# SOLITON SOLUTIONS AND GROUP ANALYSIS OF A NEW COUPLED $(2+1)$-DIMENSIONAL BURGERS EQUATIONS*

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This paper focuses on a new coupled $(2+1)$-dimensional Burgers equations. The shock wave solution is obtained by the aid of Ansatz method. There are several constraint conditions which guarantee the existence of the derived solutions. Subsequently, the simplified Hirota bilinear method, established by Hereman, is applied to construct soliton solutions to the equation. Finally, the classic Lie symmetry analysis is employed to generate a class of new solutions to the equation based on the solutions obtained earlier by Ansatz and simplified Hirota bilinear methods.

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## 1. Introduction

Nonlinear phenomena come up in a variety of scientific fields, such as solid-state physics, plasma physics, fluid dynamics, mathematical biology, chemical kinetics, *etc.* Nonlinear evolution equations can model nonlinear phenomena that appear in nature quite frequently. Therefore, searching for

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exact solutions of these equations is of paramount importance. A great deal of literature concerning these equations and methods to obtain solutions is available in the text. Amongst the several methods which are in use to handle these equations, the well known are: the inverse scattering method, the Baklund transformation method, Darboux transformation, Painleve analysis method, Exp-function method, the Hirota bilinear method, simplified Hirota bilinear method, Lie group method and, solitary wave Ansatz method. In addition, the computer symbolic systems, such as Maple and Mathematica, help us to deal with complicated and tedious calculations [1–10].

It is well known that traveling waves appear in solitary waves theory in distinct features such as solitons, kinks, *etc.* The soliton theory is an important part of the nonlinear science. Soliton-like particles can travel over long-distances without attenuation and changes of wave shapes due to the balance of the interplay between dispersion and nonlinearity ([11] and papers cited therein).

It is generally known that the Burgers equation describes the coupling between diffusion $u_{xx}$ and the convection process $uu_x$. The Burger's types of equations have been applied to various physical phenomena, such as: fluid dynamics, gas dynamics, traffic flow, *etc.* The Burgers equation (BE) is one of the fundamental model equations that is in existence for a very long time. Many researchers have been investigated these types of equations with different configurations [12–21].

In this work, our aim is to investigate a new coupled $(2+1)$-dimensional Burgers equations given through

$$\begin{aligned} u_t - 2uu_x - u_y v - uv_y - u_{xx} - u_{yy} &= 0\,, \\ v_t - 2vv_y - u_x v - uv_x - v_{xx} - v_{yy} &= 0\,, \end{aligned} \tag{1}$$

where a mixed partial derivative terms $uv_y$ and $u_x v$ are added to generalize coupled $(2+1)$-dimensional Burgers equations to a more common situation. The two additional terms, one in each equation, represent divergences with respect to $y$ and $x$, respectively. Thus, it is expected that the symmetry group and the conservation laws would take different forms.

Here, we will study Eq. (1) via Ansatz method, simplified Hirota bilinear method and Lie symmetry analysis. The layout of the paper is as follows. In Section 2, the Ansatz method is applied to get the topological soliton solutions of Eq. (1). In Section 3, Eq. (1) will be studied using simplified Hirota bilinear method. In Section 4, we perform Lie symmetry analysis on Eq. (1), whereas last section is for conclusions.

## 2. Ansatz method

In this section, we will investigate the new coupled BE given by Eq. (1) using the Ansatz approach. The shock wave solutions that are also known as topological solitons in theoretical physics are derived. In order to get the shock wave solution, we first give the following hypothesis [10, 19, 22, 23]

$$u(x,y,t) = A_1 \tanh^{p_1} \tau \tag{2}$$

and

$$v(x,y,t) = A_2 \tanh^{p_2} \tau\,, \tag{3}$$

where

$$\tau = B_1 x + B_2 y - ct\,, \tag{4}$$

and $A_j$, $B_j$ for $j = 1,2$ are free parameters and $c$ is the speed of the shock wave. Meanwhile, the unknown exponent $p_1$ and $p_2$ will be further fixed. Thus, substituting (2) and (3) into (1) yields the following equations:

$$\begin{aligned} &p_1 A_1 c \left(\tanh^{p_1+1}\tau - \tanh^{p_1-1}\tau\right) - 2p_1 A_1^2 B_1 \left(\tanh^{2p_1-1}\tau - \tanh^{2p_1+1}\tau\right) \\ &-(p_1+p_2) A_1 A_2 B_2 \left(\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\right) - A_1 p_1 \left(B_1^2 + B_2^2\right) \\ &\times \left\{(p_1-1)\tanh^{p_1-2}\tau - 2p_1 \tanh^{p_1}\tau + (p_1+1)\tanh^{p_1+2}\tau\right\} = 0\,, \end{aligned} \tag{5}$$

and

$$\begin{aligned} &p_2 A_2 c \left(\tanh^{p_2+1}\tau - \tanh^{p_2-1}\tau\right) - 2p_2 A_2^2 B_2 \left(\tanh^{2p_2-1}\tau - \tanh^{2p_2+1}\tau\right) \\ &-(p_1+p_2) A_1 A_2 B_2 \left(\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\right) - A_2 p_2 \left(B_1^2 + B_2^2\right) \\ &\times \left\{(p_2-1)\tanh^{p_2-2}\tau - 2p_2 \tanh^{p_2}\tau + (p_2+1)\tanh^{p_2+2}\tau\right\} = 0\,. \end{aligned} \tag{6}$$

From (5), based on balancing principle, one can get

$$2p_1 - 1 = p_1 + p_2 - 1 = p_1\,, \tag{7}$$

in other words,

$$p_1 = p_2 = 1\,. \tag{8}$$

Also, from (6), one can get the same conclusion as in (8). Now, from (5) and (6), assuming that the coefficients of the linearly independent functions $p_j$ and $p_{j+2}$, for $j = 1,2$, equal to zero, one can arrive at

$$c = 0\,, \tag{9}$$

and

$$A_1B_1 + A_2B_2 = B_1^2 + B_2^2\,. \tag{10}$$

This implies that the shock wave solution of the equation is a stationary shock wave, which is given by

$$u(x,y,t) = A_1 \tanh\left(B_1x + B_2y\right)\,, \tag{11}$$

and

$$v(x,y,t) = A_2 \tanh\left(B_1x + B_2y\right)\,, \tag{12}$$

where $A_1, A_2$ and $B_1, B_2$ are given by (10).

**Remark 1**: It is worth noting that the terms $uv_y$ and $u_xv$ are added, the obtained results are the same as in [19].

## 3. The simplified Hereman–Nuseir method

In this section, we will deal with (1) by using the simplified Hereman–Nuseir method. The main steps of this method can be found in [7, 8, 11, 12]. Here, we adopt the procedure outlined in [12]. Since the calculations are straightforward, therefore by omitting the details of the calculations, we directly write the $N$-kink solutions for equation (1) as

$$\begin{aligned} u &= \frac{\sum_{i=1}^{N} k_i e^{k_i(x+y)+2k_i^2 t}}{1+\sum_{i=1}^{N} e^{k_i(x+y)+2k_i^2 t}}\,, \\ v &= \frac{\sum_{i=1}^{N} k_i e^{k_i(x+y)+2k_i^2 t}}{1+\sum_{i=1}^{N} e^{k_i(x+y)+2k_i^2 t}}\,. \end{aligned} \tag{13}$$

Similarly, the $N$-singular kink solutions are

$$\begin{aligned} u &= -\frac{\sum_{i=1}^{N} k_i e^{k_i(x+y)+2k_i^2 t}}{1-\sum_{i=1}^{N} e^{k_i(x+y)+2k_i^2 t}}\,, \\ v &= -\frac{\sum_{i=1}^{N} k_i e^{k_i(x+y)+2k_i^2 t}}{1-\sum_{i=1}^{N} e^{k_i(x+y)+2k_i^2 t}}\,. \end{aligned} \tag{14}$$

For the details of calculations, the reader is referred to [12].

**Remark 2:** It should be noted that $uv_y$ and $u_xv$ are added, but the obtained results are the same as in [12].

## 4. Lie symmetry analysis of (1)

In this section, we will handle Eq. (1) using Lie symmetry analysis.

On the basis of Lie group theory, a one-parameter Lie group of point transformations are given by [19]:

$$\begin{aligned} t^* &= t+\epsilon\tau(x,y,t,u,v)+O\left(\epsilon^2\right), \qquad x^* = x+\epsilon\xi(x,y,t,u,v)+O\left(\epsilon^2\right),\\ y^* &= y+\epsilon\eta(x,y,t,u,v)+O\left(\epsilon^2\right), \qquad u^* = u+\epsilon\phi(x,y,t,u,v)+O\left(\epsilon^2\right),\\ v^* &= v+\epsilon\psi(x,y,t,u,v)+O\left(\epsilon^2\right), \end{aligned} \tag{15}$$

the associated vector field is of the form

$$\begin{aligned} V &= \tau(x,y,t,u,v)\frac{\partial}{\partial t}+\xi(x,y,t,u,v)\frac{\partial}{\partial x}+\eta(x,y,t,u,v)\frac{\partial}{\partial y}\\ &\quad +\phi(x,y,t,u,v)\frac{\partial}{\partial u}+\psi(x,y,t,u,v)\frac{\partial}{\partial v}. \end{aligned} \tag{16}$$

Here, the coefficient functions $\tau(x,y,t,u,v)$, $\xi(x,y,t,u,v)$, $\eta(x,y,t,u,v)$, $\phi(x,y,t,u,v)$, and $\psi(x,y,t,u,v)$ are to be further fixed.

If the vector field (16) has to generate a symmetry of the Eq. (1), then $V$ need to satisfy the following condition

$$\mathrm{pr}^{(2)}V(\Delta_1,\Delta_2)|_{\Delta_1=0,\Delta_2=0}=0. \tag{17}$$

Here, $\Delta_1=u_t-2uu_x-u_yv-uv_y-u_{xx}-u_{yy}$, and $\Delta_2=v_t-2vv_y-u_xv-uv_x-v_{xx}-v_{yy}$ and $\mathrm{pr}^{(2)}V$ is the second prolongation of the vector field. In other words,

$$\begin{aligned} \phi^t-2u_x\phi-2u\phi^x-\phi^yv-u_y\psi-u\phi^y-\phi v_y-\phi^{xx}-\phi^{yy} &= 0,\\ \psi^t-2v_y\psi-2v\psi^y-v_x\phi-u\psi^x-\phi^xv-u_x\psi-\psi^{xx}-\psi^{yy} &= 0. \end{aligned} \tag{18}$$

In (18), we only solve the following coefficients functions [19]:

$$\begin{aligned} \phi^t &= D_t(\phi)-u_xD_t(\xi)-u_yD_t(\eta)-u_tD_t(\tau),\\ \phi^x &= D_x(\phi)-u_xD_x(\xi)-u_yD_x(\eta)-u_tD_x(\tau),\\ \phi^y &= D_y(\phi)-u_xD_y(\xi)-u_yD_y(\eta)-u_tD_y(\tau),\\ \psi^t &= D_t(\psi)-v_xD_t(\xi)-v_yD_t(\eta)-v_tD_t(\tau),\\ \psi^x &= D_x(\psi)-v_xD_x(\xi)-v_yD_x(\eta)-v_tD_x(\tau),\\ \psi^y &= D_y(\psi)-v_xD_y(\xi)-v_yD_y(\eta)-v_tD_y(\tau),\\ \phi^{xx} &= D_x(\phi^x)-u_{xt}D_x(\tau)-u_{xx}D_x(\xi)-u_{xy}D_x(\eta),\\ \phi^{yy} &= D_y(\phi^y)-u_{yt}D_y(\tau)-u_{xy}D_y(\xi)-u_{yy}D_y(\eta),\\ \psi^{xx} &= D_x(\psi^x)-v_{xt}D_x(\tau)-v_{xx}D_x(\xi)-v_{xy}D_x(\eta),\\ \psi^{yy} &= D_y(\psi^y)-v_{yt}D_y(\tau)-v_{xy}D_y(\xi)-v_{yy}D_y(\eta). \end{aligned} \tag{19}$$

Here, $D_i$ denotes the operators of total differentiation with respect to $x, y$ and $t$, respectively

$$D_i = \frac{\partial}{\partial x^i} + u_i^p \frac{\partial}{\partial u} + u_{ij}^p \frac{\partial}{\partial u_j} + \ldots\,, \qquad i = 1,2,3\,, \qquad p = 1,2\,, \tag{20}$$

and $(x^1, x^2, x^3) = (t, x, y), (u^1, u^2) = (u, v)$.

Then, considering the Lie symmetry analysis method, one can obtain

$$\begin{aligned} \xi &= \tfrac{1}{2}c_1 x - c_3 y + c_5\,, \qquad \tau = c_1 t + c_2\,, \qquad \eta = \tfrac{1}{2}c_1 y + c_3 x + c_4\,, \\ \phi &= -\tfrac{1}{2}c_1 u - c_3 v\,, \qquad \psi = c_3 u - \tfrac{1}{2}c_1 v\,, \end{aligned} \tag{21}$$

where $c_i (i = 1, 2 \ldots 5)$ are arbitrary constants. Thus, the five vector fields are given by

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}\,, \qquad V_2 = \frac{\partial}{\partial y}\,, \qquad V_3 = \frac{\partial}{\partial t}\,, \\ V_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\,, \\ V_5 &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\,. \end{aligned} \tag{22}$$

**Remark 3:** It is clear that the vector fields are narrower than the vector fields in [19], the reason is that added terms affect the properties.

It is to be noted that the symmetry generators found in (22) form a closed and five-dimensional Lie algebra. Here, they are omitted for the sake of brevity. In order to obtain the Lie symmetry group, the following initial problems need to be considered

$$\frac{d}{d\varepsilon}\left(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{v}\right) = \sigma\left(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{v}\right), \left(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{v}\right)|_{\varepsilon=0} = \left(x, y, t, u, v\right), \tag{23}$$

here, $\varepsilon$ is a parameter and

$$\sigma = \xi u_x + \tau u_t + \eta u_y + \phi u + \psi v\,. \tag{24}$$

Therefore, we can get the following Lie symmetry group:

$$g : (x, y, t, u, v) \to \left(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{v}\right)\,. \tag{25}$$

Thus, one can get the following group:

$$\begin{aligned} & g_1 : (x + \varepsilon, y, t, u, v)\,, \\ & g_2 : (x, y + \varepsilon, t, u, v)\,, \\ & g_3 : (x, y, t + \varepsilon, u, v)\,, \\ & g_4 : (e^{\varepsilon}x, e^{\varepsilon}y, e^{2\varepsilon}t, e^{-\varepsilon}u, e^{-\varepsilon}v)\,, \\ & g_5 : (x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon, t, v\cos\varepsilon - u\sin\varepsilon, v\sin\varepsilon + u\cos\varepsilon)\,. \end{aligned} \tag{26}$$

Consequently, many new solutions can be derived by applying the above groups $g_i (i = 1, \ldots 5)$:

$$
\begin{aligned}
u_1 &= f_1(x-\varepsilon, y, t)\,, & v_1 &= h_1(x-\varepsilon, y, t)\,,\\
u_2 &= f_2(x, y-\varepsilon, t)\,, & v_2 &= h_2(x, y-\varepsilon, t)\,,\\
u_3 &= f_3(x, y, t-\varepsilon)\,, & v_3 &= h_3(x, y, t-\varepsilon)\,,\\
u_4 &= e^{-\varepsilon} f_4\left(e^{-\varepsilon}x, e^{-\varepsilon}y, e^{-2\varepsilon}t\right)\,, & v_4 &= e^{-\varepsilon} h_4\left(e^{-\varepsilon}x, e^{-\varepsilon}y, e^{-2\varepsilon}t\right)\,,\\
u_5 &= f_5\left(x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon, t, v\cos\varepsilon - u\sin\varepsilon\right)\,, \\
v_5 &= h_5\left(x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon, t, v\sin\varepsilon + u\cos\varepsilon\right)\,,
\end{aligned} \tag{27}
$$

where $\varepsilon$ is an arbitrary constant.

If taking the shock wave solution of Eq. (1) given through (11) and (12), one can get new exact solutions of Eq. (1) by applying the scaling symmetry group $g_4$ as follows:

$$
\begin{aligned}
u &= e^{-\varepsilon} A_1 \tanh\left(B_1 e^{-\varepsilon}x + B_2 e^{-\varepsilon}y\right)\,,\\
v &= e^{-\varepsilon} A_2 \tanh\left(B_1 e^{-\varepsilon}x + B_2 e^{-\varepsilon}y\right)\,.
\end{aligned} \tag{28}
$$

Also, from (13) and (14), one can get new explicit solutions of Eq. (1) by applying the scaling symmetry group $g_4$ as follows:

$$
\begin{aligned}
u &= e^{-\varepsilon} \frac{\sum_{i=1}^{N} k_i e^{k_i e^{-\varepsilon}(x+y) + 2k_i^2 e^{-2\varepsilon} t}}{1 + \sum_{i=1}^{N} e^{k_i e^{-\varepsilon}(x+y) + 2k_i^2 e^{-2\varepsilon} t}}\,,\\
v &= e^{-\varepsilon} \frac{\sum_{i=1}^{N} k_i e^{k_i e^{-\varepsilon}(x+y) + 2k_i^2 e^{-2\varepsilon} t}}{1 + \sum_{i=1}^{N} e^{k_i e^{-\varepsilon}(x+y) + 2k_i^2 e^{-2\varepsilon} t}}\,,
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
u &= -e^{-\varepsilon} \frac{\sum_{i=1}^{N} k_i e^{k_i e^{-\varepsilon}(x+y) + 2k_i^2 e^{-2\varepsilon} t}}{1 - \sum_{i=1}^{N} e^{k_i e^{-\varepsilon}(x+y) + 2k_i^2 e^{-2\varepsilon} t}}\,,\\
v &= -e^{-\varepsilon} \frac{\sum_{i=1}^{N} k_i e^{k_i e^{-\varepsilon}(x+y) + 2k_i^2 e^{-2\varepsilon} t}}{1 - \sum_{i=1}^{N} e^{k_i e^{-\varepsilon}(x+y) + 2k_i^2 e^{-2\varepsilon} t}}\,.
\end{aligned} \tag{30}
$$

**Remark 4**: A class of new invariant solutions can be found by utilizing the different groups of Eq. (1).

## 5. Conclusions

This paper addresses a new coupled $(2+1)$-dimensional Burgers equations. By using the Ansatz method, the topological 1-soliton solution is derived for this equation for the first time. It is also shown that the shock wave solution of the coupled BE is a stationary shock wave. Then, the equation was investigated for multiple-soliton solutions and multiple singular soliton solutions. The simplified form of the Hirota's method is used to obtain these solutions. At last, the Lie symmetry analysis have been applied to give an additional display of solutions. These solutions may be useful to further investigate the complicated nonlinear physical phenomena.

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