THE EXISTENCE OF BOGOMOLNY DECOMPOSITIONS FOR GAUGED $O(3)$ NONLINEAR "SIGMA" MODEL AND FOR GAUGED BABY SKYRME MODELS

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The Bogomolny decompositions (Bogomolny equations) for the gauged $O(3)$ nonlinear "sigma" model (in this paper, we call it shortly as: gauged $O(3)$ "sigma" model) and for the gauged baby Skyrme models: restricted and full one, in $(2+0)$-dimensions, are derived, for some general classes of the potentials. The conditions, which must be satisfied by the potentials, for each of these mentioned models, are also derived.

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1. Introduction

The $O(3)$ nonlinear "sigma" model (called sometimes also as "$O(3)$ nonlinear σ model", "$O(3)$ model", "CP^1 model" and so on) plays an important role in physics. Its version in $(2+0)$-dimensions is integrable and describes static field configurations in the Heisenberg ferromagnet (the dynamics of the field configurations in planar ferromagnet has been studied in many papers, *e.g.* in [1, 2]). The Bogomolny equations for it have been found in [3]. In [4], the Bogomolny equations of nonlinear "sigma" model with a suitable choice of the potential were derived and the interaction of their soliton solutions was investigated in [5]. In [6, 7], some chiral σ-model in $(3+1)$ has been investigated. In [8], toroidal soliton solutions for $O(3)^N$ nonlinear "sigma" model were obtained and investigated. Some hopfions in CP^n model were investigated in [9]. The baby Skyrme model appeared firstly as an analogon (on plane) to the Skyrme model in three-dimensional space. The last one, was introduced by Skyrme in [10–12]. It is being used for a description of the physics of strong interactions in the case of low energies [13]. The target space of Skyrme model is SU(2) [10–13], and for baby Skyrme model, the target space is S^2. In these both models, the topological classification of the static field configurations, by their winding numbers, can be done. Similarly to the Skyrme model, the following terms appear in the baby Skyrme model: the term of nonlinear $O(3)$ "sigma" model, the quartic term — the analogon of the Skyrme term and the potential. The potential, in baby Skyrme model, must occur, for existence of static solutions with finite energy. However, the form of the potential is not restricted. Many different forms of the potentials were investigated, for example in [14–18]. In [19], noncommutative baby Skyrmions were investigated and in [20], exact BPS bound for noncommutative baby Skyrme model has been obtained. The problem of peakons and Q-balls in the baby Skyrme model was studied in [21]. The Bogomolny bound and Bogomolny equations for gauged $O(3)$ "sigma" model, for some special form of the potential, were derived in [22]. In [23], the existence of soliton solutions of Bogomolny kind, in gauged linear "sigma" model in $(2+1)$-dimensions, was proved. The Bogomolny equations (Bogomol'nyi equations) for Abelian gauged $O(3)$ "sigma" model with some other specific form of the potential (and for generic nonminmimal coupling constant) have been derived in [24]. The vortex solutions for them have also been obtained there. The Bogomolny equations and their vortex solutions for the gauged "sigma" model with Kähler domain have been obtained in [25]. In [26], it was shown that the Bogomolny bound of $(1+1)$-dimensional gauged "sigma" models can be written down by using terms of two conserved charges, similarly to the Bogomolny bound of the BPS dyons in $(3+1)$-dimensions. Some new Dirac–Born–Infeld extension of BPS Skyrme model was done in [27]. Gauged version of the Faddeev–Skyrme model in $(3+1)$-dimensions, with Maxwell term, was discussed in [28]. In [29] BPS vortices in $(1+1)$-dimensional $\mathcal{N}=(2,2)$ supersymmetric gauged "sigma" model were studied. In [30], some soliton solutions (in the case of $V(S^i)=1-\vec{n}\cdot\vec{S}$, $(i=1,2,3), \vec{n}=[0,0,1]$) for gauged full baby Skyrme model were studied. The Lagrangian of the mentioned gauged full baby Skyrme model in $(2+1)$-dimensions, with some specific form of V, is the sum of [30, 31]: $O(3)$-like ("sigma") term $D_\mu\vec{S}\cdot D^\mu\vec{S}$, Skyrme term $(D^\mu\vec{S}\times D^\nu\vec{S})^2$, usual Maxwell term $F^2_{\mu\nu}$ and the potential $V(\vec{S})$, where $\vec{S}$ is three-component vector field, such that $|\vec{S}|^2=1$, $\lambda>0$ is a coupling constant, $D_\mu\vec{S}=\partial_\mu\vec{S}+A_\mu(\vec{n}\times\vec{S})$ is the covariant derivative of vector field $\vec{S}$, $F_{\mu\nu}$ is field strength, called also as the curvature and $\vec{n}=[0,0,1]$ is an unit vector and $\mu,\nu=0,1,2$. The baby Skyrme model has simpler structure than three-dimensional Skyrme model and so owing to it, we have an opportunity of better studying of the solutions of Skyrme model in $(3+1)$-

dimensions. However, on the other hand, even in the ungauged version of this model, it is still nonintegrable, topologically nontrivial and nonlinear field theory, difficult for an exploration. These reasons cause that it is difficult to make analytical studies of this model and so, the investigations of baby Skyrmions are very often numerical. Therefore, the simplification, but of course, keeping us in the class of Skyrme-like models and simultaneously, giving an opportunity for analytical calculations, is important. One may, for example, simplify the problem of solving field equations, by deriving the Bogomolny equations for the models mentioned above. All solutions of the Bogomolny equations are also the solutions of the Euler–Lagrange equations (their order is bigger than the order of Bogomolny equations). The Bogomolny equations for the ungauged restricted baby Skyrme model with the special form of the potential $V = V(S^3)$ were derived in [17].

In [32], Bogomolny decompositions for both ungauged models: restricted and full baby Skyrme one were derived. There was also showed that in the case of ungauged restricted baby Skyrme model, Bogomolny decomposition existed for arbitrary potential (in [18], the Bogomolny equations had been obtained for the potential, which was a square of some non-negative function with isolated zeroes, but by another way than used in [32]). Next, in [32], it was also showed that for the case of ungauged full baby Skyrme model, the set of the solutions of corresponding Bogomolny equations was some subset of the set of the solutions of Bogomolny equations for ungauged restricted baby Skyrme model.

The technique used in [31], for derivation of the Bogomolny equations for gauged restricted baby Skyrme model in $(2+0)$-dimensions (in the case of $V(S^i) = 1 - \vec{n} \cdot \vec{S}$, $(i = 1, 2, 3)$), was firstly applied by Bogomolny in [33], among others, for the non-Abelian gauge theory. Independently, the results, similar to some results obtained in [33], were obtained in [34] and [35] — in the context of the Bogomolny equations, this last paper has been cited only in [36]. This method is based on some proper separation of the terms in the functional of energy. The solutions of Bogomolny equations, found in this way, minimalize the energy functional and saturate Bogomolny bound (Bogomolny bound is an inequality connecting energy functional and topological charge).

In [31], the Bogomolny equations for the gauged restricted baby Skyrme model, in $(2 + 0)$-dimensions, but for the potentials of the form $V(S^3)$, have been derived and some nontrivial solutions of these equations have been obtained. Independently, in [37], the Bogomolny decomposition for the gauged restricted baby Skyrme model, for the potential $V(S^3)$ (written down in stereographical variables), obtained by applying the so-called concept of strong necessary conditions, has been presented. In [38], a novel BPS bound for some gauged BPS submodel was investigated. Some topological duality between vortices and planar skyrmions in BPS theories with the so-called APD symmetries (also for the case of the U(1) gauged versions of the models) was established in [39].

In this paper, we derive the Bogomolny equations (we call them the Bogomolny decomposition) for the gauged $O(3)$ "sigma" model and for the both gauged baby Skyrme models: restricted and full one, in $(2+0)$-dimensions, for some general form of the potential. The gauged restricted baby Skyrme model is characterized by absence of $O(3)$-like term ("sigma" term) in the Lagrangian of gauged full baby Skyrme model. We investigate here the case of the more general form of the potentials V (than these ones, investigated in [31] and [37]), *i.e.* we look for: the Bogomolny decomposition and the condition, which must be satisfied by the potential V, in order to existence of the Bogomolny decomposition.

We derive Bogomolny decompositions, for the gauged models: $O(3)$ "sigma", restricted baby Skyrme (this paper contains among others, some generalization of the results presented in [37]) and full baby Skyrme model, by applying (in contrary to [30, 31] and [38]) just the concept of strong necessary conditions, firstly presented in [40] and extended in [41, 42]. We derive also the condition, which must be satisfied by the potentials of the form V, for which the Bogomolny decomposition exists. The results, included in this paper and concerning the gauged baby Skyrme models, have been included in [37].

The procedure of deriving the Bogomolny decomposition, from the extended concept of strong necessary conditions, has been presented in [43, 44] and developed in [45].

This paper is organized as follows. In the next subsections of this section, we briefly describe the gauged models investigated in this paper. We assume at the beginning, the dependency of their potentials V, on the gauge field A_k, $(k = 1, 2)$. Obviously, the Lagrangian needs to be gauge invariant, however, we want to investigate, whether the conditions for the potentials in these models, in the case of existing of Bogomolny decomposition, will permit the dependency of the gauge field A_k, $(k = 1, 2)$ and whether we obtain some model similar to the Proca theory [46] (or to the theory of a massive vector field [47]). In Subsection 1.3, the concept of strong necessary conditions is presented. At the beginning of Section 2, we derive the most general (in the case of the topology, appropriate for the models investigated here) expressions (in stereographical variables or in their real and imaginary part) of the density of the topological invariant, needed for our computations. Next, we derive the Bogomolny decompositions for the gauged models: $O(3)$ "sigma", restricted baby Skyrme and full baby Skyrme model, by using the concept of strong necessary conditions. There are derived also the conditions for the potentials of these gauged models, which must be satisfied, in the case of Bogomolny decompositions. Section 3 contains a summary.

1.1. Gauged $O(3)$ "sigma" model

The Lagrangian of gauged $O(3)$ "sigma" model has the form

$$\mathcal{L} = D_\mu \vec{S} \cdot D^\mu \vec{S} + F_{\mu\nu}^2 + V\,, \tag{1}$$

where $\vec{S}$ is the three-component vector, such that $|\vec{S}\,|^2 = 1$ and $D_\mu \vec{S} = \partial_\mu \vec{S} + A_\mu(\vec{n} \times \vec{S}\,)$ is covariant derivative of vector field $\vec{S}$. The form of dependency of the potential V, on the dependent variables, has not been specified, obviously, it depends on its arguments such that it is a real Lorentzian scalar.

In this paper, we consider gauged $O(3)$ "sigma" model in $(2+0)$-dimensions, with the energy functional of the following form (*cf.* [22]):

$$H = \frac{1}{2}\int d^2x\ \mathcal{H} = \frac{1}{2}\int d^2x \left(\lambda_0 D_i \vec{S} \cdot D^i \vec{S} + F_{kl}^2 + \gamma^2 V\right)\,, \tag{2}$$

where $x_1 = x,\ x_2 = y$ and $i, k, l = 1, 2$. We make the stereographic projection

$$\vec{S} = \left[\frac{\omega+\omega^*}{1+\omega\omega^*}, \frac{-i\cdot(\omega-\omega^*)}{1+\omega\omega^*}, \frac{1-\omega\omega^*}{1+\omega\omega^*}\right]\,, \qquad i.e. \quad \omega = \frac{S_1+iS_2}{1+S_3}\,, \tag{3}$$

where $\omega = \omega(x,y) \in \mathbb{C}$, $x, y \in \mathbb{R}$ and $\omega(x,y) = u(x,y) + iv(x,y), u, v \in \mathbb{R}$.

After making the transformation (3), the density of the energy functional (2) has the form

$$\begin{aligned}\mathcal{H} \;=\; & \lambda_{00}\frac{\left(A_1^2+A_2^2\right)\cdot\left(u^2+v^2\right) - 2A_1\cdot\left(u_{,x}v - uv_{,x}\right) - 2A_2\cdot\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^2} \\ & +\lambda_{00}\frac{u_{,x}^2+u_{,y}^2+v_{,x}^2+v_{,y}^2}{\left(1+u^2+v^2\right)^2} + \lambda_2\cdot\left(A_{2,x}-A_{1,y}\right)^2 + V\left(u,v,A_1,A_2\right)\,,\end{aligned} \tag{4}$$

where after rescalling, the constants $\lambda_{00} = 4\lambda_0, \lambda_2$ have appeared, $u_{,x} \equiv \frac{\partial u}{\partial x}$ *etc.* The constant γ has been included in V and $u(x,y) = \Re(\omega(x,y)), v(x,y) = \Im(\omega(x,y)) \in \mathbb{R}$. The Euler–Lagrange equations for the gauged $O(3)$ "sigma" model have the form

$$\begin{aligned}& \frac{d}{dx}\left\{2\lambda_{00}\frac{-A_1v+u_{,x}}{\left(1+u^2+v^2\right)^2}\right\} + \frac{d}{dy}\left\{2\lambda_{00}\frac{-A_2v+u_{,y}}{\left(1+u^2+v^2\right)^2}\right\} \\ & -2\lambda_{00}\frac{\left[\left(A_1^2+A_2^2\right)u + A_1v_{,x} + A_2v_{,y}\right]}{\left(1+u^2+v^2\right)^2} \\ & +4\lambda_{00}u\frac{\left(A_1^2+A_2^2\right)\cdot\left(u^2+v^2\right) - 2A_1\cdot\left(u_{,x}v-uv_{,x}\right) - 2A_2\cdot\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^3} \\ & +4\lambda_{00}u\frac{u_{,x}^2+u_{,y}^2+v_{,x}^2+v_{,y}^2}{\left(1+u^2+v^2\right)^3} - V_{,u} = 0\,,\end{aligned}$$

the analogical equation, following from varying the energy functional with respect to v,

$$
\begin{aligned}
-2\lambda_2 \frac{d}{dy}\left(A_{2,x}-A_{1,y}\right) - 2\lambda_{00}\frac{A_1\cdot\left(u^2+v^2\right)-\left(u_{,x}v-uv_{,x}\right)}{\left(1+u^2+v^2\right)^2} - V_{,A_1} &= 0\,,\\
2\lambda_2 \frac{d}{dx}\left(A_{2,x}-A_{1,y}\right) - 2\lambda_{00}\frac{A_2\cdot\left(u^2+v^2\right)-\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^2} - V_{,A_2} &= 0\,, \qquad (5)
\end{aligned}
$$

where $V_{,u} \equiv \frac{\partial V}{\partial u}$ *etc.*

1.2. Gauged baby Skyrme models

In this paper, we consider also the gauged baby Skyrme models: full and restricted one. The Lagrangian of gauged full baby Skyrme model has the form (in the Lagrangian of gauged restricted baby Skyrme model, the $O(3)$-like term is absent), *cf.* [31, 37]

$$
\mathcal{L} = D_\mu \vec{S}\cdot D^\mu \vec{S} + \frac{\lambda^2}{4}\left(D_\mu \vec{S}\times D_\nu \vec{S}\right)^2 + F_{\mu\nu}^2 + V\,, \qquad (6)
$$

where $\vec{S}$ is three-component vector such that $|\vec{S}\,|^2 = 1$ and $D_\mu \vec{S} = \partial_\mu \vec{S} + A_\mu(\vec{n}\times\vec{S}\,)$ is covariant derivative of vector field $\vec{S}$, and the form of dependency of the potential V on the dependent variables has not been specified, obviously, it depends on his arguments such that it is a real Lorentzian scalar.

The gauged full baby Skyrme model (in $(2+0)$-dimensions), considered in this paper, has the energy functional of the following form:

$$
H = \frac{1}{2}\int d^2x\, \mathcal{H} = \frac{1}{2}\int d^2x \left(\lambda_0 D_i\vec{S}\cdot D^i\vec{S} + \frac{\lambda_1^2}{4}\left(\epsilon_{kl} D_k\vec{S}\times D_l\vec{S}\right)^2 + F_{kl}^2 + \gamma^2 V\right), \qquad (7)
$$

where $x_1 = x$, $x_2 = y$ and $i,k,l = 1,2$. We make the stereographic projection

$$
\vec{S} = \left[\frac{\omega+\omega^*}{1+\omega\omega^*}, \frac{-i\cdot(\omega-\omega^*)}{1+\omega\omega^*}, \frac{1-\omega\omega^*}{1+\omega\omega^*}\right], \qquad i.e. \quad \omega = \frac{S_1 + iS_2}{1+S_3}\,, \qquad (8)
$$

where $\omega = \omega(x,y) \in \mathbb{C}$, $x,y\in\mathbb{R}$ and $\omega(x,y) = u(x,y) + iv(x,y)$, $u,v\in\mathbb{R}$.

The density of energy functional (7), but without $O(3)$ term (this is the Hamiltonian of gauged restricted baby Skyrme model), has the following form after the stereographic projection [37]

$$
\begin{aligned}
\mathcal{H} \;=\; & 4\lambda_1 \frac{\left[i\cdot\left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1\cdot\left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right) + A_2\cdot\left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right]^2}{(1+\omega\omega^*)^4}\\
& + \lambda_2\cdot\left(A_{2,x} - A_{1,y}\right)^2 + V\left(\omega,\omega^*,A_1,A_2\right)\,, \qquad (9)
\end{aligned}
$$

where after rescalling, the constants λ_1, λ_2 have appeared instead of λ and γ has been included in V and $\omega_{,x} \equiv \frac{\partial\omega}{\partial x}$, *etc.*

The Euler–Lagrange equations for this model are as follows [37]:

$$\begin{aligned}
&\frac{d}{dx}\left[N_1 \cdot \left(i\omega^*_{,y}+A_2\omega^*\right)\right]+\frac{d}{dy}\left[N_1 \cdot \left(-i\omega^*_{,x} - A_1\omega^*\right)\right]+\frac{1}{4\lambda_1}N_1^2\omega^*(1+\omega\omega^*)^3 \\
&-N_1 \cdot \left(-A_1\omega^*_{,y} + A_2\omega^*_{,x}\right) - V_{,\omega} = 0\,, \\
&\text{c.c.} \\
&-2\lambda_2\frac{d}{dy}\left(A_{2,x} - A_{1,y}\right) + N_1 \cdot \left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right) - V_{,A_1} = 0\,, \\
&2\lambda_2\frac{d}{dx}\left(A_{2,x} - A_{1,y}\right) - N_1 \cdot \left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right) - V_{,A_2} = 0\,, \qquad (10)
\end{aligned}$$

where

$$\begin{aligned}
N_1 \;=\; &\frac{8\lambda_1}{(1+\omega\omega^*)^4}\left[i \cdot \left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1 \cdot \left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right)\right. \\
&\left.+A_2 \cdot \left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right].
\end{aligned}$$

After making the transformation (8), the density of the energy functional (7) has the form (this is the Hamiltonian of gauged full baby Skyrme model — it can be obtained, after adding the Skyrme term to the Hamiltonian of gauged "sigma" model) [37]

$$\begin{aligned}
\mathcal{H} \;=\; &\lambda_{00}\frac{\left(A_1^2 + A_2^2\right) \cdot \left(u^2 + v^2\right) - 2A_1 \cdot \left(u_{,x}v - uv_{,x}\right) - 2A_2 \cdot \left(u_{,y}v - uv_{,y}\right)}{\left(1 + u^2 + v^2\right)^2} \\
&+\lambda_{00}\frac{u_{,x}^2 + u_{,y}^2 + v_{,x}^2 + v_{,y}^2}{\left(1 + u^2 + v^2\right)^2} \\
&+\lambda_{11}\frac{\left[\left(u_{,x}v_{,y} - u_{,y}v_{,x}\right) - A_1 \cdot \left(uu_{,y} + vv_{,y}\right) + A_2 \cdot \left(uu_{,x} + vv_{,x}\right)\right]^2}{\left(1 + u^2 + v^2\right)^4} \\
&+\lambda_2 \cdot \left(A_{2,x} - A_{1,y}\right)^2 + V\left(u, v, A_1, A_2\right)\,, \qquad (11)
\end{aligned}$$

where after rescalling, the constants $\lambda_{00} = 4\lambda_0, \lambda_{11} = 16\lambda_1, \lambda_2$ have appeared. The constant γ has been included in V and $u(x,y) = \Re(\omega(x,y))$, $v(x,y) = \Im(\omega(x,y)) \in \mathbb{R}$. Of course, in these both gauged baby Skyrme models: restricted and full one, the potentials depend on their arguments such that they are real Lorentzian scalars.

The Euler–Lagrange equations for the gauged full baby Skyrme model, have the form [37]

$$
\begin{aligned}
&\frac{d}{dx}\left\{2\lambda_{00}\frac{-A_1 v+u_{,x}}{\left(1+u^2+v^2\right)^2}+N_2\cdot\left(v_{,y}+A_2 u\right)\right\} \\
&+\frac{d}{dy}\left\{2\lambda_{00}\frac{-A_2 v+u_{,y}}{\left(1+u^2+v^2\right)^2}+N_2\cdot\left(-v_{,x}-A_1 u\right)\right\} \\
&-2\lambda_{00}\frac{\left[\left(A_1^2+A_2^2\right)u+A_1 v_{,x}+A_2 v_{,y}\right]}{\left(1+u^2+v^2\right)^2}-N_2\cdot\left(-A_1 u_{,y}+A_2 u_{,x}\right) \\
&+\frac{2}{\lambda_{11}}uN_2^2\cdot\left(1+u^2+v^2\right)^3 \\
&+4\lambda_{00}u\frac{\left(A_1^2+A_2^2\right)\cdot\left(u^2+v^2\right)-2A_1\cdot\left(u_{,x}v-uv_{,x}\right)-2A_2\cdot\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^3} \\
&+4\lambda_{00}u\frac{u_{,x}^2+u_{,y}^2+v_{,x}^2+v_{,y}^2}{\left(1+u^2+v^2\right)^3}-V_{,u}=0\,,
\end{aligned}
$$

the analogical equation, following from varying the energy functional with respect to v,

$$
\begin{aligned}
&-2\lambda_2\frac{d}{dy}\left(A_{2,x}-A_{1,y}\right)-2\lambda_{00}\frac{A_1\cdot\left(u^2+v^2\right)-\left(u_{,x}v-uv_{,x}\right)}{\left(1+u^2+v^2\right)^2} \\
&+N_2\cdot\left(uu_{,y}+vv_{,y}\right)-V_{,A_1}=0\,, \\
&2\lambda_2\frac{d}{dx}\left(A_{2,x}-A_{1,y}\right)-2\lambda_{00}\frac{A_2\cdot\left(u^2+v^2\right)-\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^2} \\
&-N_2\cdot\left(uu_{,x}+vv_{,x}\right)-V_{,A_2}=0\,,
\end{aligned}
$$

where

$$
N_2=\frac{2\lambda_{11}}{\left(1+u^2+v^2\right)^4}\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\cdot\left(uu_{,y}+vv_{,y}\right)+A_2\cdot\left(uu_{,x}+vv_{,x}\right)\right]\,.
$$

As we have mentioned in Introduction, it is obvious that the Lagrangians of these models need to be gauge invariant, however, we have assumed at the beginning, the dependency of their potentials V, on the gauge field A_k, $(k=1,2)$. This is because we want to investigate, whether the conditions for the potentials in the models investigated in this paper, in the case of existing of the Bogomolny decomposition for any of these models, will permit presence (in the Lagrangian) of the terms of the kind $A_k A^k, k=1,2$ (the Lagrangian must be real Lorentzian scalar), so, whether we obtain some model similar to the Proca theory [46] or to the theory of a massive vector field [47].

1.3. The concept of strong necessary conditions

The idea of the concept of strong necessary conditions is such that instead of considering of the Euler–Lagrange equations,

$$F_{,u} - \frac{d}{dx}F_{,u_{,x}} - \frac{d}{dt}F_{,u_{,t}} = 0\,, \tag{12}$$

following from the extremum principle applied to the functional,

$$\Phi[u] = \int\limits_{E^2} F(u, u_{,x}, u_{,t})\; dxdt\,, \tag{13}$$

we consider strong necessary conditions [40–42],

$$F_{,u} \;=\; 0\,, \tag{14}$$

$$F_{,u_{,t}} \;=\; 0\,, \tag{15}$$

$$F_{,u_{,x}} \;=\; 0\,, \tag{16}$$

where $F_{,u} \equiv \frac{\partial F}{\partial u}$, *etc.*

Obviously, all solutions of the system of equations (14)–(16) satisfy the Euler–Lagrange equation (12). However, these solutions, if they exist, are very often trivial. So, in order to avoid such situation, we make gauge transformation of the functional (13)

$$\Phi \to \Phi + \mathrm{Inv}\,, \tag{17}$$

where Inv is such functional that its local variation with respect to $u(x,t)$ vanishes: $\delta\,\mathrm{Inv} \equiv 0$.

By virtue of this feature, we have the equivalence of: the Euler–Lagrange equations (12) and the Euler–Lagrange equations resulting from requiring of the extremum of $\Phi + \mathrm{Inv}$. On the other hand, there is not the invariance of the strong necessary conditions (14)–(16), with respect to the gauge transformation (17) and so, we may expect to obtain nontrivial solutions. As one can notice, the strong necessary conditions (14)–(16) constitute the system of the partial differential equations of the order less than the order of Euler–Lagrange equations (12).

We will use, among others, the so-called divergent invariants [41, 42]: $\frac{d}{dx}f$, $\frac{d}{dy}g$, where $f = f(u, v, A_1, A_2)$, $g = g(u, v, A_1, A_2)$ are some functions, which are to be determined later. Let us notice here that by using the Lagrangian gauged on among others divergent invariants, we can obtain the same Euler–Lagrange equations, as these ones obtained by using Lagrangian ungauged on these invariants, even in that case, when the divergent invariants are not invariant under gauge transformations of the field A_k, $k = 1, 2$.

2. Bogomolny decompositions for gauged models: $O(3)$ "sigma", restricted baby Skyrme and full baby Syrme

2.1. Derivation of the general expressions for the density of the topological invariant

The important step is to construct the general form of the density of the topological invariant for the case of the topology of this model. Some construction of the density of this topological invariant has been given in [22, 48]

$$I_1 = \vec{S} \cdot D_1\vec{S} \times D_2\vec{S} + F_{12} \cdot \left(1 - \vec{n} \cdot \vec{S}\right) , \tag{18}$$

where $D_i\vec{S} = \partial_i\vec{S} + A_i\vec{n} \times \vec{S}$, $(i = 1, 2)$ is covariant derivative of vector field $\vec{S}$ and $F_{12} = \partial_1 A_2 - \partial_2 A_1$ is magnetic field.

After making the stereographic projection (8), we have

$$\begin{aligned} I_1 \ &= \ \frac{1}{(1+\omega\omega^*)^2}\Big[2\big(i \cdot \left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1 \cdot \left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right) \\ &\quad + A_2 \cdot \left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\big)\Big] + \frac{2\omega\omega^*}{1+\omega\omega^*}\left(A_{2,x} - A_{1,y}\right) . \end{aligned} \tag{19}$$

It is useful to generalize the above expression such that there we place some real functions (differentiable at least once) $R_j = R_j(\omega, \omega^*, A_1, A_2)$, $(j = 1, 2)$

$$\begin{aligned} I_1 \ &= \ \lambda_3 \cdot \Big\{R_1\left(\omega, \omega^*, A_1, A_2\right) \cdot \left[i \cdot \left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1 \cdot \left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right)\right. \\ &\quad \left. + A_2 \cdot \left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right] + R_2\left(\omega, \omega^*, A_1, A_2\right) \cdot \left(A_{2,x} - A_{1,y}\right)\Big\} . \end{aligned} \tag{20}$$

We make the functions R_j $(j = 1, 2)$, as dependent not only on ω, ω^*, but also on A_k $(k = 1, 2)$, in order to get the most general form of I_1, as it is possible. Next, we look for such conditions for the functions R_j $(j = 1, 2)$, that the expression (20) is the density of the topological invariant *i.e.* its variations with respect to ω, ω^*, A_k $(k = 1, 2)$ always vanish.

As it turns out, $R_1 = G_1'$ and $R_2 = G_1$, hence, above expression has the following form [37]:

$$\begin{aligned} I_1 \ &= \ \lambda_3 \cdot \Big\{G_1' \cdot \left[i \cdot \left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1 \cdot \left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right)\right. \\ &\quad \left. + A_2 \cdot \left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right] + G_1 \cdot \left(A_{2,x} - A_{1,y}\right)\Big\} , \end{aligned} \tag{21}$$

where $\lambda_3 = \text{const.}$ $G_1 = G_1(\omega\omega^*) \in \mathbb{R}$ is some arbitrary function differentiable at least twice. G_1' denotes the derivative of the function G_1

with respect to its argument: $\omega\omega^*$. As we see, here is the generalization in comparison with [37], where $G_1 = G_1(\frac{2\omega\omega^*}{1+\omega\omega^*})$. This generalization makes possible deriving of the Bogomolny decomposition for more wide class of the potentials.

When we need to express (21) in real functions $u = \Re(\omega), v = \Im(\omega)$, then [37]

$$\begin{aligned} I_1 \;=\; & \lambda_3 \cdot \Big\{ G_1' \cdot \big[(u_{,x} v_{,y} - u_{,y} v_{,x}) - A_1 \cdot (u u_{,y} + v v_{,y}) \\ & + A_2 \cdot (u u_{,x} + v v_{,x}) \big] + \tfrac{1}{2} G_1 \cdot (A_{2,x} - A_{1,y}) \Big\} , \end{aligned} \tag{22}$$

where $\lambda_3 = \text{const.}$ $G_1 = G_1(u^2 + v^2) \in \mathbb{R}$ is some arbitrary function differentiable at least twice. G_1' denotes here the derivative of the function G_1 with respect to its argument: $u^2 + v^2$.

When we investigate gauged restricted baby Skyrme model, we will use (21) as the form of the density of the topological invariant, and when we investigate gauged $O(3)$ "sigma" model and gauged full baby Skyrme model, we will use (22). In the next subsections, the symbol "$\cdot$" will be neglected, for simplicity.

2.2. Bogomolny decomposition for gauged $O(3)$ "sigma" model

We make (according to Subsection 1.3) gauge transformation of (4) by using the sum of the invariants $\sum_{k=1}^{3} I_k$: $\mathcal{H} \to \tilde{\mathcal{H}}$

$$\begin{aligned} \tilde{\mathcal{H}} \;=\; & \lambda_{00} \frac{\left(A_1^2 + A_2^2\right)\left(u^2 + v^2\right) - 2A_1\left(u_{,x} v - u v_{,x}\right) - 2A_2\left(u_{,y} v - u v_{,y}\right)}{\left(1 + u^2 + v^2\right)^2} \\ & + \lambda_{00} \frac{u_{,x}^2 + u_{,y}^2 + v_{,x}^2 + v_{,y}^2}{\left(1 + u^2 + v^2\right)^2} + \lambda_2 \left(A_{2,x} - A_{1,y}\right)^2 + V\left(u, v, A_1, A_2\right) \\ & + \lambda_3 \left\{ G_1' \left[\left(u_{,x} v_{,y} - u_{,y} v_{,x}\right) - A_1\left(u u_{,y} + v v_{,y}\right) + A_2\left(u u_{,x} + v v_{,x}\right)\right] \right. \\ & \left. + \tfrac{1}{2} G_1 \left(A_{2,x} - A_{1,y}\right)\right\} + D_x G_3 + D_y G_4 , \end{aligned} \tag{23}$$

where I_1 is topological invariant of the form (22), $I_2 = D_x G_2(u, v, A_1, A_2)$, $I_3 = D_y G_3(u, v, A_1, A_2)$, $D_x \equiv \frac{d}{dx}, D_y \equiv \frac{d}{dy}$. The functions $G_1 = G_1(u^2{+}v^2)$ and $G_{n+1} = G_{n+1}(u, v, A_1, A_2), (n = 1, 2)$ are differentiable at least twice and they are to be determined later. G_1' means the derivative of G_1 with respect to its argument: $(u^2 + v^2)$.

The strong necessary conditions for (23) have the form

$$\tilde{\mathcal{H}}_{,u}:\ \lambda_{00}\frac{\left[2\left(A_1^2+A_2^2\right)u+2A_1v_{,x}+2A_2v_{,y}\right]}{\left(1+u^2+v^2\right)^2}$$
$$-4\lambda_{00}u\frac{\left(A_1^2+A_2^2\right)\left(u^2+v^2\right)-2A_1\left(u_{,x}v-uv_{,x}\right)-2A_2\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^3}$$
$$-4\lambda_{00}u\frac{u_{,x}^2+u_{,y}^2+v_{,x}^2+v_{,y}^2}{\left(1+u^2+v^2\right)^3}$$
$$+V_{,u}+\lambda_3\left\{G'_{1,u}\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\right.$$
$$\left.+G'_1\left(-A_1u_{,y}+A_2u_{,x}\right)+\tfrac{1}{2}G_{1,u}(A_{2,x}-A_{1,y})\right\}+D_xG_{3,u}+D_yG_{4,u}=0\,,\quad (24)$$

$$\tilde{\mathcal{H}}_{,v}:\ \lambda_{00}\frac{\left[2\left(A_1^2+A_2^2\right)v-2A_1u_{,x}-2A_2u_{,y}\right]}{\left(1+u^2+v^2\right)^2}$$
$$-4\lambda_{00}v\frac{\left(A_1^2+A_2^2\right)\left(u^2+v^2\right)-2A_1\left(u_{,x}v-uv_{,x}\right)-2A_2\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^3}$$
$$-4\lambda_{00}v\frac{u_{,x}^2+u_{,y}^2+v_{,x}^2+v_{,y}^2}{\left(1+u^2+v^2\right)^3}$$
$$+V_{,v}+\lambda_3\left\{G'_{1,v}\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\right.$$
$$\left.+G'_1\left(-A_1v_{,y}+A_2v_{,x}\right)+\tfrac{1}{2}G_{1,v}\left(A_{2,x}-A_{1,y}\right)\right\}+D_xG_{3,v}+D_yG_{4,v}=0\,,\quad (25)$$

$$\tilde{\mathcal{H}}_{,A_1}:\ \lambda_{00}\frac{2A_1\left(u^2+v^2\right)-2\left(u_{,x}v-uv_{,x}\right)}{\left(1+u^2+v^2\right)^2}+V_{,A_1}$$
$$-\lambda_3G'_1\left(uu_{,y}+vv_{,y}\right)+D_xG_{3,A_1}+D_yG_{4,A_1}=0\,,\quad (26)$$
$$\tilde{\mathcal{H}}_{,A_2}:\ \lambda_{00}\frac{2A_2\left(u^2+v^2\right)-2\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^2}+V_{,A_2}$$
$$+\lambda_3G'_1\left(uu_{,x}+vv_{,x}\right)+D_xG_{3,A_2}+D_yG_{4,A_2}=0\,,\quad (27)$$

$$\tilde{\mathcal{H}}_{,u_x}:\ 2\lambda_{00}\frac{-A_1v+u_{,x}}{\left(1+u^2+v^2\right)^2}+\lambda_3\left\{G'_1\left[v_{,y}+A_2u\right]\right\}+G_{3,u}=0\,,\quad (28)$$
$$\tilde{\mathcal{H}}_{,u_y}:\ 2\lambda_{00}\frac{-A_2v+u_{,y}}{\left(1+u^2+v^2\right)^2}+\lambda_3\left\{G'_1\left[-v_{,x}-A_1u\right]\right\}+G_{4,u}=0\,,\quad (29)$$
$$\tilde{\mathcal{H}}_{,v_x}:\ 2\lambda_{00}\frac{A_1u+v_{,x}}{\left(1+u^2+v^2\right)^2}+\lambda_3\left\{G'_1\left[-u_{,y}+A_2v\right]\right\}+G_{3,v}=0\,,\quad (30)$$
$$\tilde{\mathcal{H}}_{,v_y}:\ 2\lambda_{00}\frac{A_2u+v_{,y}}{\left(1+u^2+v^2\right)^2}+\lambda_3\left\{G'_1\left[u_{,x}-A_1v\right]\right\}+G_{4,v}=0\,,\quad (31)$$

$$\tilde{\mathcal{H}}_{,A_{1,x}} \,:\; G_{3,A_1} = 0\,, \tag{32}$$

$$\tilde{\mathcal{H}}_{,A_{1,y}} \,:\; -2\lambda_2\left(A_{2,x} - A_{1,y}\right) - \frac{\lambda_3}{2}G_1 + G_{4,A_1} = 0\,, \tag{33}$$

$$\tilde{\mathcal{H}}_{,A_{2,x}} \,:\; 2\lambda_2\left(A_{2,x} - A_{1,y}\right) + \frac{\lambda_3}{2}G_1 + G_{3,A_2} = 0\,, \tag{34}$$

$$\tilde{\mathcal{H}}_{,A_{2,y}} \,:\; G_{4,A_2} = 0\,. \tag{35}$$

Now we need to make equations (24)–(35) self-consistent. In this order, we need to reduce the number of independent equations by a proper choice of the functions $G_k, (k = 1, 2, 3)$. Very often such Änsatze exist only for some special forms of V and very often it is impossible to reduce the system of corresponding dual equations to the Bogomolny equations. However, even if we cannot make the reduction mentioned above, such system can be used to derive at least some set of solutions of Euler–Lagrange equations.

At first we put

$$u_{,x} + v_{,y} = -\frac{\left(1+u^2+v^2\right)^2}{2\lambda_{00}}G_{3,u} + A_1 v - A_2 u\,, \tag{36}$$

$$u_{,y} - v_{,x} = \frac{\left(1+u^2+v^2\right)^2}{2\lambda_{00}}G_{3,v} + A_1 u + A_2 v\,, \tag{37}$$

$$A_{2,x} - A_{1,y} = -\frac{1}{2\lambda_2}\left(\frac{\lambda_3}{2}G_1 + G_{3,A_2}\right)\,, \tag{38}$$

$$G_1' = \frac{2\lambda_{00}}{\lambda_3\left(1+u^2+v^2\right)^2}\,, \qquad G_{3,uA_1} = 0\,, \qquad G_{4,uA_2} = 0\,, \tag{39}$$

where G_1' denotes the derivative of the function G_1, with respect to its argument: $1 + u^2 + v^2$.

Then, it turns out that

$$G_{3,u} = G_{4,v}\,, \qquad G_{3,v} = -G_{4,u}\,, \qquad G_{4,A_1} = -G_{3,A_2}\,. \tag{40}$$

Hence, from (32) and (35)

$$G_3 = f(u,v) + c_2 A_2\,,\; G_4 = f(u,v) - c_2 A_1\,,\; c_2 = \text{const}\,,\; f_{,uu} + f_{,vv} = 0 \tag{41}$$

and equations (28)–(35) become the tautologies.

Equations (26), (27), after taking into account (32)–(35), (36)–(39), (41) and the fact that the potential V should be a Lorentzian scalar, implicate that $V_{,A_k} = 0,\; (k = 1, 2)$. Hence, after eliminating all expressions including the derivatives of the fields u, v, A_1, A_2, from equations (24)–(27), by using (36)–(39) (after taking into account (41)), we obtain some system of the partial differential equations for $V(u,v)$ and $f(u,v)$. The solutions of this system are: $f(u,v) = \mathrm{const}$ and the condition for the potential has the following form

$$V(u,v) = \frac{1}{2}\frac{2c_1\lambda_2\left(1+u^2+v^2\right)^2 - \lambda_{00}\left(c_2\left(1+u^2+v^2\right) - \frac{\lambda_{00}}{2}\right)}{\lambda_2\left(1+u^2+v^2\right)^2}\,, \tag{42}$$

where $c_1 = \mathrm{const}, c_2 = \mathrm{const}$.

Hence, the Bogomolny decomposition for gauged $O(3)$ "sigma" model in $(2+0)$-dimensions, has the form

$$\begin{aligned} u_{,x} + v_{,y} &= A_1 v - A_2 u\,, \\ u_{,y} - v_{,x} &= A_1 u + A_2 v\,, \\ A_{2,x} - A_{1,y} &= \frac{1}{2\lambda_2}\left(\frac{\lambda_{00}}{1+u^2+v^2} - c_2\right)\,, \end{aligned} \tag{43}$$

where the potential $V(u,v)$ needs to satisfy the conditon (42). As we see, this is some generalization of the result (Bogomolny equations obtained for the potential $V = (1 - \vec{n}\cdot\vec{S})^2$, where $\vec{n} = [0,0,1]$) included in [22].

2.3. The Bogomolny decomposition for gauged restricted baby Skyrme model

Now, we start to investigate gauged restricted baby Skyrme model.

We make the following gauge transformation of $\mathcal{H}$, on the sum of the invariants $\sum_{n=1}^{3} I_n$ [37]:

$$\begin{aligned} \mathcal{H} \longrightarrow \tilde{\mathcal{H}} = 4\lambda_1 &\frac{\left[i\left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1\left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right) + A_2\left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right]^2}{\left(1+\omega\omega^*\right)^4} \\ &+\lambda_2\left(A_{2,x} - A_{1,y}\right)^2 + V\left(\omega,\omega^*,A_1,A_2\right) + \lambda_3\left\{G_1'\left[\left(i\left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right)\right.\right.\right. \\ &\left.\left.\left. - A_1\left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right) + A_2\left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right)\right] + G_1(A_{2,x} - A_{1,y})\right\} \\ &+ D_x G_2 + D_y G_3\,, \end{aligned} \tag{44}$$

where I_1 is given by (21), $I_2 = D_x G_2(\omega,\omega^*,A_1,A_2)$, $I_3 = D_y G_3(\omega,\omega^*,A_1,A_2)$, $D_x \equiv \frac{d}{dx}$, $D_y \equiv \frac{d}{dy}$. $G_1 = G_1(\omega\omega^*)$ and $G_{k+1} = G_{k+1}(\omega,\omega^*,A_1,A_2)$, $(k=1,2)$ are some functions (differentiable at least twice), which are to be determinated later.

After applying the concept of strong necessary conditions to (44), we obtain the so-called dual equations [37]

$$\tilde{\mathcal{H}}_{,\omega} : -16\lambda_1 \frac{\left[i\left(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x}\right)-A_1\left(\omega_{,y}\omega^*+\omega\omega^*_{,y}\right)+A_2\left(\omega_{,x}\omega^*+\omega\omega^*_{,x}\right)\right]^2\omega^*}{\left(1+\omega\omega^*\right)^5}$$
$$+\frac{8\lambda_1\left[i\left(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x}\right)-A_1\left(\omega_{,y}\omega^*+\omega\omega^*_{,y}\right)+A_2\left(\omega_{,x}\omega^*+\omega\omega^*_{,x}\right)\right]}{\left(1+\omega\omega^*\right)^4}$$
$$\times\left(-A_1\omega^*_{,y}+A_2\omega^*_{,x}\right)+V_{,\omega}+\lambda_3\left\{G_1''\omega^*\left[i\left(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x}\right)\right.\right.$$
$$\left.-A_1(\omega_{,y}\omega^*+\omega\omega^*_{,y})+A_2\left(\omega_{,x}\omega^*+\omega\omega^*_{,x}\right)\right]+G_1'\left(-A_1\omega^*_{,y}+A_2\omega^*_{,x}\right)$$
$$\left.+G_1'\omega^*(A_{2,x}-A_{1,y})\right\}+D_xG_{2,\omega}+D_yG_{3,\omega}=0\,, \tag{45}$$

$$\tilde{\mathcal{H}}_{,\omega^*} : -16\lambda_1 \frac{\left[i\left(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x}\right)-A_1\left(\omega_{,y}\omega^*+\omega\omega^*_{,y}\right)+A_2(\omega_{,x}\omega^*+\omega\omega^*_{,x})\right]^2\omega}{\left(1+\omega\omega^*\right)^5}$$
$$+\frac{8\lambda_1\left[i\left(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x}\right)-A_1\left(\omega_{,y}\omega^*+\omega\omega^*_{,y}\right)+A_2\left(\omega_{,x}\omega^*+\omega\omega^*_{,x}\right)\right]}{\left(1+\omega\omega^*\right)^4}$$
$$\times\left(-A_1\omega_{,y}+A_2\omega_{,x}\right)+V_{,\omega^*}+\lambda_3\left\{G_1''\omega\left[i\left(\omega_{,x}\omega^*_{,y}-\omega_{,y}\omega^*_{,x}\right)\right.\right.$$
$$\left.-A_1\left(\omega_{,y}\omega^*+\omega\omega^*_{,y}\right)+A_2\left(\omega_{,x}\omega^*+\omega\omega^*_{,x}\right)\right]+G_1'\left(-A_1\omega_{,y}+A_2\omega_{,x}\right)$$
$$\left.+G_1'\omega\left(A_{2,x}-A_{1,y}\right)\right\}+D_xG_{2,\omega^*}+D_yG_{3,\omega^*}=0\,, \tag{46}$$

$$\tilde{\mathcal{H}}_{,A_1} : \; N_3\left(-\omega_{,y}\omega^*-\omega\omega^*_{,y}\right)$$
$$+V_{,A_1}+\lambda_3G_1'\left(-\omega_{,y}\omega^*-\omega\omega^*_{,y}\right)+D_xG_{2,A_1}+D_yG_{3,A_1}=0\,, \tag{47}$$
$$\tilde{\mathcal{H}}_{,A_2} : \; N_3\left(\omega_{,x}\omega^*+\omega\omega^*_{,x}\right)$$
$$+V_{,A_2}+\lambda_3G_1'\left(\omega_{,x}\omega^*+\omega\omega^*_{,x}\right)+D_xG_{2,A_2}+D_yG_{3,A_2}=0\,, \tag{48}$$

$$\tilde{\mathcal{H}}_{,\omega_{,x}} : \; N_3\left(i\omega^*_{,y}+A_2\omega^*\right)+\lambda_3G_1'\left(i\omega^*_{,y}+A_2\omega^*\right)+G_{2,\omega}=0\,, \tag{49}$$
$$\tilde{\mathcal{H}}_{,\omega_{,y}} : \; N_3\left(-i\omega^*_{,x}-A_1\omega^*\right)+\lambda_3G_1'\left(-i\omega^*_{,x}-A_1\omega^*\right)+G_{3,\omega}=0\,, \tag{50}$$
$$\tilde{\mathcal{H}}_{,\omega^*_{,x}} : \; N_3\left(-i\omega_{,y}+A_2\omega\right)+\lambda_3G_1'\left(-i\omega_{,y}+A_2\omega\right)+G_{2,\omega^*}=0\,, \tag{51}$$
$$\tilde{\mathcal{H}}_{,\omega^*_{,y}} : \; N_3\left(i\omega_{,x}-A_1\omega\right)+\lambda_3G_1'\left(i\omega_{,x}-A_1\omega\right)+G_{3,\omega^*}=0\,, \tag{52}$$

$$\tilde{\mathcal{H}}_{,A_{1,x}} : \; G_{2,A_1}=0\,, \tag{53}$$
$$\tilde{\mathcal{H}}_{,A_{1,y}} : \; -2\lambda_2\left(A_{2,x}-A_{1,y}\right)-\lambda_3G_1+G_{3,A_1}=0\,, \tag{54}$$
$$\tilde{\mathcal{H}}_{,A_{2,x}} : \; 2\lambda_2\left(A_{2,x}-A_{1,y}\right)+\lambda_3G_1+G_{2,A_2}=0\,, \tag{55}$$
$$\tilde{\mathcal{H}}_{,A_{2,y}} : \; G_{3,A_2}=0\,, \tag{56}$$

where

$$N_3 = \frac{8\lambda_1\left[i\left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1\left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right) + A_2\left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right]}{(1+\omega\omega^*)^4}$$

and G'_1, G''_1 denote the derivatives of the function G_1 with respect to its argument: $\omega\omega^*$.

Now, we consider $\omega, \omega^*, A_i, (i = 1,2), G_k, \ (k = 1,2,3)$ as equivalent dependent variables governed by the system of equations (45)–(56). We make two operations (similar operations were made firstly in [32], in the cases of ungauged baby Skyrme models: full and restricted one).

Namely, as we see, after putting [37]

$$G'_1 = -\frac{8\lambda_1\left[i\left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1\left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right) + A_2\left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right]}{\lambda_3(1+\omega\omega^*)^4}, \tag{57}$$

$$A_{2,x} - A_{1,y} = -\frac{1}{2\lambda_2}\left(\lambda_3 G_1 + G_{2,A_2}\right), \tag{58}$$

$$G_{3,A_1} = -G_{2,A_2}, \quad G_2 = c_2 A_2, \quad G_3 = -c_2 A_1, \quad c_2 = \text{const}, \tag{59}$$

equations (49)–(56) become the tautologies and the candidate for the Bogomolny decomposition is [37]

$$\begin{aligned} &\frac{8\lambda_1\left[i\left(\omega_{,x}\omega^*_{,y} - \omega_{,y}\omega^*_{,x}\right) - A_1\left(\omega_{,y}\omega^* + \omega\omega^*_{,y}\right) + A_2\left(\omega_{,x}\omega^* + \omega\omega^*_{,x}\right)\right]}{\lambda_3(1+\omega\omega^*)^4} = -G'_1, \\ &2\lambda_2(A_{2,x} - A_{1,y}) + \lambda_3 G_1 + c_2 = 0. \end{aligned} \tag{60}$$

Now, the next step is checking, when equations (45)–(48) are satisfied, if (60) hold. Thus, we insert (59) and (60), into (45)–(48). From (47)–(48), we obtain that $V_{,A_k} = 0, \ (k = 1,2)$. Hence, we get some system of partial differential equations for $V(\omega, \omega^*)$ and the solution of it is [37]

$$\begin{aligned} V(\omega,\omega^*) = {} & \frac{\lambda_3}{8\lambda_1\lambda_2}\int\Big(\lambda_2\lambda_3(1+\omega\omega^*)^4 G''_1 + 2\lambda_2\lambda_3(1+\omega\omega^*)^3 G'_1 + 4\lambda_1(\lambda_3 G_1 \\ & + c_2)\Big) G'_1\omega^* d\omega + \frac{\lambda_3}{8\lambda_1\lambda_2}\int\Big(\lambda_2\lambda_3(1+\omega\omega^*)^4 G''_1 + 2\lambda_2\lambda_3(1+\omega\omega^*)^3 G'_1 \\ & + 4\lambda_1(\lambda_3 G_1 + c_2)\Big) G'_1\omega d\omega^* + \int\frac{1}{8\lambda_1\lambda_2}\Bigg[\lambda_3\Bigg(-\Bigg(\int\Bigg(\lambda_3(8\lambda_2(\omega\omega^*)^3 \\ & + 18\lambda_2(\omega\omega^*)^2 + 4(\lambda_1 + 3\lambda_2)\omega\omega^* + 2\lambda_2)G'^2_1 + (\lambda_2(9\omega\omega^* + 1)(\omega\omega^* + 1)^3\lambda_3 G''_1 \\ & + \omega\omega^*\lambda_2\lambda_3(\omega\omega^* + 1)^4 G'''_1 + 4\lambda_1(\lambda_3 G_1 + c_2))G'_1 + G''_1\omega^*(\lambda_2\lambda_3(\omega\omega^* + 1)^4 G''_1 \\ & + 4\lambda_1(\lambda_3 G_1 + c_2))\omega\Bigg) d\omega\Bigg)\Bigg)\Bigg] d\omega^*. \end{aligned} \tag{61}$$

So, equations (60) are the Bogomolny decomposition for gauged restricted baby Skyrme model in (2+0)-dimensions, for the potential $V(\omega,\omega^*)$, satisfying (61), where $G_1 = G_1(\omega\omega^*) \in \mathcal{C}^3$ and G_1', G_1'', G_1''' denote the derivatives of the function G_1 with respect to its argument $\omega\omega^*$.

2.4. The Bogomolny decomposition for gauged full baby Skyrme model

We make gauge transformation of (11) by using two topological invariants of the form (22) on the sum of the invariants $\sum_{n=1}^{4} I_n$ [37] $\mathcal{H} \to \tilde{\mathcal{H}}$

$$\begin{aligned}
\tilde{\mathcal{H}} &= \lambda_{00}\frac{\left(A_1^2+A_2^2\right)\left(u^2+v^2\right)-2A_1\left(u_{,x}v-uv_{,x}\right)-2A_2\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^2}\\
&+\lambda_{00}\frac{u_{,x}^2+u_{,y}^2+v_{,x}^2+v_{,y}^2}{\left(1+u^2+v^2\right)^2}\\
&+\lambda_{11}\frac{\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]^2}{\left(1+u^2+v^2\right)^4}\\
&+\lambda_2\left(A_{2,x}-A_{1,y}\right)^2+V\left(u,v,A_1,A_2\right)\\
&+\lambda_3\left\{F_1'\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\right.\\
&\left.+\tfrac{1}{2}F_1\left(A_{2,x}-A_{1,y}\right)\right\}+\lambda_4\{F_2'\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)\right.\\
&\left.+A_2\left(uu_{,x}+vv_{,x}\right)\right]+\tfrac{1}{2}F_2\left(A_{2,x}-A_{1,y}\right)\}+D_xG_3+D_yG_4\,, \qquad (62)
\end{aligned}$$

where: I_k $(k=1,2)$ are topological invariants of the form (22), $I_3 = D_xG_3(u,v,A_1,A_2), I_4 = D_yG_4(u,v,A_1,A_2), D_x \equiv \frac{d}{dx}, D_y \equiv \frac{d}{dy}$. $F_k = F_k(u^2+v^2), (k=1,2)$ and $G_{n+1} = G_{n+1}(u,v,A_1,A_2), (n=2,3)$, are some functions (differentiable at least twice), which are to be determined later and F_k' means the derivative of F_k, with respect to its argument: (u^2+v^2).

The strong necessary conditions for (62) have the form [37]

$$\tilde{\mathcal{H}}_{,u}:\ \lambda_{00}\frac{\left[2\left(A_1^2+A_2^2\right)u+2A_1v_{,x}+2A_2v_{,y}\right]}{\left(1+u^2+v^2\right)^2}$$
$$-4\lambda_{00}u\frac{\left(A_1^2+A_2^2\right)\left(u^2+v^2\right)-2A_1\left(u_{,x}v-uv_{,x}\right)-2A_2\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^3}$$
$$-4\lambda_{00}u\frac{u_{,x}^2+u_{,y}^2+v_{,x}^2+v_{,y}^2}{\left(1+u^2+v^2\right)^3}$$
$$+2\lambda_{11}\frac{[(u_{,x}v_{,y}-u_{,y}v_{,x})-A_1(uu_{,y}+vv_{,y})+A_2(uu_{,x}+vv_{,x})](-A_1u_{,y}+A_2u_{,x})}{\left(1+u^2+v^2\right)^4}$$
$$-8\lambda_{11}u\frac{\left[(u_{,x}v_{,y}-u_{,y}v_{,x})-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]^2}{\left(1+u^2+v^2\right)^5}+V_{,u}$$
$$+\lambda_3\left\{F'_{1,u}\left[(u_{,x}v_{,y}-u_{,y}v_{,x})-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\right.$$
$$\left.+F'_1\left(-A_1u_{,y}+A_2u_{,x}\right)+\tfrac{1}{2}F_{1,u}\left(A_{2,x}-A_{1,y}\right)\right\}$$
$$+\lambda_4\left\{F'_{2,u}\left[(u_{,x}v_{,y}-u_{,y}v_{,x})-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\right.$$
$$\left.+F'_2\left(-A_1u_{,y}+A_2u_{,x}\right)+\tfrac{1}{2}F_{2,u}\left(A_{2,x}-A_{1,y}\right)\right\}+D_xG_{3,u}+D_yG_{4,u}=0\,, \tag{63}$$

$$\tilde{\mathcal{H}}_{,v}:\ \lambda_{00}\frac{\left[2\left(A_1^2+A_2^2\right)v-2A_1u_{,x}-2A_2u_{,y}\right]}{\left(1+u^2+v^2\right)^2}$$
$$-4\lambda_{00}v\frac{\left(A_1^2+A_2^2\right)\left(u^2+v^2\right)-2A_1\left(u_{,x}v-uv_{,x}\right)-2A_2\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^3}$$
$$-4\lambda_{00}v\frac{u_{,x}^2+u_{,y}^2+v_{,x}^2+v_{,y}^2}{\left(1+u^2+v^2\right)^3}$$
$$+2\lambda_{11}\frac{[(u_{,x}v_{,y}-u_{,y}v_{,x})-A_1(uu_{,y}+vv_{,y})+A_2(uu_{,x}+vv_{,x})](-A_1v_{,y}+A_2v_{,x})}{\left(1+u^2+v^2\right)^4}$$
$$-8\lambda_{11}v\frac{\left[(u_{,x}v_{,y}-u_{,y}v_{,x})-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]^2}{\left(1+u^2+v^2\right)^5}+V_{,v}$$
$$+\lambda_3\left\{F'_{1,v}\left[(u_{,x}v_{,y}-u_{,y}v_{,x})-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\right.$$
$$\left.+F'_1\left(-A_1v_{,y}+A_2v_{,x}\right)+\tfrac{1}{2}F_{1,v}\left(A_{2,x}-A_{1,y}\right)\right\}$$
$$+\lambda_4\left\{F'_{2,v}\left[(u_{,x}v_{,y}-u_{,y}v_{,x})-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\right.$$
$$\left.+F'_2\left(-A_1v_{,y}+A_2v_{,x}\right)+\tfrac{1}{2}F_{2,v}\left(A_{2,x}-A_{1,y}\right)\right\}+D_xG_{3,v}+D_yG_{4,v}=0\,, \tag{64}$$

$$\tilde{\mathcal{H}}_{,A_1}:\ \lambda_{00}\frac{2A_1\left(u^2+v^2\right)-2\left(u_{,x}v-uv_{,x}\right)}{\left(1+u^2+v^2\right)^2}$$
$$-2\lambda_{11}\frac{\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\left(uu_{,y}+vv_{,y}\right)}{\left(1+u^2+v^2\right)^4}$$
$$+V_{,A_1}-\lambda_3F_1'\left(uu_{,y}+vv_{,y}\right)-\lambda_4F_2'\left(uu_{,y}+vv_{,y}\right)+D_xG_{3,A_1}+D_yG_{4,A_1}=0\,, \tag{65}$$

$$\tilde{\mathcal{H}}_{,A_2}:\ \lambda_{00}\frac{2A_2\left(u^2+v^2\right)-2\left(u_{,y}v-uv_{,y}\right)}{\left(1+u^2+v^2\right)^2}$$
$$+2\lambda_{11}\frac{\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\left(uu_{,x}+vv_{,x}\right)}{\left(1+u^2+v^2\right)^4}$$
$$+V_{,A_2}+\lambda_3F_1'\left(uu_{,x}+vv_{,x}\right)+\lambda_4F_2'\left(uu_{,x}+vv_{,x}\right)+D_xG_{3,A_2}+D_yG_{4,A_2}=0\,, \tag{66}$$

$$\tilde{\mathcal{H}}_{,u_x}:\ 2\lambda_{00}\frac{-A_1v+u_{,x}}{\left(1+u^2+v^2\right)^2}$$
$$+2\lambda_{11}\frac{\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\left(v_{,y}+A_2u\right)}{\left(1+u^2+v^2\right)^4}$$
$$+\lambda_3\left\{F_1'\left[v_{,y}+A_2u\right]\right\}+\lambda_4\left\{F_2'\left[v_{,y}+A_2u\right]\right\}+G_{3,u}=0\,, \tag{67}$$

$$\tilde{\mathcal{H}}_{,u_y}:\ 2\lambda_{00}\frac{-A_2v+u_{,y}}{\left(1+u^2+v^2\right)^2}$$
$$+2\lambda_{11}\frac{\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\left(-v_{,x}-A_1u\right)}{\left(1+u^2+v^2\right)^4}$$
$$+\lambda_3\left\{F_1'\left[-v_{,x}-A_1u\right]\right\}+\lambda_4\left\{F_2'\left[-v_{,x}-A_1u\right]\right\}+G_{4,u}=0\,, \tag{68}$$

$$\tilde{\mathcal{H}}_{,v_x}:\ 2\lambda_{00}\frac{A_1u+v_{,x}}{\left(1+u^2+v^2\right)^2}$$
$$+2\lambda_{11}\frac{\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\left(-u_{,y}+A_2v\right)}{\left(1+u^2+v^2\right)^4}$$
$$+\lambda_3\left\{F_1'\left[-u_{,y}+A_2v\right]\right\}+\lambda_4\left\{F_2'\left[-u_{,y}+A_2v\right]\right\}+G_{3,v}=0\,, \tag{69}$$

$$\tilde{\mathcal{H}}_{,v_y}:\ 2\lambda_{00}\frac{A_2u+v_{,y}}{\left(1+u^2+v^2\right)^2}$$
$$+2\lambda_{11}\frac{\left[\left(u_{,x}v_{,y}-u_{,y}v_{,x}\right)-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)\right]\left(u_{,x}-A_1v\right)}{\left(1+u^2+v^2\right)^4}$$
$$+\lambda_3\left\{F_1'\left[u_{,x}-A_1v\right]\right\}+\lambda_4\left\{F_2'\left[u_{,x}-A_1v\right]\right\}+G_{4,v}=0\,, \tag{70}$$

$$\tilde{\mathcal{H}}_{,A_{1,x}} : \; G_{3,A_1} = 0 \,, \tag{71}$$

$$\tilde{\mathcal{H}}_{,A_{1,y}} : \; -2\lambda_2 \left(A_{2,x} - A_{1,y}\right) - \frac{\lambda_3}{2} F_1 - \frac{\lambda_4}{2} F_2 + G_{4,A_1} = 0 \,, \tag{72}$$

$$\tilde{\mathcal{H}}_{,A_{2,x}} : \; 2\lambda_2 \left(A_{2,x} - A_{1,y}\right) + \frac{\lambda_3}{2} F_1 + \frac{\lambda_4}{2} F_2 + G_{3,A_2} = 0 \,, \tag{73}$$

$$\tilde{\mathcal{H}}_{,A_{2,y}} : \; G_{4,A_2} = 0 \,. \tag{74}$$

Now, we need to make equations (63)–(74) self-consistent. In this order, at first we put [37]

$$u_{,x} + v_{,y} = -\frac{\left(1 + u^2 + v^2\right)^2}{2\lambda_{00}} G_{3,u} + A_1 v - A_2 u \,, \tag{75}$$

$$u_{,y} - v_{,x} = \frac{\left(1 + u^2 + v^2\right)^2}{2\lambda_{00}} G_{3,v} + A_1 u + A_2 v \,, \tag{76}$$

$$u_{,x} v_{,y} - u_{,y} v_{,x} - A_1 \left(u u_{,y} + v v_{,y}\right) + A_2 \left(u u_{,x} + v v_{,x}\right) = -\frac{\lambda_4}{2\lambda_{11}} \left(1 + u^2 + v^2\right)^4 F_2' \,, \tag{77}$$

$$A_{2,x} - A_{1,y} = -\frac{1}{2\lambda_2} \left(\frac{\lambda_3}{2} F_1 + \frac{\lambda_4}{2} F_2 + G_{3,A_2} \right) , \tag{78}$$

$$F_1' = \frac{2\lambda_{00}}{\lambda_3 \left(1 + u^2 + v^2\right)^2} \,, \qquad G_{3,uA_1} = 0 \,, \qquad G_{4,uA_2} = 0 \,, \tag{79}$$

where F_1' denotes the derivative of the function F_1, with respect to its argument: $1 + u^2 + v^2$ and F_2' denotes the derivative of the function F_2, with respect to its argument: $u^2 + v^2$.

Then it has turned out that

$$G_{3,u} = G_{4,v} \,, \qquad G_{3,v} = -G_{4,u} \,, \qquad G_{4,A_1} = -G_{3,A_2} \,. \tag{80}$$

Hence, from (71) and (74)

$$G_3 = f(u,v) + c_2 A_2 \,, \quad G_4 = f(u,v) - c_2 A_1 \,, \quad c_2 = \text{const}, \quad f_{,uu} + f_{,vv} = 0 \tag{81}$$

and equations (67)–(74) become the tautologies.

Equations (65)–(66), after taking into account (71)–(74), (75)–(79), (81) and the fact that the potential V should be a Lorentzian scalar, implicate that $V_{,A_k} = 0, \; (k = 1, 2)$. Hence, after eliminating all expressions including the derivatives of the fields u, v, A_1, A_2, from equations (63)–(66), by using (75)–(79) (after taking into account (81)), we obtain the system of the partial differential equations for $V(u,v)$ and $f(u,v)$. The solutions of this system are: $f(u,v) = \mathrm{const}$ and the condition for the potential [37],

$$
\begin{aligned}
V(u,v) = {} & \int \frac{1}{\lambda_{11}\lambda_2\left(1+u^2+v^2\right)^3}\Big(\Big(2\lambda_2\lambda_4^2\left(1+u^2+v^2\right)^6 F_2'^2 \\
& +\lambda_4\Big(\lambda_2\lambda_4\left(1+u^2+v^2\right)^5 F_2'' + \tfrac{1}{2}\left(\tfrac{1}{2}\lambda_4\left(1+u^2+v^2\right)F_2\right. \\
& \left.+c_2\left(1+u^2+v^2\right)-\lambda_{00}\right)\lambda_{11}\Big)\left(1+u^2+v^2\right)^2 F_2' + \left(\tfrac{1}{2}\lambda_4\left(1+u^2+v^2\right)F_2\right. \\
& \left.+c_2\left(1+u^2+v^2\right)-\lambda_{00}\right)\lambda_{00}\lambda_{11}\Big)\,u\Big)\,du \\
& +\int \frac{1}{\lambda_{11}\lambda_2\left(1+u^2+v^2\right)^3}\Big\{\Big[-\frac{1}{2}\left(1+u^2+v^2\right)^3\Big(\int\frac{1}{\left(1+u^2+v^2\right)^4} \\
& \times\Big(4\Big(6\lambda_4^2\Big(\lambda_2 u^4+2\lambda_2\left(1+v^2\right)u^2+2\lambda_2 v^2+\frac{1}{24}\lambda_{11}+\left(1+v^4\right)\lambda_2\Big)\left(1+u^2+v^2\right)^4 F_2'^2 \\
& +\lambda_4\Big(8\lambda_2\lambda_4\left(1+u^2+v^2\right)^5 F_2''+\lambda_2\lambda_4\left(1+u^2+v^2\right)^6 F_2'''+\lambda_{00}\lambda_{11}\Big) \\
& \times\left(1+u^2+v^2\right)^2 F_2' + \lambda_2\lambda_4^2\left(1+u^2+v^2\right)^8 F_2''^2 \\
& +\frac{1}{2}\lambda_{11}\lambda_4\left(1+u^2+v^2\right)^3\Big(\frac{\lambda_4}{2}\left(1+u^2+v^2\right)F_2+c_2\left(1+u^2+v^2\right)-\lambda_{00}\Big)F_2'' \\
& -2\lambda_{00}\lambda_{11}\Big(\frac{\lambda_4}{2}\left(1+u^2+v^2\right)F_2+c_2\left(1+u^2+v^2\right)-\frac{3}{2}\lambda_{00}\Big)\Big)\,u\Big)\,du\Big) \\
& +2\lambda_2\lambda_4^2\left(1+u^2+v^2\right)^6 F_2'^2+\lambda_4\Big(\lambda_2\lambda_4\left(1+u^2+v^2\right)^5 F_2'' \\
& +\frac{\lambda_{11}}{2}\Big(\frac{\lambda_4}{2}\left(1+u^2+v^2\right)F_2+c_2\left(1+u^2+v^2\right)-\lambda_{00}\Big)\Big)\left(1+u^2+v^2\right)^2 F_2' \\
& +\lambda_{00}\lambda_{11}\Big(\frac{\lambda_4}{2}\left(1+u^2+v^2\right)F_2+c_2\left(1+u^2+v^2\right)-\lambda_{00}\Big)\Big]\,v\Big\}\,dv+c_1\,, \qquad (82)
\end{aligned}
$$

where $c_1 = \mathrm{const}, c_2 = \mathrm{const}, F_2 = F_2(u^2+v^2) \in \mathcal{C}^3$ and F_2', F_2'', F_2''' denote the derivatives of the function F_2, with respect to its argument: u^2+v^2.

Hence, the Bogomolny decomposition for gauged full baby Skyrme model in $(2+0)$-dimensions has the form [37]

$$u_{,x}+v_{,y} = A_1 v - A_2 u\,, \qquad (83)$$

$$u_{,y}-v_{,x} = A_1 u + A_2 v\,, \qquad (84)$$

$$u_{,x}v_{,y}-u_{,y}v_{,x}-A_1\left(uu_{,y}+vv_{,y}\right)+A_2\left(uu_{,x}+vv_{,x}\right)=-\frac{\lambda_4}{2\lambda_{11}}\left(1+u^2+v^2\right)^4 F_2'\,, \qquad (85)$$

$$A_{2,x}-A_{1,y} = -\frac{1}{2\lambda_2}\left(\frac{\lambda_4}{2}F_2-\frac{\lambda_{00}}{1+u^2+v^2}+c_2\right)\,, \qquad (86)$$

where $F_2 = F_2(u^2 + v^2)$ and F_2' denotes the derivative of the function F_2, with respect to its argument: $u^2 + v^2$, and the potential $V(u, v)$ needs to satisfy the conditon (82).

3. Summary

We started from finding the most general form of the functions $R_j = R_j(\omega, \omega^*, A_1, A_2)$, $(j = 1, 2)$, in the density of the topological invariant (20), written down in complex field variables, and the most general form of these functions in the density of the topological invariant (an analogon to (20)), written down in real field variables u, v. It has turned out that $R_1 = G_1'$ and $R_2 = G_1$, where $G_1 = G_1(\omega\omega^*)$ (or $G_1 = G_1(u^2 + v^2)$, then the factor $1/2$ appears, by the function $G_1(u^2 + v^2)$, in the density of the corresponding topological invariant). The form of the dependency of the function G_1, on the field variables ω, ω^* (or u, v) and the independence of G_1 on A_k $(k = 1, 2)$ have the influence on the dependency of the potential V on these field variables. As we explained it in Introduction and later, the necessity of gauge-invariance of the Lagrangian is obvious, but we planned to investigate whether the conditions for the potentials in these models, in the case of existing of the Bogomolny decomposition, would permit the dependency of the gauge field A_k, $(k = 1, 2)$ and then whether we would obtain some model similar to the Proca theory [46] (or to the theory of a massive vector field [47]).

Next, we applied the concept of strong necessary conditions for the gauged models in $(2+0)$-dimensions: $O(3)$ "sigma", restricted baby Skyrme and full baby Skyrme model. In result, we obtained the Bogomolny decomposition, *i.e.* the Bogomolny equations, for each of them: (43), (60) and (86), correspondingly, for wide class of the potentials: $V(\omega\omega^*)$ (or $V(u^2+v^2)$). We derived also the conditions (for the potentials) of existence of this Bogomolny decomposition, these conditions have the forms: (42), (61) and (82), correspondingly. In the case of gauged $O(3)$ "sigma" model and gauged restricted baby Skyrme model, obtained results are some generalizations of the results obtained in [22] and [31], [37], correspondingly. Moreover, at the beginning of this paper, we have assumed that for the gauged $O(3)$ "sigma" model and for the gauged baby Skyrme models: restricted and full one, the potentials in their Hamiltonians, depend on $\omega, \omega^*, A_1, A_2$ or u, v, A_1, A_2. Next, it has turned out that the most general forms of the topological invariant for these models, are built on, among others, function G_1 and its derivative with respect to the argument of G_1: $\omega\omega^*$ and $u^2 + v^2$, respectively. Finally, this function and its derivatives have been included into the expression, that V needs to be equal to, if we want to get Bogomolny decomposition. On the other hand, it has turned out that in the case of existence of the Bogomolny decompositions for any of the gauged models, investigated in this paper, the potential V does not depend on $A_k, k = 1, 2$. Hence, in the case of the Bogomolny decompositions for these models, the potentials of them cannot include the expression $A_k A^k$ ($k = 1, 2$), which occurs in the potential in the Proca theory [46] or in the theory of a massive vector field [47].

As it has turned out, the BPS equations (or zero-pressure equations) of BPS baby Skyrme model can be extended to first order equations with a non-zero pressure [49, 50]. Their solutions are topologically nontrivial solutions of Euler–Lagrange equations, however then, the pressure does not vanish — this has been showed for the BPS Skyrme model in [51]. As we think, applying the concept of strong necessary conditions (CSNC) for the case of non-zero pressure configurations would be possible. Obviously, as usual, if we apply CSNC, we need to possess a complete set of the invariants appropriate to the present topology. In the case of the BPS Skyrme model, the homotopy group is $\pi_3(S^3)$ (and the topological charge is here identified with the baryon number [52]). After deriving dual equations, probably, the next steps will be analogical to the ones presented in this paper or in [32], among others, it will be interesting to see what the contributions from the densities of the invariants will be, to the condition for the potential. Another interesting matter is the existence of the solutions of Euler–Lagrange equations and Bogomolny equations of the gauged BPS baby Skyrme model. In [31] and [53], some analytical arguments and numerical exploration have been included, according to which, the gauged BPS baby Skyrme model with a double vacuum potential does not support any baby skyrmions even in the non-BPS sector (*i.e.* solving the full Euler–Lagrange equations). The exploration of the problems of: applying of CSNC in the case of non-zero pressure configurations and existence of the solutions in BPS sector, especially, for some more wide class of the potentials, is in proceed [54].

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