NONCOMMUTATIVE AND DYNAMICAL ANALYSIS IN A CURVED PHASE-SPACE

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In this work, we have analyzed the dynamics of the model of a free particle over a 2-sphere in a noncommutative (NC) phase-space. Besides, we have shown that the solution of the equations of motion allows one to show the equivalence between the movement of the particle upon a 2-sphere and the one described by a central field. We have considered the effective force felt by the particle as being caused by the curvature of the space. We have analyzed the NC Poisson algebra of classical observables in order to obtain the NC corrections to Newton's second law analogous to the one caused by a central field. We have also discussed the relation between affine connection and Dirac brackets, as they describe the proper evolution of the model over the surface of constraints in the Lagrangian and Hamiltonian formalisms, respectively. As an application, we have treated the so-called *Zitterbewegung* of the Dirac electron. Since it is assumed to be an observable effect, then we have traced its physical origin by assuming that the electron has an internal structure.

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1. Introduction

The great interest in noncommutative (NC) space-time theories nowadays had inspired the analysis of both several models/theories and the behavior of the divergences in this "new" regime. The last one, by the way, was the original motivation that made Snyder to publish the first paper on the subject [1]. However, since it was demonstrated that the tame of the infinities of QFT was not accomplished [2], the NC formalism was put to sleep for more than fifty years. The string formalism [3] has awaked the noncommutativity procedure and since then we have witnessed the growing interest on noncommutativity [4]. One of these interests is to observe what is the contribution of this formalism concerning both classical mechanics and general relativity. There are several ways to introduce noncommutativity in a physical system. One of them is through the introduction of the so-called NC parameter, which has area dimension and its value is within Planck scale, we can say that, when introduced in classical theories, it could mean a kind of semi-classical approach. Therefore, considering general relativity, noncommutativity is one candidate to obtain a path to quantize gravitation. Considering other classical systems, the introduction of noncommutativity can be realized as a link to Planck scales, like quantum mechanics and its possible relation to UV/IR mixing [5].

In the specific case of classical mechanics considered here, one can analyze the contribution of noncommutativity in order to add a perturbation in Newton's second law for the systems considered [6]. Namely, since the equations of motions are modified, when treated in a NC space, we can ask about the effects in the acceleration coordinate [5, 6]. Here, we will describe classically our system, a free particle in a curved space, *i.e.*, a 2-sphere.

Classical mechanics is one of the most enlightening starting points for introducing many distinct mathematical tools such as differential equations, symplectic structures [7] and, in particular, the basic concepts of differential geometry. For example, in [8], the author used a potential motion to construct the corresponding geometric setting. In this way, some notions such as Riemann metric space, Christoffel symbols, parallel transport and covariant derivative were introduced. We extend this idea here. Instead of treating a potential motion, we will describe a free particle constrained to a curved surface. By constructing its corresponding Lagrangian, we are naturally led to a free motion in a Riemann space. Definitions of metric and Christoffel symbols appear in the course of constructing the dynamics of the model.

We will analyze in details the movement of a particle over a 2-sphere, which is the analog to the nonlinear sigma model problem, which was intensely studied in the past (see [9] and references within). Solution of the equations of motion are given in two different ways. Firstly, we will explore the geometrical properties of the model and after that, we will use the Noether charges to decouple the equations of motion. Moreover, due to the symmetrical structure of the 2-sphere, we will establish the equivalence between the motion in a central field and the free particle over the 2-sphere. It turns out that the central potential is proportional to the curvature of the surface. Then, constrained systems may be also a suitable analogue formalism to introduce general relativity, once Einstein interpreted gravity as a deformation of space-time due to the presence of mass [10]. We will also treat the corresponding hamiltonization of the free particle over the 2-sphere according to the Dirac algorithm for constrained systems [11], which enables one to establish the intrinsic relation between the Dirac brackets and Christoffel symbols, since both of them are supposed to provide the proper evolution over the surface where the model is defined, the first in the phase space and the former, in the configuration space. Although all the calculations are performed classically, we will discuss an application in the quantum realm. We set one possible interpretation of the so-called *Zitterbewegung*, a quivering motion predicted by Schrödinger when he scrutinized the Dirac equation [12]. The time evolution of the electron position operators may be separated in two parts: one in a rectilinear movement and the other oscillates in a ellipse as trajectory, resembling the physical variables of a free particle over a 2-sphere. Thus, the *Zitterbewegung* may be interpreted as a position variable constrained to a 2-sphere if we assume an internal structure to the electron.

The paper is organized as follows. In Section 2, we will discuss an alternative way to introduce a constraint into a Lagrangian. We show the equivalence between the formulation to describe the model in terms of physical variables and the one where the constraint is inserted via Lagrange multipliers. Sections 3 and 4 will be dedicated to a detailed analysis of a particle over a 2-sphere. The construction of the action in terms of physical variables and its limits according to the principle of least action lead naturally to the concepts of metric and affine connection. We will also obtain the general solution of the equations of motion. In Section 5, we will establish the equivalence between the movement in a central field and the one taken by the physical variables of our particle over a 2-sphere. In Section 6, we will provide the hamiltonization of the constrained system described in the previous sections. The application concerning the electron *Zitterbewegung* will be discussed in Section 7. In Section 8, we will introduce noncommutativity in the system and through the equations of motion, via symplectic framework and Poisson brackets, we will analyze the results. Section 9 will be dedicated to the conclusions and perspectives.

2. Constrained systems: the basic formalism

The basic path to introduce a constraint into a Lagrangian is via Lagrange multipliers. Equivalently, knowing *a priori* the constraint of the model, one may find one of the variables in terms of the others and include it into the Lagrangian, leading to a new formulation in terms of physical variables, *i.e.*, whose dynamics is independent of the remaining ones. Our first step in this notes is to show the equivalence between the new and former formulations. Besides, this section also fixes the notation which shall be used throughout the paper.

Let us consider a free particle constrained to the surface

$$\Phi\left(x^i\right) = 0\,, \tag{1}$$

where $x^i = x^i(t)$; $i = 1, \ldots, N$ are the coordinates of the system. There are technical conditions satisfied by the function Φ where we can find one of the variables, say x^1, in terms of the others,

$$\Phi\left(x^i\right) = 0 \Leftrightarrow x^1 = f\left(x^\alpha\right)\,; \qquad \alpha = 2, \ldots, N\,. \tag{2}$$

From now on in this section, Greek letters mean the values $2, \ldots, N$. In this case, x^1 is a nonphysical degree of freedom because its dynamics is dependent of the remaining variables x^α. If $L = L(x^i, \dot{x}^i)$ is the Lagrangian of the free particle in the absence of the constraint (1), then the prescription to construct an action in terms of the physical variables x^α is the following

$$S_1 = \int dt L\left(x^i, \dot{x}^i\right)\Big|_{x^1 = f(x^\alpha)}\,, \tag{3}$$

where we have denoted $\dot{x}^i \equiv \frac{dx^i}{dt}$. We can also write that

$$L\left(x^i, \dot{x}^i\right)\Big|_{x^1=f(x^\alpha)} = L\left(x^1 = f(x^\alpha), \dot{x}^1 = \frac{\partial f}{\partial x^\beta}\dot{x}^\beta, x^\alpha, \dot{x}^\alpha\right) \equiv \bar{L}(x^\alpha, \dot{x}^\alpha)\,. \tag{4}$$

The notation $\bar{L}$ indicates the substitution of $x^1 = f(x^\alpha)$ in (3) and repeated indexes mean summation, as usual. To obtain the Euler–Lagrange equations of (3), we evaluate separately the derivatives of the expression (4)

$$\frac{\partial \bar{L}\left(x^\alpha, \dot{x}^\alpha\right)}{\partial x^\gamma} = \frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial x^1}\bigg|\frac{\partial f}{\partial x^\gamma} + \frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial \dot{x}^1}\bigg|\frac{\partial^2 f}{\partial x^\gamma \partial x^\beta}\dot{x}^\beta + \frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial x^\gamma}\bigg|\,, \tag{5}$$

$$\frac{\partial \bar{L}\left(x^\alpha, \dot{x}^\alpha\right)}{\partial \dot{x}^\gamma} = \frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial \dot{x}^1}\bigg|\frac{\partial f}{\partial x^\gamma} + \frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial \dot{x}^\gamma}\bigg|\,, \tag{6}$$

where $|$ corresponds to the substitution expressed in (3). It will be used in subsequent calculations. Hence, the equations of motion given by

$$\frac{\delta S_1}{\delta x^\gamma} = \frac{\partial \bar{L}\left(x^\alpha, \dot{x}^\alpha\right)}{\partial x^\gamma} - \frac{d}{dt}\left(\frac{\partial \bar{L}\left(x^\alpha, \dot{x}^\alpha\right)}{\partial \dot{x}^\gamma}\right) = 0 \tag{7}$$

provide, after rearranging the terms,

$$\left.\frac{\delta S_1}{\delta x^\gamma}\right| + \left.\frac{\delta S_1}{\delta x^1}\right| \frac{\partial f}{\partial x^\gamma} = 0\,. \tag{8}$$

The idea here is to show that one may insert the constraint $\Phi(x^i) = 0$ into the initial Lagrangian leading to an equivalent description. Let us consider the following action

$$S_2 = \int dt \tilde{L}\left(x^i, \dot{x}^i, \lambda\right)\,, \tag{9}$$

defined in an extended configuration space parametrized by x^i and λ, where

$$\tilde{L}\left(x^i, \dot{x}^i, \lambda\right) = L\left(x^i, \dot{x}^i\right) + \lambda \Phi\left(x^i\right)\,. \tag{10}$$

The functions L and Φ are the same as the initial construction and λ is a Lagrange multiplier. Hence, the Euler–Lagrange equations are

$$\frac{\delta S_2}{\delta x^1} = 0 \Rightarrow \frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial x^1} + \lambda \frac{\partial \Phi}{\partial x^1} = \frac{d}{dt}\left(\frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial \dot{x}^1}\right)\,, \tag{11}$$

$$\frac{\delta S_2}{\delta x^\gamma} = 0 \Rightarrow \frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial x^\gamma} + \lambda \frac{\partial \Phi}{\partial x^\gamma} = \frac{d}{dt}\left(\frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial \dot{x}^\gamma}\right)\,, \tag{12}$$

$$\frac{\delta S_2}{\delta \lambda} = 0 \Rightarrow \Phi\left(x^i\right) = 0\,. \tag{13}$$

From (11), we can find that

$$\lambda = -\left(\frac{\partial \Phi}{\partial x^1}\right)^{-1}\left[\frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial \dot{x}^i}\right)\right]\,. \tag{14}$$

The substitution of (14) in (12) eliminates the λ-dependence of equations of motion

$$\begin{aligned} &\frac{\partial L(x^i, \dot{x}^i)}{\partial x^\gamma} - \frac{d}{dt}\left(\frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial \dot{x}^\gamma}\right) \\ &- \left(\frac{\partial \Phi}{\partial x^1}\right)^{-1} \frac{\partial \Phi}{\partial x^\gamma}\left[\frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L\left(x^i, \dot{x}^i\right)}{\partial \dot{x}^1}\right)\right] = 0\,. \end{aligned} \tag{15}$$

Finally, from (13) and according to (2)

$$\Phi\left(x^{i}\right)=0 \Leftrightarrow x^{1}=f\left(x^{\alpha}\right) . \tag{16}$$

Substitution of $x^1 = f(x^\alpha)$ into the constraint $\Phi(x^i) = 0$ gives the identity

$$\Phi\left(x^{1}=f\left(x^{\alpha}\right), x^{\alpha}\right) \equiv 0\,, \tag{17}$$

whose derivative provides

$$0=\frac{d}{dx^{\gamma}} \Phi\left(x^{1}=f(x^{\alpha}), x^{\alpha}\right)=\left.\frac{\partial \Phi\left(x^{i}\right)}{\partial x^{1}}\right|\frac{\partial f}{\partial x^{\gamma}}+\left.\frac{\partial \Phi\left(x^{i}\right)}{\partial x^{\gamma}}\right| . \tag{18}$$

Then, we have that

$$\frac{\partial f}{\partial x^{\gamma}}=-\left[\left.\frac{\partial \Phi\left(x^{i}\right)}{\partial x^{1}}\right|\right]^{-1}\left.\frac{\partial \Phi\left(x^{1}\right)}{\partial x^{\gamma}}\right| . \tag{19}$$

This expression appears in (15), which is now rewritten by eliminating x^1

$$\begin{aligned}&\left[\frac{\partial L\left(x^{i}, \dot{x}^{i}\right)}{\partial x^{\gamma}}-\frac{d}{dt}\left(\frac{\partial L\left(x^{i}, \dot{x}^{i}\right)}{\partial \dot{x}^{\gamma}}\right)\right]\Bigg| \\ &+\left[\frac{\partial L\left(x^{i}, \dot{x}^{i}\right)}{\partial x^{1}}-\frac{d}{dt}\left(\frac{\partial L\left(x^{i}, \dot{x}^{i}\right)}{\partial \dot{x}^{1}}\right)\right]\Bigg|\frac{\partial f}{\partial x^{\gamma}}=0\,.\end{aligned} \tag{20}$$

Since $\frac{d}{dt}\left(L|\right)=\frac{dL}{dt}|$, we arrive at

$$\left.\frac{\delta S_{1}}{\delta x^{\gamma}}\right|+\left.\frac{\delta S_{1}}{\delta x^{1}}\right|\frac{\partial f}{\partial x^{\gamma}}=0\,. \tag{21}$$

These are the same equations of motion of the initial formulation, see (8). The equivalence between both constructions that have been developed so far becomes clearer if we compare the number of degrees of freedom in each description. The initial construction described by $\bar{L}=\bar{L}(x^\gamma, \dot{x}^\gamma)$ was formulated by eliminating x^1 with the previous knowledge of the constraint surface the model is immersed in. We are left with $N-1$ degrees of freedom. On the other hand, the second one starts with $N+1$ variables. First, we have excluded λ from the description by using (11). Then, with the help of (13), x^1 was eliminated, see (16). These two steps left us with $N+1-2=N-1$ degrees of freedom, as expected. This concludes the equivalence between S_1 and S_2. An application will be treated in the next section, when we consider the example of a particle over a 2-sphere.

3. A concrete example of constrained dynamics: particle over a 2-sphere

We will now discuss an application of the result found in the last section. Actually, the main aim of these notes is the classical and NC descriptions of a free particle over a 2-sphere. Besides, the example of the particle over a 2-sphere will be used for a classical description of the Dirac spinning electron, see Section 7.

Let m be the mass of the particle and $x^i = x^i(t)$, $i = 1, 2, 3$, its spatial coordinates. Since we want to formulate the particle evolution constrained to a 2-sphere, we take the following action,

$$S_\lambda = \int dt \left[\frac{m}{2} \delta_{ij} \dot{x}^i \dot{x}^j + \lambda \left(\delta_{ij} x^i x^j - a^2 \right) \right] , \tag{22}$$

where δ_{ij} stands for the delta Kronecker symbol and λ is a Lagrange multiplier. S_λ has manifest SO(3)-invariance, which guarantees, for example, conservation of angular momentum. The equation of motion for λ gives the desired constraint

$$\delta_{ij} x^i x^j = a^2 . \tag{23}$$

So, Eq. (22), in fact, describes a free particle over a 2-sphere of radius a. On the other hand, we could exclude one of the variables with the help of (23)

$$x^3 = \pm \sqrt{a^2 - \delta_{ij} x^i x^j} , \tag{24}$$

where i, j run the values 1 and 2. Concerning the parametrization of the 2-sphere, we take the upper half plane $x^3 > 0$. Then, according to (3), we substitute (24) into the action for the free particle in a flat 3-dimensional space leading to

$$S_{\mathrm{ph}} = \int dt \frac{m}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta , \tag{25}$$

where 'ph' refers to the physical variables and we have that

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta} + \frac{x_\alpha x_\beta}{a^2 - \delta_{ij} x^i x^j} . \tag{26}$$

The action was named S_{ph} since we have eliminated the spurious degree of freedom x^3, obtaining an equivalent description of the particle over a 2-sphere in terms of physical variables x^1, x^2. It has a simple interpretation: since the particle is constrained to a 2-sphere, (25) describes a free particle in a Riemann space whose metric is given by $g_{\alpha\beta}$ [13]. The elimination of

x^3 naturally led us to the concept of first fundamental form (or metric) [14]. In the limit $a \to +\infty$, we have a free particle in a flat bi-dimensional space. Namely, $g_{\alpha\beta} \to \delta_{\alpha\beta}$ and the Lagrangian originated from (25) becomes the kinetic energy of the particle,

$$\frac{m}{2} g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \to \frac{m}{2} \left[\left(\dot{x}^1\right)^2 + \left(\dot{x}^2\right)^2 \right] . \tag{27}$$

We now turn our attention to the time evolution of the model. The dynamics is governed by the principle of least action. The minimization $\delta S_{\rm ph} = 0$ gives the equation of motion

$$\ddot{x}^\alpha = G^\alpha{}_{\sigma\beta} \dot{x}^\sigma \dot{x}^\beta , \tag{28}$$

where

$$G^\alpha{}_{\sigma\beta} = g^{\alpha\gamma} \left(\tfrac{1}{2} \partial_\gamma g_{\sigma\beta} - \partial_\sigma g_{\gamma\beta} \right) . \tag{29}$$

$g^{\alpha\gamma}$ corresponds to the inverse of the metric: $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha{}_\gamma$ and $\partial_\gamma \equiv \frac{\partial}{\partial x^\gamma}$. Explicit calculation of G gives

$$G^\alpha{}_{\sigma\beta} = \frac{1}{2} \frac{x_\sigma \delta^\alpha{}_\beta - x_\beta \delta^\alpha{}_\sigma}{a^2 - \delta_{\gamma\rho} x^\gamma x^\rho} - \frac{x^\alpha g_{\sigma\beta}}{a^2} . \tag{30}$$

The first term of G is antisymmetric on $\sigma \leftrightarrow \beta$. Then, it vanishes when contracted with the symmetric factor $\dot{x}^\sigma \dot{x}^\beta$ of (28). We are finally left with

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 , \tag{31}$$

and Γ is given by

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{x^\alpha}{a^2} g_{\beta\gamma} , \tag{32}$$

where (31) is the equation of a geodesic line: the particle chooses the trajectory with the shortest length. Moreover, the principle of least action gave us the Christoffel symbol or affine connection $\Gamma^\alpha{}_{\beta\gamma}$. Once again, the "static" concepts of differential geometry (geodesic line and second fundamental form Γ) were discovered via a dynamical realization. In the limit $a \to +\infty$, the equation of motion tends to

$$\ddot{x}^\alpha = 0 , \tag{33}$$

which corresponds to the motion of a free particle (in flat bi-dimensional space) since $\Gamma^\alpha{}_{\beta\gamma} \to 0$, in accordance with our intuition.

In the next section, we will solve the equations of motion (31). It will be accomplished in two different ways. The first one is by exploring the geometric setup that the model was constructed and the second one is by using the conserved currents obtained from the Noether theorem [15].

4. Solution to equations of motion

Let us now obtain the solutions of the equations of motion (31) in the commutative plane. It will be obtained via two different approaches. In the first one, we will use the geometric structure of the problem, *i.e.*, since the particle is free, it is supposed to describe a circumference of radius a with constant angular velocity. Besides, we will also use the Noether theorem which provides two integrals of motion, which allow us to find the general solution of equations of motion.

4.1. Solving equations of motion: geometrical point of view

There is a standard way to solve the equations of motion in different models: if we know a particular solution, the general one is obtained by applying a transformation group in which the model is based on. For example, in [16], the author finds general spinors connected with an arbitrary state of motion of the Dirac electron by boosting plane wave solutions of the Dirac equation for a particle at rest. We will use the same prescription here. Initially, we take the following particular solution,

$$x^i(t) = \begin{pmatrix} 0 \\ a\sin\omega t \\ a\cos\omega t \end{pmatrix}, \tag{34}$$

that describes our free particle with constant (and arbitrary) angular velocity ω constrained to the 2-sphere of radius a. A direct calculation shows that it satisfies (31). We have restricted the motion to the plane x^2x^3. The general solution is achieved by three successive passive rotations around x^1, x^2 and x^3 axes. The rotations introduce three new and arbitrary parameters which, combined with ω, complete the necessary number of four constants of integration concerning the second order equation (31). Denoting $\mathcal{R}_{x^i}(\theta_j)$ the rotation around x^i-axis by an angle θ_j, we have

$$x^i(t) = [\mathcal{R}_{x^3}(\theta_3)]^i{}_j\,[\mathcal{R}_{x^2}(\theta_2)]^j{}_k\,[\mathcal{R}_{x^1}(\theta_1)]^k{}_l\, y^l(t)\,, \tag{35}$$

where, for example,

$$\mathcal{R}_{x^1}(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix}. \tag{36}$$

The other matrices $\mathcal{R}_{x^2}(\theta_2)$ and $\mathcal{R}_{x^3}(\theta_3)$ are well-known from the SO(3)-group. The parameters θ_i are the Euler angles, taken in the $x^1x^2x^3$ convention. For different representations of the Euler angles, see for example, [17, 18].

So, for the general solution one can obtain that

$$x^i(t) = \begin{pmatrix} a\sin\theta_2\cos\theta_3\cos(\omega t+\theta_1) + a\sin\theta_3\sin(\omega t+\theta_1) \\ -a\sin\theta_2\sin\theta_3\cos(\omega t+\theta_1) + a\cos\theta_3\sin(\omega t+\theta_1) \\ a\cos\theta_2\cos(\omega t+\theta_1) \end{pmatrix} . \tag{37}$$

In Section 3, we have withdrawn the variable x^3 from the description. One may check that the expression above obeys the identity

$$x^3(t) \equiv \sqrt{a^2 - (x^1(t))^2 - (x^2(t))^2}\,. \tag{38}$$

Then, the physical solution is given by the projection of $x^i = x^i(t)$ onto the plane x^1x^2. On this plane, the trajectory is an ellipse. In fact, with no loss of generality[1], we take to the solution

$$\tilde{x}^i(t) = [\mathcal{R}_{x^2}(\theta_2)]^i{}_k\,[\mathcal{R}_{x^1}(\theta_1)]^k{}_l\,y^l(t) \tag{39}$$

in the plane x^1x^2,

$$\tilde{x}^\alpha(t) = \begin{pmatrix} a\sin\theta_2\cos(\omega t+\theta_1) \\ a\sin(\omega t+\theta_1) \end{pmatrix} . \tag{40}$$

The trajectory is obtained by excluding the time of the parametric equations (40). It is given by

$$\frac{\left(\tilde{x}^1\right)^2}{a^2\sin^2\theta_2} + \frac{\left(\tilde{x}^2\right)^2}{a^2} = 1\,, \tag{41}$$

which is the equation of an ellipse.

Finally, the general solution that we were looking for is given by the projection of (37) in the plane x^1x^2,

$$x^\alpha(t) = \begin{pmatrix} a\sin\theta_2\cos\theta_3\cos(\omega t+\theta_1) + a\sin\theta_3\sin(\omega t+\theta_1) \\ -a\sin\theta_2\sin\theta_3\cos(\omega t+\theta_1) + a\cos\theta_3\sin(\omega t+\theta_1) \end{pmatrix} , \tag{42}$$

whose trajectory is an ellipse. One then can ask about the possibility of interpreting this movement as generated by a central field. It will be discussed in Section 5. Our next step consists of finding $x^\alpha = x^\alpha(t)$ with the help of conserved quantities.

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[1] The only effect of the last rotation $\mathcal{R}_{x^3}(\theta_3)$ is to make the semi-axes of the ellipse not coincident with the coordinate axes x^1 and x^2. Thus, for simplicity, we obtain the trajectory by looking to the solution $\tilde{x}^\alpha$ in (40).

4.2. Solving equations of motion: conserved quantities

One of the most impressive results in classical mechanics is the Noether theorem: if an action is invariant under a global transformation, then there is a related integral of motion, known as Noether charge. In our case, we may look at (22) or (25) since they are equivalent. Considering that (22) has a global SO(3)-invariance,

$$x^i \to x'^i = R^i{}_j x^j\,; \qquad \text{where} \quad R^T = R^{-1}\,. \tag{43}$$

It implies the conservation of angular momentum

$$L_i = m\varepsilon_{ijk} x^j \dot{x}^k \Rightarrow \frac{dL_i}{dt} = 0\,. \tag{44}$$

One may also look at the expression (25), which is invariant under time translations

$$t \to t' = t + \tau\,. \tag{45}$$

In this case, the corresponding conserved quantity is

$$E = \frac{m}{2} g_{\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta\,, \tag{46}$$

where E is considered as the energy of the particle. We now turn our attention to the equation of motion (31). It is immediately decoupled if we use (46)

$$\ddot{x}^\alpha + \frac{x^\alpha}{a^2} g_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0 \Rightarrow \ddot{x}^\alpha + \frac{2E}{ma^2} x^\alpha = 0\,. \tag{47}$$

Thus, the solution of (47) can promptly be written

$$x^\alpha(t) = A^\alpha \sin(\Omega t + \varphi_\alpha)\,; \qquad \Omega = \frac{\sqrt{2mE}}{ma}\,, \tag{48}$$

where A^α and φ_α are arbitrary constants of integration. Substitution of solution (48) in (44) and (46) gives, respectively,

$$\frac{L_3}{m\Omega} = -A^1 A^2 \sin(\varphi_2 - \varphi_1)\,, \tag{49}$$

$$\left(A^1\right)^2 + \left(A^2\right)^2 = a^2 + \frac{L_3^2}{2mE}\,, \tag{50}$$

and (49) means that the angle between $x^1(t)$ and $x^2(t)$ is $\varphi_2 - \varphi_1$. If we assume that $\varphi_2 - \varphi_1 = \frac{\pi}{2}$, then the general solution may be achieved by rotating the particular solution with this restriction. So, first if we substitute (49) in (50), we have that

$$\left(A^1\right)^2 + \left(A^2\right)^2 = a^2 + \frac{\left(A^1\right)^2 \left(A^2\right)^2}{a^2} \Rightarrow A^1 = a \Rightarrow A^2 = -\frac{L_3}{\sqrt{2mE}} . \quad (51)$$

We have then a particular solution $x_p^\alpha = x_p^\alpha(t)$, where x_p^1 and x_p^2 are perpendicular

$$x_p^\alpha(t) = \begin{pmatrix} a\sin(\Omega t + \varphi_1) \\ -\frac{L_3}{\sqrt{2mE}}\cos(\Omega t + \varphi_1) \end{pmatrix} . \quad (52)$$

A final general solution can be obtained by rotating the particular solution above in an active way,

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cos\varphi_2 & \sin\varphi_2 \\ -\sin\varphi_2 & \cos\varphi_2 \end{pmatrix} \begin{pmatrix} x_p^1 \\ x_p^2 \end{pmatrix} , \quad (53)$$

that is,

$$x^\alpha(t) = \begin{pmatrix} a\cos\varphi_2\sin(\Omega t + \varphi_1) - \frac{L_3}{\sqrt{2mE}}\sin\varphi_2\cos(\Omega t + \varphi_1) \\ -a\sin\varphi_2\sin(\Omega t + \varphi_1) - \frac{L_3}{\sqrt{2mE}}\cos\varphi_2\cos(\Omega t + \varphi_1) \end{pmatrix} . \quad (54)$$

As expected, we have four constants of integration: $\varphi_{1,2}$, E and L_3. Equivalence between the two solutions (42) and (54) is manifest if we write

$$\begin{aligned} \omega &= \Omega , \\ \theta_1 &= \varphi_1 , \\ \theta_3 &= \varphi_2 + \frac{\pi}{2} , \\ a\sin\theta_2 &= \frac{L_3}{\sqrt{2mE}} . \end{aligned} \quad (55)$$

In the next section, we will discuss a possible interpretation of the solution of the equations of motion in terms of an effective central potential induced by the space curvature.

5. Equivalence between a central force problem and the particle over a 2-sphere

The movement of the particle over the 2-sphere was completely described so far by the physical variables $x^\alpha(t)$, $\alpha = 1, 2$, see (42) or (54). Since the trajectory is an ellipse, one may think that it could be derived by a central field. So, the objective of this section is to show that the solution $x^\alpha(t)$ is equivalent to the one described by an isotropic harmonic oscillator. We already know the time evolution of the particle. The idea is, instead of solving a differential equation of motion, we would like to obtain it. For that, we will use polar coordinates $(x^1, x^2) \leftrightarrow (r, \theta)$

$$\begin{array}{l} x^1 = r\cos\theta \\ x^2 = r\sin\theta \end{array} \quad \Leftrightarrow \quad \begin{array}{l} r = \sqrt{(x^1)^2 + (x^2)^2} \\ \theta = \arctan\left(\frac{x^2}{x^1}\right). \end{array} \tag{56}$$

For simplicity, we have used the solution (40). Let us construct the differential equations obeyed by the coordinates θ and r. We have that

$$\theta(t) = \arctan\left(\frac{x^2(t)}{x^1(t)}\right) = \arctan\left(\frac{\sin\Delta}{\sin\theta_2\cos\Delta}\right), \tag{57}$$

$$r(t) = a\sqrt{\sin^2\theta_2\cos^2\Delta + \sin^2\Delta}, \tag{58}$$

where we have used the shorthand notation $\Delta = \omega t + \theta_1$. First time derivative of (57) gives

$$\dot{\theta}(t) = \frac{\omega a^2\sin\theta_2}{r^2(t)} = \frac{L_3}{mr^2(t)}, \tag{59}$$

since the angular momentum L_3 is given by

$$L_3 = m\left(\dot{x}^2x^1 - \dot{x}^1x^2\right) = m\omega a^2\sin\theta_2\,. \tag{60}$$

We turn our attention to the radial variable. It is a tedious but rather direct calculation to obtain the second order time derivative of Eq. (58). We have

$$\dot{r}(t) = \frac{\omega a^2\sin 2\Delta\left(1 - \sin^2\theta_2\right)}{2r}. \tag{61}$$

The second time derivative reads

$$\ddot{r} = \frac{\omega^2a^2\cos 2\Delta\left(1 - \sin^2\theta_2\right)}{r} - \frac{\omega a^2\sin 2\Delta\left(1 - \sin\theta_2\right)}{2r^2}\dot{r}\,. \tag{62}$$

Substituting $\dot{r}(t)$ into the expression above, one can find after rearranging the terms,

$$\ddot{r} = \frac{\omega^2a^4}{r^3}\left[-\left(\cos^2\Delta\sin^2\theta_2 + \sin^2\Delta\right)^2 + \sin^2\theta_2\left(\sin^2\Delta + \cos^2\Delta\right)^2\right], \tag{63}$$

which multiplied by the mass of the particle becomes

$$m\ddot{r} = -\omega^2 r + \frac{L_3^2}{m^2 r^3} \Rightarrow m\ddot{r} = -m\omega^2 r + \frac{L_3^2}{mr^3}\,. \tag{64}$$

Equations (59) and (64) are exactly the ones which rule the movement of a particle in a central field [17]. Equation (64) may be seen as the second Newton's law for a particle in a isotropic harmonic oscillator. The term $\frac{L_3^2}{mr^3}$ corresponds to the centrifugal force always present when one writes a central force in polar coordinates. The first term, that has been associated with the harmonic oscillator, may be considered as an effective force due to the curved space the particle is constrained to. In fact, we construct the scalar or total curvature of the surface

$$R = g^{\alpha\beta}\left(\partial_\gamma \Gamma^\gamma{}_{\alpha\beta} - \partial_\beta \Gamma^\gamma{}_{\alpha\gamma} + \Gamma^\gamma{}_{\alpha\beta}\Gamma^\lambda{}_{\lambda\gamma} - \Gamma^\gamma{}_{\alpha\lambda}\Gamma^\lambda{}_{\beta\gamma}\right)\,. \tag{65}$$

Using the Christoffel symbols (32) and the inverse of the metric

$$g^{\alpha\beta} = \delta^{\alpha\beta} + \frac{x^\alpha x^\beta}{a^2}\,, \tag{66}$$

one obtains

$$R = \frac{2}{a^2}\,. \tag{67}$$

It turns out that the constant force of Newton's second law (64) is proportional to the total curvature

$$k = m\omega^2 = m\frac{2mE}{m^2 a^2} = RE\,. \tag{68}$$

Thus the movement of the free particle over a 2-sphere projected in x^1x^2-plane is equivalent to the movement described by a particle in a central effective potential

$$V_{\text{eff}}(r) = \frac{RE}{2}r^2 + \frac{L_3}{2mr^2}\,, \tag{69}$$

and both potentials, $V(r) \sim \frac{1}{r}$ and $V(r) \sim r^2$ produce the same trajectory, *i.e.*, an ellipse.

6. Hamiltonization of constrained systems: interpretation of the Dirac brackets based on geometric grounds

Since our discussion on the dynamics of a constrained system has been restricted to the Lagrangian formalism, the objective of this section is based on the hamiltonization of the Lagrangian L_λ. At the time when Dirac proposed his formalism, it was not completely understood how to introduce constraints into the Hamiltonian formalism [11], which is a solved problem nowadays [13, 19–21]. Hamiltonization of L_λ leads to the so-called Dirac brackets and we will provide its geometric interpretation. The construction of the Hamiltonian concerning (22) begins with the definition of the conjugate momenta

$$p_A \equiv \frac{\partial L}{\partial \dot{q}^A}\,, \tag{70}$$

where we wrote collectively $q^A = (x^i, \lambda)$. According to the formalism, we can use the expression of conjugate momenta to obtain the maximum number of velocities as functions of momenta and configuration variables

$$p_A = \frac{\partial L}{\partial \dot{q}^A} \Leftrightarrow \left\{ \begin{array}{l} p_i = \frac{\partial L}{\partial \dot{x}^i} \Leftrightarrow \dot{x}^i = \frac{1}{m} p_i \\ p_\lambda = \frac{\partial L}{\partial \dot{\lambda}} \Rightarrow p_\lambda = 0 \end{array} \right. . \tag{71}$$

Let us define $T_1 \equiv p_\lambda = 0$ and call it as primary constraint. The complete Hamiltonian is defined in extended phase space q^A, p_A, v

$$H = p_A \dot{q}^A - L + v p_\lambda = \frac{1}{2m} p_i^2 - \lambda \left[\left(x^i\right)^2 - a^2 \right] + v p_\lambda\,, \tag{72}$$

where v is a Lagrange multiplier and all velocities enter into H according to (71). Let us write the equations of motion via Poisson brackets again such that

$$\begin{aligned} \dot{q}^A &= \left\{q^A, H\right\} \Rightarrow \left\{ \begin{array}{l} \dot{x}^i = \frac{1}{m} p_i \\ \dot{\lambda} = v \end{array} \right. , & (73) \\ \dot{p}_i &= \{p_i, H\} = 2\lambda x^i\,. & (74) \end{aligned}$$

Since a constraint must be constant, one obtains the following chain of secondary constraints

$$\begin{aligned} T_2 &= \dot{p}_\lambda = \{p_\lambda, H\} \Rightarrow T_2 = \left(x^i\right)^2 - a^2 = 0\,, & (75) \\ T_3 &= \dot{T}_2 = \{T_2, H\} \Rightarrow T_3 = x^i p_i = 0\,, & (76) \\ T_4 &= \dot{T}_3 = \{T_3, H\} \Rightarrow T_4 = \frac{1}{m} p_i^2 + 2\lambda \left(x^i\right)^2\,. & (77) \end{aligned}$$

Finally, the evolution in time of T_4 allows us to find the Lagrange multiplier v

$$v = 0 \,. \tag{78}$$

The matrix $T_{ab} = \{T_a, T_b\}$; $a, b = 1, 2, 3, 4$ is invertible, then according to the Dirac terminology, the constraints are called second class (actually, the existence of T_{ab}^{-1} is the reason why all multipliers have been found [13]). The Dirac brackets are

$$\{A, B\}^* = \{A, B\} - \{A, T_a\} T_{ab}^{-1} \{T_b, B\} \,. \tag{79}$$

So, the equations of motion are defined over the constraint surface and one may forget about the equations $T_a = 0$. They read

$$\dot{Y} = \{Y, H_0\}^* \,, \tag{80}$$

where $Y = (x^i, p_i)$ since the sector (λ, p_λ) may be omitted and $H_0 \equiv H - v p_\lambda$. The basic Dirac brackets for the (x^i, p_i)-sector have the form

$$\left\{x^i, x^j\right\}^* = 0 \,, \tag{81}$$

$$\left\{x^i, p^j\right\}^* = \delta^{ij} - \frac{x^i x^j}{a^2} \,, \tag{82}$$

$$\left\{p^i, p^j\right\}^* = -\frac{1}{a^2}\left(x^i p^j - x^j p^i\right) \,. \tag{83}$$

Since the equations of motion described via Lagrangian formalism give the proper time evolution of the particle over the surface as well as the Lagrangian and Hamiltonian formulations are equivalent [19], one expects a relation between Christoffel symbols and the Dirac bracket. To see this, first we decouple the equation for x^i

$$m\dot{x}^i = p^i \Rightarrow m\ddot{x}^i = 2\lambda x^i \,. \tag{84}$$

With the help of the constraints T_2, T_4 and (73), we obtain that

$$\lambda = -\frac{m\left(\dot{x}^i\right)^2}{2a^2} \,. \tag{85}$$

Then,

$$\ddot{x}^i = -\frac{\left(\dot{x}^i\right)^2}{a^2} x^i \,. \tag{86}$$

On the other hand, we may write

$$\ddot{x}^i = \frac{1}{m}\left\{p^i, H_0\right\}^* | \,, \tag{87}$$

where | denotes substitution of p_j in terms of position and velocity variables, see (71). The α-sector ($\alpha = 1, 2$) of equation (86) coincides with equations of motion (31) of the Lagrangian formalism. Comparing it with (87), one finds

$$\{H_0, p^\alpha\}^*| = m\Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = -\ddot{x}^\alpha \,. \tag{88}$$

This calculation that compares equations of motion in both Lagrangian and Hamiltonian formalisms shows the intrinsic relation between Christoffel symbols and Dirac brackets, as these structures are the ones responsible for the time evolution of the particle in each formalism.

7. Application: spinning particle

The complete understanding of electron spin was accomplished in the realm of quantum electrodynamics. If we consider the Dirac equation

$$i\hbar\partial_t\Psi = \hat{H}\Psi; \qquad \hat{H} = c\alpha^i\hat{p}_i + mc^2\beta \tag{89}$$

as one-particle equation in Relativistic Quantum Mechanics then, in the Heisenberg picture, the position operators experience a quivering motion [22]

$$x^i = a^i + bp^it + c^i \exp\left\{-\frac{2iH}{\hbar}t\right\} \tag{90}$$

that may be considered a superposition of a rectilinear movement with an harmonic one, with high frequency $\frac{2H}{\hbar} \sim \frac{2mc^2}{\hbar}$. This harmonic oscillation was named *Zitterbewegung* by Schrödinger [12]. In recent literature, it has been proposed a model with commuting variables that produces the Dirac equation through quantization [23]. Analysis of the classical counterpart of the model leads to the so-called *Zitterbewegung*, also experienced by spin variables. In order to provide space-time interpretation for the evolution of the classical position and spin coordinates, they were combined to produce configuration coordinates whose dynamics is given by (see details in [24])

$$\tilde{x}^i(t) \;=\; x^i + c\frac{p^i}{p^0}t\,, \tag{91}$$

$$J^i(t) \;=\; \frac{1}{2|p|}\left(A^i\cos\omega t - B^i\sin\omega t\right)\,, \tag{92}$$

with A^i, B^i, p^μ being some constants, $|p| \equiv \sqrt{-p_\mu p^\mu}$ and ω has the same order of magnitude as the Compton frequency. They evolve similarly to the center-of-mass and relative position of two-body problem in a central field.

The potential turns out to be $V(J) \sim J^2$; $J = |J^i|$. Assuming that (91) and (92) are the position variables for the electron, then J^i describes an ellipse with restricted size (a particular feature of the model restricts the magnitude of A^i and B^i as well as their direction, since $p_i A^i = p_i B^i = 0$, center-of-mass moves perpendicularly to the plane of oscillations). According to the previous sections, we interpret J^i as the physical variables for the motion over a 2-sphere. This may explain the physical origin of the *Zitterbewegung* if we assume that the electron has an internal structure [25]. It seems that Dirac himself believed that the electron was not an elementary particle, see [26]. The formalization of the idea developed in this section is in progress.

The idea of a composed electron goes back to the seminal paper by Dirac on the unitary irreducible particle representations of the anti-de Sitter group [27]. Actually, in his work, he found two remarkable representations of SO(2,3), the isometry group of anti-de Sitter space AdS_4. Those representations do not have a counterpart in Poincaré group; they are typical of SO(2,3). This means that, whenever the (Riemann) curvature of AdS_4 goes to zero, these two representations may be combined in order to construct one of the unitary irreducible representations of the Poincaré group in terms of one-particle states. He called these representations singletons. These days, singleton physics is an active research area [28]. Moreover, preons appear as "point-like" particles and are conceived as being subcomponents of quarks and leptons. This term was coined by Pati and Salam in their 1974 paper [29]. Preon models set out as an attempt to describe particle physics in a more fundamental level than the Standard Model [30]. In these preonic models, one postulates a set of fewer fundamental particles than those of the Standard Model, together with the interactions governing the dynamics of these fundamental particles. Based on these laws, preon models try to explain some physics beyond the Standard Model, often producing new particles and a number of phenomena which do not belong to the Standard Model.

8. Noncommutative classical mechanics in a curved phase-space

The discussion of noncommutative (NC) theories has attracted the lights of the theoretical physics over the last few years since the work of Seiberg and Witten [3], where the algebra of string theory embedded in a magnetic background has shown NC features. Since then the study of NC spaces brought interesting results [4]. One of the motivations to study noncommutativity (NCY) is the belief that in some theories, including gravity, the framework of space-time must change at short distances. We can mention, for example, besides string theory, the quantum Hall effect, which presents NCY in the canonical coordinates and momenta [31].

The NC idea is that, in order to describe a NC space, we would have the commutation relations obeyed by their coordinate operators such as

$$\left[\hat{x}^i,\, \hat{x}^j\right] \,=\, i\,\theta^{ij}\,; \qquad \left[\hat{x}^i,\, \hat{p}_j\right] \,=\, i\,\delta^i_j\,; \qquad \left[\hat{p}_i,\, \hat{p}_j\right] \,=\, 0\;, \tag{93}$$

where we are using that $\hbar = 1$ and the θ^{ij}s are c-numbers with the dimensionality of (length)^2. Let us assume that this so-called NC parameter is within the Planck's area order, *i.e.*, $l_{\mathrm{P}}^2 = \hbar G/c^3$, so we have that the tensor θ^{ij} must be of the G/c^3 order. Hence, in the classical limit, the symplectic framework will not have $\hbar$ [6]. This result agrees with this kind of limit. At the classical level, the quantum mechanical commutator is substituted by the Poisson bracket via

$$\left[\hat{A},\,\hat{B}\right] \longrightarrow i\,\{A,\,B\} \tag{94}$$

and consequently, the classical limit of (93) is

$$\left\{x^i,\, x^j\right\} \,=\, \theta^{ij}\,; \qquad \left\{x^i,\, p_j\right\} \,=\, \delta^i_j\,; \qquad \{p_i,\, p_j\} \,=\, 0\,, \tag{95}$$

and the Poisson bracket must have the same properties as the quantum mechanical commutator (bilinear, antisymmetric, Leibniz rules, Jacobi identity). In this section, we will assume a symplectic structure given by (95) in order to obtain the corresponding equations of motion. It is important to say that there are NC formulations where the momenta commutator (Poisson bracket) is not zero. But we will not analyze it here.

We will assume a symplectic structure for the classical mechanics of a particle in a curved phase-space. The target geometry is the 2-sphere described above. We will demonstrate that there is a correction term added to Newton's second law thanks to the curved configuration of the phase-space, which shows that the space configuration alone can bring consequences to the result. On the other hand, we will see that in a flat space, what causes a NC correction is the potential function, which is a standard result in NC classical mechanics. In the 2-sphere curved space, we will see that there is a NC correction without the existence of a potential effect over the particle. This result is coherent with the one obtained here that established analogy between the curvatura of a 2-sphere and a central field.

Let us begin by describing the origin of the NC contribution in generalized (without a specific potential) Newton's second law [5, 6, 32]. We can define a theory as being formulated by a set of canonical variables ξ^a, where $a = 1, \ldots, 2n$ combined with a symplectic structure $\{\xi^a, \xi^b\}$. This structure can be extended in order to accommodate arbitrary function of ξ^a such as

$$\{F, G\} = \left\{\xi^a, \xi^b\right\} \frac{\partial F}{\partial \xi^a} \frac{\partial G}{\xi^b}\,, \tag{96}$$

where repeated indices are summed from now on. Equation (96) can be used, of course, in classical mechanical systems [5, 6, 32] as the one we will analyze in this work.

In Hamiltonian systems, we can use the structure given in (96) to write the equations of motion for a Hamiltonian given by $H = H(\xi^a)$ such that

$$\dot{\xi}^a = \{\xi^a, H\} \tag{97}$$

and for a generalized function F defined in this space, we can write that

$$\dot{F} = \{F, H\}\,. \tag{98}$$

In our case, we will consider a phase-space given by the physical variables x and y and so, $\xi = (x, p_x, y, p_y)$. The algebra between these coordinates are

$$\{x, y\} = \theta\,, \qquad \{x, p_x\} \;=\; \{y, p_y\} = \mathbb{1}\,, \qquad \{p_x, p_y\} = 0\,, \tag{99}$$

where θ, as we said before, must have dimension of area. Let us consider two arbitrary functions F and G, defined on the phase-space. Using Eqs. (96) and (99), we have that

$$\{F, G\} \;=\; \theta^{ij}\,\frac{\partial F}{\partial x^i}\frac{\partial G}{x^j} \;+\; \frac{\partial F}{\partial x^i}\frac{\partial G}{p_i} \;-\; \frac{\partial F}{\partial p_i}\frac{\partial G}{x^i}\,, \tag{100}$$

where $i, j \;=\; x, y$. For example, if we have a Hamiltonian of the standard form with $\xi \;=\; (x^i, p_i)$ such that

$$H \;=\; \frac{p_i p^i}{2m} \;+\; V(x) \tag{101}$$

using (98) and (100), we have the equations of motion given by

$$\dot{x}^i = \big\{x^i, H\big\} = \theta\frac{\partial H}{\partial x_i} \;+\; \frac{\partial H}{\partial p_i} \Longrightarrow \dot{x}^i \;=\; \frac{p^i}{m} \;+\; \theta^{ij}\,\frac{\partial V}{\partial x^j}$$

and analogously

$$\dot{p}_i \;=\; -\,\frac{\partial V}{\partial x^i}\,. \tag{102}$$

Notice from these both equations that an obvious conclusion is that if $V = 0$ (free particle) we have $p_i \;=$ constant and x^i is a linear function of time. Hence, the second term of $\dot{x}^i$ is connected to V, an external field. We can understand that the dynamics of the framework is ruled by the perturbation caused by this external field in the NC phase-space. Newton's second law can be obtained analogously (from Eq. (98)) and the result is

$$m\,\ddot{x}^i \;=\; -\,\frac{\partial V}{\partial x_i} \;+\; m\theta^{ij}\,\frac{\partial^2 V}{\partial x^j \partial x_k}\dot{x}_k\,. \tag{103}$$

This result was used to investigate several models in physics [33]. Here, we want to verify how the phase-space curvature affects the NC contribution. We can see that this new force can be understood, analogously to (102), as the result of a perturbation in the classical phase-space as a consequence of an external field.

In our case, we want to discuss the NC approach for the free particle in a flat 3D space which has the Lagrangian given by

$$L_{\text{ph}} \;=\; \frac{m}{2} g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}\,, \tag{104}$$

where $g_{\alpha\beta}$ is given in (26). From (104), we have that

$$\begin{aligned} p_x \;&=\; m\dot{x} \,+\, \frac{mx(x\dot{x}+y\dot{y})}{a^2-x^2-y^2}\,, \\ p_y \;&=\; m\dot{y} \,+\, \frac{my(x\dot{x}+y\dot{y})}{a^2-x^2-y^2}\,, \end{aligned} \tag{105}$$

where we have used that $x_1 = x$ and $x_2 = y$. From Eqs. (104) and (105), the Hamiltonian is given by

$$H = \frac{1}{2ma^2}\left[\left(a^2-x^2\right)p_x^2 + \left(a^2-y^2\right)p_y^2 \,-\, 2p_xp_yxy\right] \tag{106}$$

and our set of symplectic variables is given by $\xi = (x, y, p_x, p_y)$, as we said before. Using Eqs. (98)–(100) and the Hamiltonian in (106), we have the NC equations of motion

$$\begin{aligned} \dot{x} \;&=\; \frac{1}{ma^2}\left[\left(a^2-x^2\right)p_x - xyp_y - \theta\left(yp_y^2 + xp_xp_y\right)\right]\,, \\ \dot{y} \;&=\; \frac{1}{ma^2}\left[\left(a^2-y^2\right)p_x - xyp_x - \theta\left(xp_x^2 + yp_xp_y\right)\right]\,, \\ \dot{p_x} \;&=\; \frac{1}{ma^2}\left(xp_x^2 + p_xp_yy\right)\,, \\ \dot{p_y} \;&=\; \frac{1}{ma^2}\left(yp_y^2 \,+\, p_xp_yx\right)\,. \end{aligned} \tag{107}$$

Notice that when $\theta = 0$, we have the standard commutative phase-space equations of motion. Secondly, from (76), we can see the effect of a curved phase-space. For a free particle, we must have $\dot{p}_x = \dot{p}_y = 0$, and this is the result of a free particle in a flat phase-space. However, before the calculation of $\dot{p}_x$ or $\dot{p}_y$, we can see the curvature effect already in $\dot{x}$ and $\dot{y}$. In other words, we do not need the values of $\dot{p}_x$ and $\dot{p}_y$ to know that the curvature plays a kind of potential in order to perturb the NC calculations.

It is important to say that if we have NCY in the momentum bracket of Eqs. (95) and (98), we would have a θ-term in the momentum dynamics of (107).

After a long algebra the NC Newton's second law for our particle on the 2-sphere is

$$m\ddot{x} = -\frac{1}{ma^4}\left[x\left(a^2 - x^2\right)p_x^2 - 2x^2 y p_x p_y - x\left(a^2 + y^2\right)p_y^2\right] - \frac{\theta}{ma^2}\left(p_x^2 + p_y^2\right)p_y \tag{108}$$

and

$$m\ddot{y} = -\frac{1}{ma^4}\left[y\left(a^2 - y^2\right)p_y^2 - 2xy^2 p_x p_y - y\left(a^2 + x^2\right)p_x^2\right] - \frac{\theta}{ma^2}\left(p_x^2 + p_y^2\right)p_x \tag{109}$$

and curiously we saw that in (103) the NC correction depends on the background space through the θ^{ij} parameter and also on the variations of the potential. This result could lead us to think that for our free particle, the NC corrections would be zero, as the expression obtained in [6] (Eq. (103)) could indicate. However, we can see in (108)–(109) that the curvature of the space originates a NC correction as well, in spite of a zero potential. In other words, we understand Eqs. (108) and (109) as a new NC Newton's second law. At the final terms of Eqs. (108) and (109), we can realize the correction due to the NC rule. This correction term relies on the background space through the NC θ-parameter. However, we can see the 2-sphere term represented by a, which is an expected result.

9. Conclusions and perspectives

To investigate some ingredients of the formalism that can lead us to work in the Planck energy scale means to discuss the physics of the early Universe, for instance, where quantum mechanics and general relativity were combined and quantum gravity is formed. This is one of the main motivations to study mechanisms that introduces Planck scale parameters in classical systems. And this is one of the main motivations to use NCY in order to introduce this so-called Planck scale parameter. In this work, we have analyzed the free movement of a particle upon a 2-sphere considering NC classical mechanics approach. In this scenario, we can consider a semi-classical approach where the Planck constant was substituted by the NC parameter.

The NC Newton's second law have shown us that the curvature of the space acted the same way as if there was a potential since the particle flat space acceleration has the NC contribution given by the potential, namely, the NC contribution would be zero but it is not. In the 2-sphere free particle dynamics, the NC additional term is different from zero, which means that its origin is the curvature of the system.

The introduction of the NC contribution make us also ask what would be the nature of the potential effect caused by the curvature. In other words, since in a free particle flat space system the NC contribution is connected with a potential such that if $V = 0$, we have no contribution, and in the curved space this effect does not happen, what is the physical meaning of this potential-type effect brought by the curvature? And in the case of curved space and $V \neq 0$? Where would the NC contribution appear?

Furthermore, we have also introduced some basic ideas of classical mechanics and differential geometry. We started by formulating the procedure of introducing constraints into the Lagrangian formalism: they were inserted via Lagrange multipliers and we have demonstrated that this procedure leads to the same number of degrees of freedom and equations of motion if we have obtained one of the variables of the known constraint and substitute it in the free Lagrangian. After that, we have given a detailed analysis of a particle constrained over a 2-sphere.

Basic notions of differential geometry, such as the metric and Christoffel symbols, appear as a consequence of the description of a constrained Lagrangian system and its corresponding principle of least action. A solution of the equations of motion was given based on geometric grounds and with the help of the Noether theorem. It was also shown that physical position variables of the model evolve over an ellipse. We have proposed a central force problem whose solution for position variables are the same as those of the particle over a 2-sphere. One can be led to interpret the curvature of the space where the particle lives as an origin of an effective potential. This example may be a starting point for studying general relativity. We have also naively discussed the relation between both the Dirac brackets and Christoffel symbols, since both of them are supposed to describe the correct evolution of a particle constrained to a surface.

Finally, as an example, we have treated the so-called *Zitterbewegung* of the Dirac electron. It may be seen as the effective motion of a particle over a 2-sphere, assuming that the electron bears an internal structure.

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