\mathrm{dr}} + \widehat{\chi}_{\mathrm{rl}} = \widehat{\chi}_{\mathrm{dr}} + L^{-1}\frac{dL}{dt}, \tag{12}$$

$$\widehat{\chi}_{\mathrm{dr}}{}^{A}{}_{B} = \left(L^{-1}\right)^{A}{}_{F}\Gamma^{F}{}_{DC}L^{D}{}_{B}L^{C}{}_{E}v^{E}. \tag{13}$$

The last formula is quite analogous to (9). Just like there, $\widehat{\chi}$ contains the "drive" term built of the connection coefficients. It is only the L-rotation that is coupled in this way to spatial geometry; the R-rotation is geometry-independent.

We conclude that (10) is structurally identical with the corresponding formula for extended affine bodies [9] with the proviso, however, that $\widehat{\chi}$ contains the "drive" term. The expression for $\widehat{\vartheta}$ is free of such a correction. Everything that has to do with (M, Γ, g)-geometry is absorbed by the $\widehat{\chi}$-term.

3. Special two-dimensional cases

Let us now consider some instructive special examples, namely, the two-dimensional test affinely-rigid body moving in constant-curvature spaces like the spherical space $S^2(0, R)$ and pseudo-spherical Lobachevsky space $H^{2,2,+}(0, R)$. If no gyroscopic constraints are imposed and the internal motion is affine, then, of course, there are four internal degrees of freedom; together with translational motion one obtains six degrees of freedom. We use the same, just as in [1], parametrization of these worlds, *i.e.*, (r, φ) coordinates. We consider highly symmetric systems, when the internal inertia is isotropic, so we can use the two-polar decomposition. When expressed in terms of the two-polar decomposition, *i.e.*,

$$\phi = LDR^{-1} \in \mathrm{GL}(2, \mathbb{R}),$$

then ϕ is parameterized by generalized coordinates $\alpha, \beta, \lambda, \mu$, where

$$L(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad R(\beta) = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},$$

$$D(\lambda, \mu) = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.$$

From now on, all angular velocities become one-dimensional objects, denoted by scalar factors χ, ϑ, more precisely, they are equal to $\chi\epsilon$ and $\vartheta\epsilon$, where $\epsilon := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, *i.e.*,

$$\widehat{\chi} = L^{-1}\frac{dL}{dt} = \chi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \widehat{\vartheta} = R^{-1}\frac{dR}{dt} = \vartheta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and χ is given by the following expressions:

(i) sphere:

$$\chi = \chi_{\mathrm{rl}} + \chi_{\mathrm{dr}} = \frac{d\alpha}{dt} + \cos\frac{r}{R}\frac{d\varphi}{dt},$$

(ii) pseudosphere:

$$\chi = \chi_{\mathrm{rl}} + \chi_{\mathrm{dr}} = \frac{d\alpha}{dt} + \cosh\frac{r}{R}\frac{d\varphi}{dt},$$

but ϑ has no “drive” term, *i.e.*,

$$\vartheta = \frac{d\beta}{dt}.$$

The internal kinetic energy is given by

$$T_{\mathrm{int}} = \frac{I}{2}\left(\left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{d\mu}{dt}\right)^2\right) + \frac{I\left(\lambda^2+\mu^2\right)}{2}\chi^2 + \frac{I\left(\lambda^2+\mu^2\right)}{2}\vartheta^2 - 2I\lambda\mu\chi\vartheta,$$

and the translational part of the kinetic energy T_{tr} has the form [1]:

(i) sphere:

$$T_{\mathrm{tr}} = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2\sin^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right),$$

(ii) pseudosphere:

$$T_{\mathrm{tr}} = \frac{m}{2}\left(\left(\frac{dr}{dt}\right)^2 + R^2 \sinh^2\frac{r}{R}\left(\frac{d\varphi}{dt}\right)^2\right).$$

It turns out that to avoid some embarrassing cross-terms, it is convenient to introduce the "mixed" coordinates

$$x := \frac{1}{\sqrt{2}}(\lambda - \mu), \qquad y := \frac{1}{\sqrt{2}}(\lambda + \mu), \qquad \gamma := \alpha + \beta, \qquad \delta := \alpha - \beta.$$

The inverse rules read that

$$\lambda = \frac{1}{\sqrt{2}}(x + y), \qquad \mu = \frac{1}{\sqrt{2}}(y - x), \qquad \alpha = \frac{1}{2}(\gamma + \delta), \qquad \beta = \frac{1}{2}(\gamma - \delta).$$

3.1. Spherical case

Let us order our generalized coordinates q^i, $i = \overline{1,6}$, as follows:

$$r, \varphi, \gamma, \delta, x, y.$$

As usual in analytical mechanics, the kinetic energy may be identified with some Riemannian structure on the configuration space

$$T = \frac{m}{2} G_{ij}(q)\frac{dq^i}{dt}\frac{dq^j}{dt},$$

where for the above ordering of variables, the matrix $[G_{ij}]$ of the metric tensor G consists of three blocks subsequently placed along the diagonal (looking from the top to bottom):

— the 1×1 block M_1, *i.e.*,

$$M_1 = [1],$$

— the 3×3 block M_2 given as follows:

$$M_2 = \begin{bmatrix} R^2\sin^2\frac{r}{R} + \frac{I}{m}\left(x^2 + y^2\right)\cos^2\frac{r}{R} & \frac{I}{m}x^2\cos\frac{r}{R} & \frac{I}{m}y^2\cos\frac{r}{R} \\ \frac{I}{m}x^2\cos\frac{r}{R} & \frac{I}{m}x^2 & 0 \\ \frac{I}{m}y^2\cos\frac{r}{R} & 0 & \frac{I}{m}y^2 \end{bmatrix},$$

— the 2×2 isotropic block M_3, *i.e.*,

$$M_3 = \frac{I}{m}I_2 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m} \end{bmatrix},$$

where, obviously, I_2 denotes the 2×2 identity matrix. Explicitly, the block matrix $[G_{ij}]$ is given as follows:

$$[G_{ij}] = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} .$$

For the system with deformative degrees of freedom as above, the geodetic model is not physical because it admits unlimited expansion and contraction. Therefore, some potential must be assumed. Just as in the gyroscopic case, we restrict ourselves to some special class of potentials, assuming in particular that all angles φ, α, β (equivalently φ, γ, δ) are cyclic variables. Let us assume that the potential energy separates explicitly with respect to a cyclic variables, *i.e.*,

$$V(r,x,y) = V_r(r) + V_x(x) + V_y(y) .$$

The r-geodetic model with $V_r(r) = 0$ is obviously well formulated. But in d'Alembert models, the (x,y)-geodetic case ($V_x(x) = 0$, $V_y(y) = 0$) would be quite not physical because of admitting unlimited expansion and contraction of the body. This is not the case in affine models where the "elastic vibrations" may be encoded in the very kinetic energy [10].

We consider a special case, when the translational part of the potential energy $V(r)$ has the Bertrand structure [1]:

(a) oscillatory potentials:

$$V_r(r) = \frac{\xi}{2} R^2 \tan^2 \frac{r}{R} , \tag{14}$$

(b) Kepler–Coulomb potentials:

$$V_r(r) = -\frac{\Upsilon}{R} \cot \frac{r}{R} . \tag{15}$$

Of course, with the spherical topology also the geodetic problem belongs here:

(c) $V_r(r) = 0$, *i.e.*, (in a sense) the special case of (a) or (b) when $\xi = 0$, $\Upsilon = 0$.

The mentioned Bertrand models lead to completely integrable and maximally degenerate (hyperintegrable) problems. But even for the simplest, *i.e.*, geodetic, models with the internal degrees of freedom the situation drastically changes. There exist interesting and practically applicable integrable models, but as a rule interaction with internal degrees of freedom reduces or completely removes degeneracy [1, 11–13].

Let us tell a few words about the motion on the plane of deformation variables x, y. There exist "universaly separable" potentials and they have the form

$$V(x,y) = \frac{A}{x^2} + \frac{B}{y^2} + C\left(x^2 + y^2\right) ,$$

where A, B, C are constants. By an appropriate choice of A, B, C (more generally, some arbitrary one-variable functions might be used instead of them), one can obtain the potential with a local minimum at the reference configuration and in a certain neighbourhood of this configuration some phenomenological conditions, known from elasticity theory, will be satisfied. Thus, we consider the following model of potential

$$V(x,y) = \frac{\varkappa}{y^2} + \frac{\varkappa}{2}\left(x^2 + y^2\right) , \qquad \varkappa > 0 , \tag{16}$$

where $\varkappa$ is a constant. The first term prevents any kind of collapse of the two-dimensional body: to the point or to the straight line. The second term of the "harmonic oscillator" type prevents the unlimited expansion.

Just as in the flat-space problems, there exist reasonably-looking models separable in other coordinates in the space of deformation invariants, moreover, separable simultaneously in several systems of coordinates in this space, thus, probably degenerate (hyperintegrable) ones.

An interesting class of separable models is obtained when one uses the polar coordinates (ϱ, ε) in the space of deformation invariants

$$x = \varrho \sin \varepsilon , \qquad y = \varrho \cos \varepsilon . \tag{17}$$

Then, the kinetic energy

$$T = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt} \frac{dq^j}{dt}$$

with coordinates ordered like

$$\left(q^1, q^2, q^3, q^4, q^5, q^6\right) = (r, \varphi, \gamma, \delta, \varrho, \varepsilon)$$

has the block matrix of the metric components

$$[G_{ij}] = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix} ,$$

where

$$K_1 = [1]\,,$$

$$K_2 = \begin{bmatrix} R^2\sin^2\frac{r}{R} + \frac{I}{m}\varrho^2\cos^2\frac{r}{R} & \frac{I}{m}\varrho^2\sin^2\varepsilon\cos\frac{r}{R} & \frac{I}{m}\varrho^2\cos^2\varepsilon\cos\frac{r}{R} \\ \frac{I}{m}\varrho^2\sin^2\varepsilon\cos\frac{r}{R} & \frac{I}{m}\varrho^2\sin^2\varepsilon & 0 \\ \frac{I}{m}\varrho^2\cos^2\varepsilon\cos\frac{r}{R} & 0 & \frac{I}{m}\varrho^2\cos^2\varepsilon \end{bmatrix},$$

$$K_3 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m}\varrho^2 \end{bmatrix} = \frac{I}{m}\begin{bmatrix} 1 & 0 \\ 0 & \varrho^2 \end{bmatrix}. \tag{18}$$

As previously, the system is separable (thus, completely integrable) for potentials independent of (φ, α, β). It is easily seen that such problems with cyclic variables (φ, α, β) are separable for deformation potentials of the form

$$V(\varrho, \varepsilon) = V_\varrho(\varrho) + \frac{V_\varepsilon(\varepsilon)}{\varrho^2}\,, \tag{19}$$

i.e., for the total potentials, we have that

$$V(r, \varrho, \varepsilon) = V_r(r) + V_\varrho(\varrho) + \frac{V_\varepsilon(\varepsilon)}{\varrho^2}\,. \tag{20}$$

The potentials of the form (19), (20) are very convenient from the point of view of nonlinear macroscopic elasticity. Being compatible with the very nature of deformative degrees of freedom,s they are also interesting in the theory of infinitesimal objects. Strictly speaking, we consider the following model of potential:

$$V(\varrho, \varepsilon) = \frac{\varkappa}{\varrho^2\cos^2\varepsilon} + \frac{\varkappa}{2}\varrho^2\,, \qquad \varkappa > 0\,. \tag{21}$$

Again, the first term prevents any kind of collapse of the body and the second term of the "harmonic oscillator" type prevents the unlimited expansion. The natural state (no deformation) minimizes the potential energy; it is a stable equilibrium. Extension in one direction is accompanied by contraction in the orthogonal one.

3.2. Pseudospherical case

Let us now consider a deformable top moving in the Lobachevsky space. All symbols concerning internal degrees of freedom are just those used in spherical geometry. The metric tensor G underlying the kinetic energy expression, *i.e.*,

$$T = \frac{m}{2} G_{ij}(q) \frac{dq^i}{dt} \frac{dq^j}{dt} \,,$$

has the form analogous to the spherical case with the trigonometric functions simply replaced by the hyperbolic ones without any change of sign. Thus, the matrix $[G_{ij}]$ consists of three blocks M_1, M_2, M_3, where

$$\begin{aligned} M_1 &= [1] \,, \\ M_2 &= \begin{bmatrix} R^2 \sinh^2 \frac{r}{R} + \frac{I}{m}\left(x^2+y^2\right) \cosh^2 \frac{r}{R} & \frac{I}{m} x^2 \cosh \frac{r}{R} & \frac{I}{m} y^2 \cosh \frac{r}{R} \\ \frac{I}{m} x^2 \cosh \frac{r}{R} & \frac{I}{m} x^2 & 0 \\ \frac{I}{m} y^2 \cosh \frac{r}{R} & 0 & \frac{I}{m} y^2 \end{bmatrix} , \\ M_3 &= \frac{I}{m} I_2 = \begin{bmatrix} \frac{I}{m} & 0 \\ 0 & \frac{I}{m} \end{bmatrix} . \end{aligned}$$

Again, we assume that the potential energy does not depend on the angles (φ, α, β), *i.e.* they are cyclic variables for the total Hamiltonian. Similarly, when the (x, y)-deformation invariants are used, the most natural separable potentials have the explicitly separated form

$$V(r, x, y) = V_r(r) + V_x(x) + V_y(y) \,,$$

where $V_r(r)$ is a Bertrand-type potential [1], *i.e.*

(a) the "harmonic oscillator"-type potential:

$$V(r) = \frac{\xi}{2} R^2 \tanh^2 \frac{r}{R} \,, \qquad \xi > 0 \,, \tag{22}$$

(b) the "attractive Kepler–Coulomb"-type one

$$V(r) = -\frac{\Upsilon}{R} \coth \frac{r}{R} \,, \qquad \Upsilon > 0 \,, \tag{23}$$

and $V(x, y)$ is given by (16).

Exactly as in the theory of deformable gyroscope, in the spherical space, it is convenient and practically useful to parameterize deformation invariants with the use of polar variables ϱ, ε (see (17)). The only formal difference is that the trigonometric functions of r/R (but not those of ε!) are replaced by the hyperbolic ones without the change of sign, thus, K_2 (18) takes on the form

$$K_2 = \begin{bmatrix} R^2 \sinh^2 \frac{r}{R} + \frac{I}{m}\varrho^2 \cosh^2 \frac{r}{R} & \frac{I}{m}\varrho^2 \sin^2\varepsilon \cosh\frac{r}{R} & \frac{I}{m}\varrho^2 \cos^2\varepsilon \cosh\frac{r}{R} \\ \frac{I}{m}\varrho^2 \sin^2\varepsilon \cosh\frac{r}{R} & \frac{I}{m}\varrho^2 \sin^2\varepsilon & 0 \\ \frac{I}{m}\varrho^2 \cos^2\varepsilon \cosh\frac{r}{R} & 0 & \frac{I}{m}\varrho^2 \cos^2\varepsilon \end{bmatrix}.$$

4. The quantized problems

After all above classical preliminaries, we can formulate the quantized version of our two-dimensional models. Before doing this, we remain briefly within the traditional Schrödinger framework, *i.e.*, with wave mechanics on differential manifolds [14].

Let Q be a configuration space, *i.e.*, a differential manifold of dimension f ($f = \dim Q$ is the number of classical degrees of freedom). If it is endowed with some positive volume measure μ, then the wave functions may be considered as complex scalar fields $\Psi : Q \to \mathbb{C}$. The corresponding scalar product is given by

$$\langle \Psi_1 | \Psi_2 \rangle = \int \overline{\Psi}_1(q) \Psi_2(q) d\mu(q)$$

and our Hilbert space is meant as $L^2(Q, \mu)$. Usually, μ comes from some Riemannian structure (Q, G) and then

$$d\mu(q) = \sqrt{|\det[G_{ij}]|} dq^1 \ldots dq^f \, .$$

For simplicity, the square-root expression will be denoted by $\sqrt{|G|}$.

Quantum operator of the kinetic energy is given by

$$\widehat{T} = -\frac{\hbar^2}{2} \Delta \, ,$$

where Δ denotes the Laplace–Beltrami operator corresponding to G,

$$\Delta = \frac{1}{\sqrt{|G|}} \sum_{i,j} \partial_i \sqrt{|G|} G^{ij} \partial_j = G^{ij} \nabla_i \nabla_j \, ,$$

and ∇ is the Levi-Civita covariant differentiation in the G-sense.

When the problem is non-geodetic and based on some potential $V(q)$, then the corresponding quantum Hamiltonian is given by

$$\widehat{H} = \widehat{T} + \widehat{V},$$

where $\hat{V}$ denotes the operator multiplying wave functions by the potential V,

$$\left(\widehat{V}\Psi\right)(q) = V(q)\Psi(q)\,,$$

usually we do not distinguish them graphically. Below, we return to our two-dimensional models.

A basis of solutions of the stationary Schrödinger equation

$$\widehat{H}\Psi = E\Psi$$

has the form:

(i) x, y-coordinates:

$$\Psi(r,\varphi,\gamma,\delta,x,y) = f_r(r)f_\varphi(\varphi)f_\gamma(\gamma)f_\delta(\delta)f_x(x)f_y(y)\,.$$

It is convenient to use the new variable $\theta = r/R$ for our calculations, then we put

$$\Psi(\theta,\varphi,\gamma,\delta,x,y) = f_\theta(\theta)f_x(x)f_y(y)e^{is\varphi}e^{ij\gamma}e^{iu\delta}\,,$$

where s, j, u are integers.

(ii) ϱ, ε-coordinates:

$$\Psi(r,\varphi,\gamma,\delta,\varrho,\varepsilon) = f_r(r)f_\varphi(\varphi)f_\gamma(\gamma)f_\delta(\delta)f_\varrho(\varrho)f_\varepsilon(\varepsilon)\,.$$

For $\theta = r/R$, we have

$$\Psi(\theta,\varphi,\gamma,\delta,\varrho,\varepsilon) = f_\theta(\theta)f_\varrho(\varrho)f_\varepsilon(\varepsilon)e^{is\varphi}e^{ij\gamma}e^{iu\delta}\,.$$

4.1. Spherical case

After some calculations, we obtain for the Laplace–Beltrami operator the expression written below. Depending on the considered coordinates, it has the following form:

(i) x, y-coordinates:

$$\begin{aligned}\Delta \;=\;& \frac{\partial^2}{\partial r^2} + \frac{\cot\frac{r}{R}}{R}\frac{\partial}{\partial r} + \frac{1}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2} + \left(\frac{m}{Ix^2} + \frac{\cot^2\frac{r}{R}}{R^2}\right)\frac{\partial^2}{\partial\gamma^2}\\ &+\left(\frac{m}{Iy^2} + \frac{\cot^2\frac{r}{R}}{R^2}\right)\frac{\partial^2}{\partial\delta^2} - \frac{2\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\gamma} - \frac{2\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\delta}\\ &+\frac{2\cot^2\frac{r}{R}}{R^2}\frac{\partial^2}{\partial\delta\partial\gamma} + \frac{m}{I}\frac{\partial^2}{\partial x^2} + \frac{m}{Ix}\frac{\partial}{\partial x} + \frac{m}{I}\frac{\partial^2}{\partial y^2} + \frac{m}{Iy}\frac{\partial}{\partial y}\,.\end{aligned}$$

(ii) ϱ, ε-coordinates:

$$\begin{aligned}\Delta \;=\; & \frac{\partial^2}{\partial r^2} + \frac{\cot\frac{r}{R}}{R}\frac{\partial}{\partial r} + \frac{1}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2} + \left(\frac{m}{I\varrho^2\sin^2\varepsilon} + \frac{\cot^2\frac{r}{R}}{R^2}\right)\frac{\partial^2}{\partial\gamma^2} \\ & + \left(\frac{m}{I\varrho^2\cos^2\varepsilon} + \frac{\cot^2\frac{r}{R}}{R^2}\right)\frac{\partial^2}{\partial\delta^2} - \frac{2\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\gamma} - \frac{2\cos\frac{r}{R}}{R^2\sin^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\delta} \\ & + \frac{2\cot^2\frac{r}{R}}{R^2}\frac{\partial^2}{\partial\delta\partial\gamma} + \frac{m}{I}\frac{\partial^2}{\partial\varrho^2} + \frac{2m}{I\varrho}\frac{\partial}{\partial\varrho} + \frac{m}{I\varrho^2}\frac{\partial^2}{\partial\varepsilon^2} + \frac{m(\cos^2\varepsilon - \sin^2\varepsilon)}{I\varrho^2\sin\varepsilon\cos\varepsilon}\frac{\partial}{\partial\varepsilon}\,.\end{aligned}$$

Hence, the stationary Schrödinger equation with an arbitrary potential V leads after the standard separation procedure to the following system of one-dimensional eigenequations:

(i) x, y-coordinates:

$$\begin{aligned}& \frac{d^2 f_x(x)}{dx^2} + \frac{1}{x}\frac{df_x(x)}{dx} - \left(\frac{(k+l)^2}{4x^2} - \frac{2I}{\hbar^2}\left(E_x(x) - V_x(x)\right)\right) f_x(x) = 0\,, \\ & \frac{d^2 f_y(y)}{dy^2} + \frac{1}{y}\frac{df_y(y)}{dy} - \left(\frac{(k-l)^2}{4y^2} - \frac{2I}{\hbar^2}\left(E_y(y) - V_y(y)\right)\right) f_y(y) = 0\,, \\ & \frac{d^2 f_\theta(\theta)}{d\theta^2} + \frac{\cot\theta}{R}\frac{df_\theta(\theta)}{d\theta} \\ & - \left(\frac{(s - k\cos\theta)^2}{R^2\sin^2\theta} - \frac{2m}{\hbar^2}\left(E - E_x(x) - E_y(y) - V_\theta(\theta)\right)\right) f_\theta(\theta) = 0\,, \quad (24)\end{aligned}$$

where E, $E_x(x)$, $E_y(y)$ are fixed values of energies. The relationship between (γ, δ) and (α, β) implies that $k = j + u$ and $l = j - u$.

(ii) ϱ, ε-coordinates:

$$\begin{aligned}& \frac{d^2 f_\varrho(\varrho)}{d\varrho^2} + \frac{2}{\varrho}\frac{df_\varrho(\varrho)}{d\varrho} - \left(\frac{2I}{\hbar^2}(C - V_\varrho(\varrho) - \frac{A}{\varrho^2}\right) f_\varrho(\varrho) = 0\,, \\ & \frac{d^2 f_\varepsilon(\varepsilon)}{d\varepsilon^2} + 2\cot 2\varepsilon\frac{df_\varepsilon(\varepsilon)}{d\varepsilon} \\ & - \left(\frac{k^2 + 2kl\cos 2\varepsilon + l^2}{\sin^2 2\varepsilon} - \frac{2I}{\hbar^2}(A - V_\varepsilon(\varepsilon)\right) f_\varepsilon(\varepsilon) = 0\,, \\ & \frac{d^2 f_\theta(\theta)}{d\theta^2} + \frac{\cot\theta}{R}\frac{df_\theta(\theta)}{d\theta} \\ & - \left(\frac{(s - k\cos\theta)^2}{R^2\sin^2\theta} - \frac{2m}{\hbar^2}\left(E - C - V_\theta(\theta)\right)\right) f_\theta(\theta) = 0\,, \quad (25)\end{aligned}$$

where A, C are separation constants and E is a fixed value of energy.

It is natural to expect that for Bertrand potentials discussed in [1] and for potentials given by equations (16), (21), the resulting Schrödinger equations should be rigorously solvable in terms of some standard special functions. The most convenient way of solving them is to use the Sommerfeld polynomial method [10, 12, 15, 16]. In this method, the solutions are expressed by the usual or confluent Riemann P-functions. They are deeply related to the hypergeometric functions (respectively, usual F or confluent F_1). If the usual convergence demands are imposed, then the hypergeometric functions become polynomials and our solutions are expressed by elementary functions. At the same time, the energy levels and separation constants are expressed by the eigenvalues of the corresponding operators. There exists some special class of potentials to which the Sommerfeld polynomial method is applicable. The restriction to solutions expressible in terms of Riemann P-functions is reasonable, because this class of functions is well investigated and many special functions used in physics may be expressed by them.

4.2. Examples

The one-dimensional eigenequations may be solved only when the explicit form of potential is specified. Here, we consider a special case, when the translational part of the potential energy $V(r)$ ($V(\theta)$) has the Bertrand structure, *i.e.*, oscillatory potential (14) and internal part is given by (16), (21) for (x, y)- and (ϱ, ε)-deformation invariants, respectively.

Applying the Sommerfeld polynomial method, we obtain the energy levels as follows:

(i) x, y-coordinates:

$$E_x(x) = \frac{\hbar\widetilde{\Omega}}{2}\left(4n_x + 2 + |k+l|\right), \tag{26}$$

$$E_y(y) = \frac{\hbar\widetilde{\Omega}}{2}\left(4n_y + 2 + \sqrt{(k-l)^2 + \frac{16\varkappa I}{\hbar^2}}\right), \tag{27}$$

and finally

$$E = \frac{1}{2}\hbar\Omega\left(\left(2n_r+1+|s-k|+\sqrt{(s+k)^2+\frac{\xi m R^4}{\hbar^2}}\right)^2 - 4k^2 - \frac{4\xi m R^4}{\hbar^2} - 1\right) + \frac{1}{2}\hbar\widetilde{\Omega}\left(4n+4+|k+l|+\sqrt{(k-l)^2+\frac{16\varkappa I}{\hbar^2}}\right), \tag{28}$$

where $\Omega = \hbar\omega/4mR^2$, $\omega = \sqrt{\xi/m}$, $\widetilde{\Omega} = \sqrt{\varkappa/I}$, $n_r, n = 0, 1, \ldots$ The energy in (28) depends on an integer combination of the quantum numbers, *i.e.*, $n = n_x + n_y$. After some calculations, we obtain the wave functions in the form:

$$f_x(x) = x^{\chi}\kappa^{\frac{1}{4}+\frac{\chi}{2}}e^{-\frac{\kappa}{2}x^2}F_1\left(-n_x; 1+\chi; \kappa x^2\right), \tag{29}$$

$$f_y(y) = y^{\iota}\kappa^{\frac{1}{4}+\frac{\iota}{2}}e^{-\frac{\kappa}{2}y^2}F_1\left(-n_y; 1+\iota; \kappa y^2\right), \tag{30}$$

$$f_r(r) = \left(\cos\frac{r}{R}\right)^{\zeta}\left(\sin\frac{r}{R}\right)^{\nu}F\left(-n_r, n_r+1+\zeta+\nu; 1+\zeta; \cos^2\frac{r}{R}\right), \tag{31}$$

where

$$\chi = \frac{1}{2}\sqrt{(k-l)^2+\frac{16\varkappa I}{\hbar^2}}, \qquad \kappa = \sqrt{\frac{\varkappa I}{\hbar^2}}, \qquad \iota = \frac{1}{2}|k+l|,$$

$$\zeta = \sqrt{(s+k)^2+\frac{\xi m R^4}{\hbar^2}}, \qquad \nu = |s-k|.$$

(ii) ϱ, ε-coordinates:

Here, the models (14), (21) may be also rigorously solved on the quantum level. We obtain the following expressions for the spectrum of eigenvalues of the constants C, A

$$C = \frac{1}{2}\hbar\widetilde{\Omega}\left(4n_{\varrho}+2+\sqrt{1+\frac{8I}{\hbar^2}A}\right), \tag{32}$$

$$A = \frac{\hbar^2}{8I}\left(\left(4n_{\varepsilon}+2+|k+l|+\sqrt{(k-l)^2+\frac{16\varkappa I}{\hbar^2}}\right)^2-1\right). \tag{33}$$

Finally, we obtain the energy spectrum in the following form:

$$E = \frac{1}{2}\hbar\Omega\left(\left(2n_r+1+|s-k|+\sqrt{(s+k)^2+\frac{\xi m R^4}{\hbar^2}}\right)^2-4k^2-\frac{4\xi m R^4}{\hbar^2}-1\right)$$
$$+\frac{1}{2}\hbar\widetilde{\Omega}\left(4n+4+|k+l|+\sqrt{(k-l)^2+\frac{16\varkappa I}{\hbar^2}}\right). \tag{34}$$

The energy in (34) depends on an integer combination of the quantum numbers, *i.e.*, $n = n_{\varrho} + n_{\varepsilon}$. The functions $f_{\varrho}(\varrho)$, $f_{\varepsilon}(\varepsilon)$ have the form:

$$f_{\varrho}(\varrho) = \varrho^{p}\kappa^{\frac{1+2p}{4}}e^{-\frac{\kappa}{2}\varrho^2}F_1\left(-n_{\varrho}; 1+p; \kappa\varrho^2\right), \tag{35}$$

$$f_{\varepsilon}(\varepsilon) = (\cos 2\varepsilon)^{\chi}(\sin 2\varepsilon)^{\iota}F\left(-n_{\varrho}, n_{\varrho}+1+\chi+\iota; 1+\chi; \cos 2\varepsilon\right), \tag{36}$$

where

$$p = \frac{1}{2}\left(1+\sqrt{1+\frac{8I}{\hbar^2}A}\right)$$

and $f_r(r)$ is given by (31).

4.3. Pseudospherical case

Let us now consider a deformable top moving in the Lobachevsky space. One can easily show that the Laplace–Beltrami operators and one-dimensional eigenequations take on the form exactly as in the theory of deformable gyroscope in the spherical space. The only formal difference is that the trigonometric functions of r/R (θ), (but not those of ε!), are replaced by the hyperbolic ones without the change of sign, thus,

(i) x, y-coordinates:

$$\begin{aligned}\Delta \;=\; & \frac{\partial^2}{\partial r^2} + \frac{\coth\frac{r}{R}}{R}\frac{\partial}{\partial r} + \frac{1}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2} + \left(\frac{m}{Ix^2}+\frac{\coth^2\frac{r}{R}}{R^2}\right)\frac{\partial^2}{\partial\gamma^2}\\ & + \left(\frac{m}{Iy^2}+\frac{\coth^2\frac{r}{R}}{R^2}\right)\frac{\partial^2}{\partial\delta^2} - \frac{2\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\gamma} - \frac{2\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\delta}\\ & + \frac{2\coth^2\frac{r}{R}}{R^2}\frac{\partial^2}{\partial\delta\partial\gamma} + \frac{m}{I}\frac{\partial^2}{\partial x^2} + \frac{m}{Ix}\frac{\partial}{\partial x} + \frac{m}{I}\frac{\partial^2}{\partial y^2} + \frac{m}{Iy}\frac{\partial}{\partial y}\,,\end{aligned}$$

and then

$$\begin{aligned}&\frac{d^2 f_x(x)}{dx^2} + \frac{1}{x}\frac{df_x(x)}{dx} - \left(\frac{(k+l)^2}{4x^2} - \frac{2I}{\hbar^2}\left(E_x(x)-V_x(x)\right)\right) f_x(x) = 0\,,\\ &\frac{d^2 f_y(y)}{dy^2} + \frac{1}{y}\frac{df_y(y)}{dy} - \left(\frac{(k-l)^2}{4y^2} - \frac{2I}{\hbar^2}\left(E_y(y)-V_y(y)\right)\right) f_y(y) = 0\,,\\ &\frac{d^2 f_\theta(\theta)}{d\theta^2} + \frac{\coth\theta}{R}\frac{df_\theta(\theta)}{d\theta}\\ &-\left(\frac{(s-k\cosh\theta)^2}{R^2\sinh^2\theta} - \frac{2m}{\hbar^2}\left(E-E_x(x)-E_y(y)-V_\theta(\theta)\right)\right) f_\theta(\theta) = 0\,. \qquad (37)\end{aligned}$$

(ii) ϱ, ε-coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\coth\frac{r}{R}}{R}\frac{\partial}{\partial r} + \frac{1}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi^2} + \left(\frac{m}{I\varrho^2\sin^2\varepsilon} + \frac{\coth^2\frac{r}{R}}{R^2}\right)\frac{\partial^2}{\partial\gamma^2}$$
$$+\left(\frac{m}{I\varrho^2\cos^2\varepsilon} + \frac{\coth^2\frac{r}{R}}{R^2}\right)\frac{\partial^2}{\partial\delta^2} - \frac{2\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\gamma} - \frac{2\cosh\frac{r}{R}}{R^2\sinh^2\frac{r}{R}}\frac{\partial^2}{\partial\varphi\partial\delta}$$
$$+\frac{2\coth^2\frac{r}{R}}{R^2}\frac{\partial^2}{\partial\delta\partial\gamma} + \frac{m}{I}\frac{\partial^2}{\partial\varrho^2} + \frac{2m}{I\varrho}\frac{\partial}{\partial\varrho} + \frac{m}{I\varrho^2}\frac{\partial^2}{\partial\varepsilon^2} + \frac{m\left(\cos^2\varepsilon - \sin^2\varepsilon\right)}{I\varrho^2\sin\varepsilon\cos\varepsilon}\frac{\partial}{\partial\varepsilon},$$

and then

$$\frac{d^2 f_\varrho(\varrho)}{d\varrho^2} + \frac{2}{\varrho}\frac{df_\varrho(\varrho)}{d\varrho} - \left(\frac{2I}{\hbar^2}\left(C - V_\varrho(\varrho) - \frac{A}{\varrho^2}\right)\right)f_\varrho(\varrho) = 0\,,$$
$$\frac{d^2 f_\varepsilon(\varepsilon)}{d\varepsilon^2} + 2\coth 2\varepsilon\frac{df_\varepsilon(\varepsilon)}{d\varepsilon}$$
$$-\left(\frac{k^2 + 2kl\cos 2\varepsilon + l^2}{\sin^2 2\varepsilon} - \frac{2I}{\hbar^2}\left(A - V_\varepsilon(\varepsilon)\right)\right)f_\varepsilon(\varepsilon) = 0\,,$$
$$\frac{d^2 f_\theta(\theta)}{d\theta^2} + \frac{\coth\theta}{R}\frac{df_\theta}{d\theta} - \left(\frac{(s-k\cosh\theta)^2}{R^2\sinh^2\theta} - \frac{2m}{\hbar^2}\left(E-C-V_\theta(\theta)\right)\right)f_\theta = 0\,. \quad (38)$$

4.4. Examples

Again, translational part of the potential energy has the Bertrand structure, *i.e.*, the "harmonic oscillator"-type potential (22) and internal part is given by (16), (21) for (x, y)- and (ϱ, ε)-deformation invariants, respectively.

We find the energy levels in the form:

(i) x, y-coordinates:

$$E = \frac{1}{2}\hbar\Omega\left(\left(2n_r+1+|s-k|+\sqrt{(s+k)^2+\frac{\xi mR^4}{\hbar^2}}\right)^2+4k^2-\frac{4\xi mR^4}{\hbar^2}-1\right)$$
$$+\frac{1}{2}\hbar\widetilde{\Omega}\left(4n+4+|k+l|+\sqrt{(k-l)^2+\frac{16\varkappa I}{\hbar^2}}\right), \quad (39)$$

and $E_x(x)$, $E_y(y)$ are given by (26) and (27), respectively. The energy in (39) depends on an integer combination of the quantum numbers, *i.e.*,

$n = n_x + n_y$. After some calculations, we obtain the wave functions $f_x(x)$, $f_y(y)$ given by (29), (30) and $f_r(r)$ is as follows:

$$f_r(r) = \left(\cosh\frac{r}{R}\right)^{\zeta}\left(\sinh\frac{r}{R}\right)^{\nu} F\left(-n_r, n_r + 1 + \zeta + \nu; 1 + \zeta; \cosh^2\frac{r}{R}\right). \tag{40}$$

(ii) ϱ, ε-coordinates:

Here, the expression for the energy levels E is as follows:

$$\begin{aligned} E \;=\; & \frac{1}{2}\hbar\Omega\left(\left(2n_r+1+|s-k|+\sqrt{(s+k)^2+\frac{\xi m R^4}{\hbar^2}}\right)^2+4k^2-\frac{4\xi m R^4}{\hbar^2}-1\right) \\ & +\frac{1}{2}\hbar\widetilde{\Omega}\left(4n+4+|k+l|+\sqrt{(k-l)^2+\frac{16\varkappa I}{\hbar^2}}\right). \end{aligned} \tag{41}$$

The corresponding spectrum of eigenvalues of the constants C, A is given by (32), (33), respectively. The energy in (41) depends on an integer combination of the quantum numbers, *i.e.*, $n = n_\varrho + n_\varepsilon$. The functions $f_\varrho(\varrho)$, $f_\varepsilon(\varepsilon)$ and $f_r(r)$ are given by (35), (36), (40), respectively.

5. Conclusions

There exist deformation potentials given by the formula

$$V = \frac{\varkappa}{y^2} + \frac{\varkappa}{2}\left(x^2 + y^2\right) = \frac{\varkappa}{\varrho^2\cos^2\varepsilon} + \frac{\varkappa}{2}\varrho^2, \qquad \varkappa > 0,$$

which could lead to the Schrödinger equation separable simultaneously in two mentioned above coordinate systems. As we know from analytical mechanics, this simultaneous separability usually has to do with hidden symmetries and degeneracy of the problem. The considered systems are one-fold degenerate. On the quantum level, this fact is reflected by the existence of five quantum numbers labeling the energy levels. They cannot be combined into a single quantum number, *i.e.*, there is no total quantum degeneracy (hyperintegrability) with respect to them.

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