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QUASI-POWER LAW ENSEMBLES

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Quasi-power law ensembles are discussed from the perspective of nonextensive Tsallis distributions characterized by a nonextensive parameter q. A number of possible sources of such distributions are presented in more detail. It is further demonstrated that the data suggest that nonextensive parameters deduced from Tsallis distributions functions $f\left(p_{\mathrm{T}}\right)$, q_1, and from multiplicity distributions (connected with Tsallis entropy), q_2, are not identical and that they are connected via $q_1 + q_2 = 2$. It is also shown that Tsallis distributions can be obtained directly from Shannon information entropy, provided some special constraints are imposed. They are connected with the type of dynamical processes under consideration (additive or multiplicative). Finally, it is shown how a Tsallis distribution can accommodate the log-oscillating behavior apparently seen in some multiparticle data.

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1. Introduction

The two most characteristic ensembles is, on the one hand, the one resulting in exponential (Boltzmann–Gibbs (BG)) distributions, $f_{\mathrm{E}}(X) \sim \exp(-X/T)$, and, on the other hand, the one with power distributions, $f_{\mathrm{P}}(X) \sim X^{-\gamma}$. Both are encountered in the realm of the high energy multiparticle production processes investigated in hadronic and nuclear collisions. They are connected there with, respectively, nonperturbative *soft* dynamics operating at small Xs and described by exponential distributions, $f_{\mathrm{E}}(X)$), and with perturbative *hard* dynamics, responsible for large Xs and described by power distributions, $f_{\rm P}(X)$. These two types of dynamics are investigated separately. It is usually assumed that they operate in distinct parts of phase space of X, separated by $X = X_0$. However, it was found recently that the new high energy data covering the whole of phase space (*cf.*, for example [1–3]) are best fitted by a simple, quasi-power law formula extrapolating between both ensembles [4–6]

$$H(X) = C\left(1+\frac{X}{nX_0}\right)^{-n} \longrightarrow \begin{cases} \exp\left(-\frac{X}{X_0}\right) & \text{for } X\to 0 \\ X^{-n} & \text{for } X\to\infty \end{cases} . \tag{1}$$

This formula coincides with the so-called Tsallis nonextensive distribution [7] for $n = 1/(q-1)$

$$h_q(X) = C_q\left[1-(1-q)\frac{X}{X_0}\right]^{\frac{1}{1-q}} \stackrel{\rm def}{=} C_q \exp_q\left(-\frac{X}{X_0}\right) \stackrel{q\to 1}{\Longrightarrow} C_1\exp\left(-\frac{X}{X_0}\right) . \tag{2}$$

This is the distribution we shall concentrate on and discuss. In Section 2, we shall discuss some examples of processes leading to such distributions. It depends on the nonextensivity parameter q and this can be different depending on whether it arises from a Tsallis distribution (q_1) or from the nonextensive Tsallis entropy (q_2). Both are connected by $q_1 + q_2 = 2$ and this relation seems to be confirmed experimentally. This is presented in Section 4. In Section 3, we shall discuss necessary conditions for obtaining a Tsallis distribution from the Shannon information entropy. Section 5 demonstrates that a Tsallis distribution can also accommodate the log-periodic oscillations apparently observed in high energy data. Our conclusions and a summary are presented in Section 6.

2. Some examples of mechanisms leading to Tsallis distributions

In many practical applications, a Tsallis distribution is derived from Tsallis statistics based on his nonextensive entropy[1]

$$S_q = -\sum p_i \ln_q p_i\,, \qquad \text{where} \qquad \ln_q x = \frac{p^{q-1}-1}{q-1} . \tag{3}$$

On the other hand, there are even more numerous examples of physical situations not based on S_q and still leading to quasi-power distributions in the Tsallis form. In what follows, we shall present some examples of such mechanisms, concentrating on those which allow for an interpretation of the parameter q.

[1] *Cf.* [8] for most recent work with references; thermodynamical consistency of such an approach can be found in [9]. We shall not discuss this point here.

2.1. Superstatistics

The first example is *superstatistics* [10] (*cf.*, also [11, 12]) based on the property that a gamma-like fluctuation of the scale parameter in exponential distribution results in the q-exponential Tsallis distribution with $q > 1$ (*cf.* Eq. (2)). The parameter q defines the strength of such fluctuations, $q = 1 + \mathrm{Var}(X)/\langle X\rangle^2$. From the thermal perspective, it corresponds to a situation in which the heat bath is not homogeneous, but has different temperatures in different parts which are fluctuating around some mean temperature T_0. It must be therefore described by two parameters: a mean temperature T_0 and the mean strength of fluctuations, given by q. As shown in [13], this allows for further generalization to cases where one also has an energy transfer to/from heat bath. The scale T in the Tsallis distribution becomes then q-dependent

$$T = T_{\mathrm{eff}} = T_0 + (q-1)T_V\,. \tag{4}$$

Here, the parameter T_V depends on the type of energy transfer, *cf.* [14, 15] for illustrative examples from, respectively, nuclear collisions and cosmic ray physics.

2.2. Preferential attachment

The second example is the *preferential attachment approach* (used in stochastic networks [16]). Here, the system under consideration exhibits correlations of the preferential attachment type (like, for example, "rich-get-richer" phenomenon in networks) and the scale parameter depends on the variable under consideration. If $x_0 \to x_0'(x) = x_0 + (q-1)x$, then the probability distribution function, $f(x)$, is given by an equation the solution of which is a Tsallis distribution (again, with $q > 1$)

$$\frac{df(x)}{dx} = -\frac{1}{x_0'(x)}f(x) \implies f(x) = \frac{2-q}{x_0}\left[1-(1-q)\frac{x}{x_0}\right]^{\frac{1}{1-q}}\,. \tag{5}$$

For $x_0'(x) = x_0$, one again gets the usual exponential distribution.

2.3. Multiplicative noise

Consider now a *Tsallis distribution from multiplicative noise* [11, 12]. We start from the Langevin equation [12],

$$\frac{dp}{dt} + \gamma(t)p = \xi(t)\,, \tag{6}$$

where $\gamma(t)$ and $\xi(t)$ denote stochastic processes corresponding to, respectively, multiplicative and additive noises. This results in the following Fokker–Planck equation for the distribution function f,

$$\frac{\partial f}{\partial t} = -\frac{\partial\left(K_1 f\right)}{\partial p} + \frac{\partial^2\left(K_2 f\right)}{\partial p^2}\,. \tag{7}$$

Stationary f satisfies

$$\frac{d\left(K_2 f\right)}{dp} = K_1 f \tag{8}$$

with

$$K_1 = \langle\xi\rangle - \langle\gamma\rangle p \qquad \text{and} \qquad K_2 = \mathrm{Var}(\xi) - 2\mathrm{Cov}(\xi,\gamma)p + \mathrm{Var}(\gamma)p^2\,. \tag{9}$$

In the case of no correlation between noises and no drift term due to additive noise (*i.e.*, for $\mathrm{Cov}(\xi,\gamma) = \langle\xi\rangle = 0$ [17]), its solution is a Tsallis distribution for p^2

$$f(p) = \left[1 + (q-1)\frac{p^2}{T}\right]^{\frac{q}{1-q}} \quad \text{with} \quad T = \frac{2\mathrm{Var}(\xi)}{\langle\xi\rangle}; \qquad q = 1 + \frac{2\mathrm{Var}(\gamma)}{\langle\gamma\rangle}\,. \tag{10}$$

However, if we insist on a solution in the form of

$$f(p) = \left[1 + \frac{p}{nT}\right]^n, \qquad n = \frac{1}{q-1}\,, \tag{11}$$

Eq. (8) has to be replaced by

$$K_2(p) = \frac{nT + p}{n}\left[K_1(p) - \frac{dK_1(p)}{dp}\right]. \tag{12}$$

One then gets $f(p)$ in the form of a Tsallis distribution, (11), but with

$$n = 2 + \frac{\langle\gamma\rangle}{\mathrm{Var}(\gamma)} \qquad \text{or} \qquad q = 1 + \frac{\mathrm{Var}(\gamma)}{\langle\gamma\rangle + 2\mathrm{Var}(\gamma)} \tag{13}$$

and with q-dependent T (reminiscent of T_{eff} from Eq. (4) discussed before, *cf.* [18])

$$T(q) = (2-q)\left[T_0 + (q-1)T_1\right] \qquad \text{with} \qquad T_0 = \frac{\mathrm{Cov}(\xi,\gamma)}{\langle\gamma\rangle}\,, \qquad T_1 = \frac{\langle\xi\rangle}{2\langle\gamma\rangle}\,. \tag{14}$$

2.4. All variables fixed

Let us now remember that the usual situation in statistical physics is that out of three variables considered, energy U, multiplicity N and temperature T, two are fixed and one fluctuates. Fluctuations are then given by gamma distributions [19] (in the case of multiplicity distributions where N are integers, they become Poisson distributions) and only in the thermodynamic limit ($N \to \infty$) does one get them in the form of Gaussian distributions, usually discussed in textbooks. In [19], we discussed in detail situations when two or all three variables fluctuate. If all are fixed, we have a distribution of the type of

$$f(E) = (1 - E/U)^{N-2} \,. \tag{15}$$

This is nothing else but a Tsallis distribution with $q = (N-3)/(N-2) < 1$ [2].

2.5. Conditional probability

For the *constrained systems*, one gets $q < 1$. For example, if we have n independent energies, $\{E_{i=1,\dots,N}\}$, then each of them is distributed according to the Boltzman distribution, $g_i(E_i) = (1/\lambda)\exp(-E_i/\lambda)$ (and their sum, $E = \sum_{i=1}^{N} E_i$, is distributed according to gamma distribution, $g_N(E) = 1/[\lambda(N-1)](E/\lambda)^{N-1}\exp(-E/\lambda))$. However, if the available energy is limited, $E = N\alpha = \mathrm{const}$, then the resulting *conditional probability* becomes a Tsallis distribution with $q < 1$ [3]

$$\begin{aligned} f(E_i|E = N\alpha) &= \frac{g_1(E_i)\,g_{N-1}(N\alpha - E_i)}{g_N(N\alpha)} = \frac{(N-1)}{N\alpha}\left(1 - \frac{1}{N}\frac{E_i}{\alpha}\right)^{N-2} \\ &= \frac{2-q}{\lambda}\left[1 - (1-q)\frac{E_i}{\lambda}\right]^{\frac{1}{1-q}} , \end{aligned} \tag{16}$$

$$q = \frac{N-3}{N-2} < 1\,, \qquad \lambda = \frac{\alpha N}{N-1}\,. \tag{17}$$

[2] Actually, such distributions emerge directly from calculus of probability for situation known as *induced partition* [20]. In short: $N-1$ randomly chosen independent points $\{U_1, \dots, U_{N-1}\}$ breaks segment $(0, U)$ into N parts, length of which is distributed according to Eq. (15). The length of the k^{th} such part corresponds to the value of energy $E_k = U_{k+1} - U_k$ (for ordered U_k). One could think of some analogy in physics to the case of random breaks of string in $N-1$ points in the energy space. Notice that induced partition differs from *successive sampling* from the uniform distribution, $E_k \in [0, U - E_1 - E_2 - \dots - E_{k-1}]$, which results in $f(E) = 1/E$ [21].

[3] One could get a Tsallis-like distribution with $q > 1$ only if the scale parameter λ would fluctuate in the same way as in the case of superstatistics, see [22].

2.6. Statistical physics

We end this part by a reminder of how Tsallis distribution with $q<1$ arises from *statistical physics considerations*. Consider an isolated system with energy $U=\mathrm{const}$ and with ν degrees of freedom (n particles). Choose single degree of freedom with energy E (*i.e.*, the remaining, or reservoir, energy is $E_{\mathrm{r}}=U-E$). If this degree of freedom is in a single, well defined state then the number of states of the whole system is $\Omega(U-E)$ and probability that the energy of the chosen degree of freedom is E is $P(E)\propto\Omega(U-E)$. Expanding (slowly varying) $\ln\Omega(E)$ around U,

$$\ln\Omega(U-E)=\sum_{k=0}^{\infty}\frac{1}{k!}\frac{\partial^{(k)}\ln\Omega}{\partial E_{\mathrm{r}}^{(k)}}\,,\qquad\text{with}\qquad\beta=\frac{1}{k_{\mathrm{B}}T}\stackrel{\mathrm{def}}{=}\frac{\partial\ln\Omega\left(E_{\mathrm{r}}\right)}{\partial E_{\mathrm{r}}}\,, \tag{18}$$

and (because $E\ll U$) keeping only the two first terms, one gets

$$\ln P(E)\propto\ln\Omega(E)\propto-\beta E\qquad\text{or}\qquad P(E)\propto\exp(-\beta E)\,, \tag{19}$$

i.e., a Boltzmann distribution (or $q=1$). On the other hand, because one usually expects that $\Omega\left(E_{\mathrm{r}}\right)\propto\left(E_{\mathrm{r}}/\nu\right)^{\alpha_1\nu-\alpha_2}$ (where $\alpha_{1,2}$ are of the order of unity and we put $\alpha_1=1$ and, to account for diminishing the number of states in the reservoir by one, $\alpha_2=2$) [23], one can write

$$\frac{\partial^k\beta}{\partial E_{\mathrm{r}}^k}\propto(-1)^k k!\frac{\nu-2}{E_{\mathrm{r}}^{k+1}}=(-1)^k k!\frac{\beta^{k-1}}{(\nu-2)^k} \tag{20}$$

and write the full series for probability of choosing energy E

$$\begin{aligned}P(E)\;&\propto\;\frac{\Omega(U-E)}{\Omega(U)}=\exp\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\frac{1}{(\nu-2)^k}(-\beta E)^{k+1}\right]\\&=\;C\left(1-\frac{1}{\nu-2}\beta E\right)^{(\nu-2)}\\&=\;\beta(2-q)[1-(1-q)\beta E]^{\frac{1}{1-q}}\,,\end{aligned} \tag{21}$$

where we have used the equality $\ln(1+x)=\sum_{k=0}^{\infty}(-1)^k[x^{k+1}/(k+1)]$. This result, with $q=1-1/(\nu-2)\leq 1$, coincides with results from conditional probability and the induced partition.

2.7. Fluctuations of multiplicity N

Constant values of U, N and T result in Eq. (15) and $q<1$. To get larger values of q, one has to allow for fluctuations of one of the variables: U,

N or T. In superstatistics [10–12], it was T that was fluctuating, let us now consider the example of fluctuating N. It means that, whereas for fixed N (to simplify notation, we changed $N-2$ to N here)

$$f_N(E) = \left(1 - \frac{E}{U}\right)^N \qquad \text{and} \qquad U = \sum E = \text{const}\,, \tag{22}$$

for N fluctuating according to some $P(N)$, the resulting distribution is

$$f(E) = \sum f_N(E)P(N)\,. \tag{23}$$

The most characteristic for our purposes are situations provided by the, respectively, *Binomial Distribution* (BD), Eq. (24), *Poissonian Distribution* (PD), Eq. (25) and by the *Negative Binomial Distributions* (NBD), Eq. (26) (*cf.* [24])

$$P_{\rm BD}(N) = \frac{M!}{N!(M-N)!}\left(\frac{\langle N\rangle}{M}\right)^N \left(1 - \frac{\langle N\rangle}{M}\right)^{M-N}\,, \tag{24}$$

$$P_{\rm PD}(N) = \frac{\langle N\rangle^N}{N!}e^{-\langle N\rangle}\,, \tag{25}$$

$$P_{\rm NBD}(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)}\left(\frac{\langle N\rangle}{k}\right)^N \left(1 + \frac{\langle N\rangle}{k}\right)^{-k-N}\,. \tag{26}$$

They lead, respectively, to Tsallis distribution with $q = 1 - 1/M < 1$, Eq. (27), to exponential Boltzmann distribution, Eq. (28) with $q = 1$, and to Tsallis distribution with $q = 1 + 1/k > 1$, Eq. (29) ($\beta = \langle N\rangle/U$)

$$f_{\rm BD}(E) = \left(1 - \frac{\beta E}{M}\right)^M\,, \tag{27}$$

$$f_{\rm PD}(E) = \exp(-\beta E)\,, \tag{28}$$

$$f_{\rm NBD}(E) = \left(1 + \frac{\beta E}{k}\right)^{-k}\,. \tag{29}$$

Note that in all three cases,

$$q - 1 = \frac{\mathrm{Var}(N)}{\langle N\rangle^2} - \frac{1}{\langle N\rangle}\,. \tag{30}$$

It is natural that for BD, where $\mathrm{Var}(N)/\langle N\rangle < 1$, one has $q < 1$, for PD with $\mathrm{Var}(N)/\langle N\rangle = 1$ also $q = 1$ and for NBD, where $\mathrm{Var}(N)/\langle N\rangle > 1$, one has $q > 1$ (*cf.*, also case $U = \text{const}$ and $T = \text{const}$ considered in [19]). Examples of occurrence $q < 1$ and $q > 1$ are presented in Fig. 1.

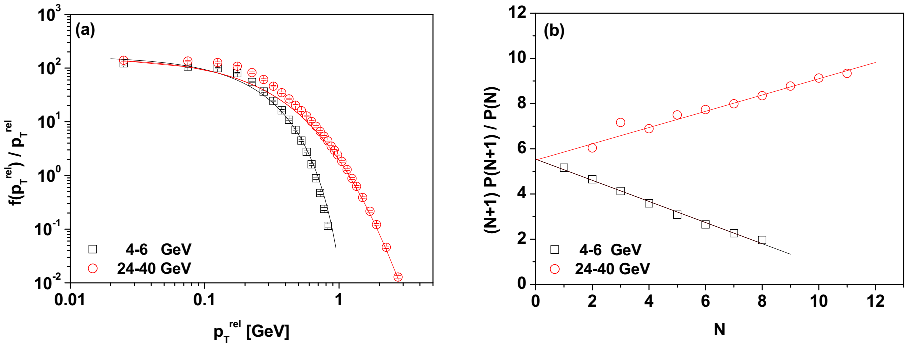

Fig. 1. (Color online) Examples of $q < 1$ or $q > 1$. (a) Distributions of $p_T^{\mathrm{rel}} = |\vec{p} \times \vec{p_{\mathrm{jet}}}| / |\vec{p}_{\mathrm{jet}}|$ for particles inside the jets with different values of $\vec{p}_{\mathrm{jet}}$. Distributions are fitted using a Tsallis distribution (2) with $T = 0.18$. (b) $(N+1)$ $P(N+1)/P(N) = a + bN$ as a function of multiplicity N in jets with different values of $\vec{p}_{\mathrm{jet}}$ as presented in (a). Depending on the range of phase space covered, the corresponding values of q are 0.885 for $\vec{p}_{\mathrm{jet}} \in (4\text{–}6)$ GeV and 1.094 for $\vec{p}_{\mathrm{jet}} \in (24\text{–}40)$ GeV. Data are from [3, 27] and from the Durham HepData Project; `http://hepdata.cedar.ac.uk/view/irn9136932`

Fluctuations of N can be translated into fluctuations of T. Notice first that [13] (*cf.* also [25]) NBD, $P(N) = \Gamma(N+k)/[\Gamma(N+1)\Gamma(k)]\,\gamma^k(1+\gamma)^{-k-N}$, arises also if in the Poisson multiplicity distribution, $P(N) = \bar{N}^N e^{-\bar{N}}/N!$, one fluctuates the mean multiplicity $\bar{N}$ using gamma distribution $f(\bar{N}) = \gamma^k \bar{N}^{k-1}/\Gamma(k)\, e^{-\gamma\bar{N}}$ with $\gamma = k/\langle\bar{N}\rangle$ [4]. Now, identifying fluctuations of mean $\bar{N}$ with fluctuations of T, one can express the above observation via fluctuations of temperature. Noticing that $\bar{\beta} = \bar{N}/U$ (*i.e.*, that $\langle\bar{N}\rangle = U\langle\bar{\beta}\rangle$ and $\gamma = k/[U\langle\bar{\beta}\rangle]$), one can rewrite $f(\bar{N})$ as

$$\begin{aligned} f\left(\bar{\beta}\right) &= \frac{k}{\langle\bar{\beta}\rangle\,\Gamma(k)}\left(\frac{k\bar{\beta}}{\langle\bar{\beta}\rangle}\right)^{k-1}\exp\left(-\frac{k\bar{\beta}}{\langle\bar{\beta}\rangle}\right) \\ &= \frac{\left(\frac{1}{q-1}\frac{\bar{\beta}}{\langle\bar{\beta}\rangle}\right)^{\frac{1}{q-1}-1}}{(q-1)\,\langle\bar{\beta}\rangle\,\Gamma\left(\frac{1}{q-1}\right)}\exp\left(-\frac{1}{q-1}\frac{\bar{\beta}}{\langle\bar{\beta}\rangle}\right). \end{aligned} \tag{31}$$

And this is just a gamma distribution describing fluctuations of $\beta = 1/T$ discussed in [11].

[4] We have two types of averages here: $\bar{X}$ means average value in a given event, whereas $\langle X\rangle$ denotes averages over events (or ensembles).

There is more to the physical meaning of q. Since $U = \langle N\rangle T$, the heat capacity C can be written as $1/C = dU/dT = \langle N\rangle$. However, because in our case $U = \text{const}$, *i.e.*, $\mathrm{Var}(N)/\langle N\rangle^2 = \mathrm{Var}(\beta)/\langle\beta\rangle^2$ (or $\langle N\rangle \sim \langle\beta\rangle$ and $\mathrm{Var}(N) \sim \mathrm{Var}(\beta)$), the formula $q-1 = \mathrm{Var}(N)/\langle N\rangle^2 - 1/\langle N\rangle$ obtained before coincides with a similar formula obtained in [26]

$$q-1 = \frac{\mathrm{Var}(\beta)}{\langle\beta\rangle^2} - \frac{1}{C} \simeq \frac{\mathrm{Var}(T)}{\langle T\rangle^2} - \frac{1}{C}\,. \tag{32}$$

This can be confronted with $q-1 = \mathrm{Var}(\bar{N})/\langle\bar{N}\rangle^2$ from [13], which, due to the form of $f(\bar{\beta})$ in Eq. (31), can be rewritten approximately as

$$q-1 = \frac{\mathrm{Var}\left(\bar{\beta}\right)}{\left\langle\bar{\beta}\right\rangle^2} \simeq \frac{\mathrm{Var}\left(\bar{T}\right)}{\left\langle\bar{T}\right\rangle^2}\,. \tag{33}$$

3. Tsallis distribution from Shannon entropy

Tsallis distribution is usually derived either from the Tsallis entropy (via the MaxEnt variational approach, not discussed here) or from some dynamical considerations, some examples of which are presented in this paper. However, it turns out that it also emerges in a quite natural (in the same MaxEnt approach) way from the Shannon entropy, provided one imposes the right constraints. In fact, as shown in [28], one can establish a transformation between these two variational problems and show that they contain the same information. This means that the two approaches seem to be equivalent, one can either use Tsallis entropy with relatively simple constraints, or the Shannon entropy with rather complicated ones (*cf.*, for example, one can get in this way a list of possible distributions [29]).

In general, Shannon entropy for some probability density $f(x)$, $S = -\int dx f(x)\ln[f(x)]$, supplied with constraint $\langle h(x)\rangle = \int dx f(x)h(x) = \text{const}$, where $h(x)$ is some function of x, subjected to the usual MaxEnt variational procedure, results in the following form of $f(x)$:

$$f(x) = \exp\left[\lambda_0 + \lambda h(x)\right]\,, \tag{34}$$

with constants λ_0 and λ calculated from the normalization of $f(x)$ and from the constraint equation. It is now straightforward to check that

$$\langle z\rangle = z_0 = \frac{q-1}{2-q}\,, \qquad \text{where} \qquad z = \ln\left[1-(1-q)\frac{E}{T_0}\right] \tag{35}$$

results in $f(z) = (1/z_0)\exp\left(-z/z_0\right)$ which translates to (remembering that $f(z)dz = f(E)dE$) a Tsallis distribution

$$f(E) = \frac{1}{(1+z_0)\,T_0}\left(1+\frac{z_0}{1+z_0}\frac{E}{T_0}\right)^{-\frac{1+z_0}{z_0}} = \frac{2-q}{T_0}\left[1-(1-q)\frac{E}{T_0}\right]^{\frac{1}{1-q}}\,. \tag{36}$$

The parameter T_0 can be deduced from the additional condition which must be imposed, namely from the assumed knowledge of the $\langle E\rangle$ (notice that in the case of BG distribution, this would be the only condition).

So far, the physical significance of the constraint (35) is not fully understood. Its form can be deduced from the idea of varying scale parameter in the form of the preferential attachment, Eq. (5), which in present notation means $T \to T(E) = T_0 + (q-1)E$. As shown in (5), it results in the Tsallis distribution (36). This suggests the use of $z = \ln[T(E)/T_0]$ constrained as in Eq. (35). In such an approach, $\ln f(E) = -[1/(q-1)]\ln[T(E)] + [(2-q)/(q-1)]\ln(T_0)$ and, because $S = -\langle \ln f(E)\rangle$, therefore $S = 1/(2-q) + \ln(T_0)$ for the Tsallis distribution becoming $S = 1 + \ln(T_0)$ for the Boltzmann–Gibbs (BG) distribution ($q = 1$).

It is interesting that the constraint (35) seems to be natural for multiplicative noise described by the Langevine equation: $dp/dt + \gamma(t)p = \xi(t)$, with traditional multiplicative noise $\gamma(t)$ and additive noise (stochastic processes) $\xi(t)$) (see [18] for details). In fact, there is a connection between the kind of noise in this process and the condition imposed in the MaxEnt approach. For processes described by an additive noise, $dx/dt = \xi(t)$, the natural condition is that imposed on the arithmetic mean, $\langle x\rangle = c + \langle\xi\rangle t$, and it results in the exponential distributions. For the multiplicative noise, $dx/dt = x\gamma(t)$, the natural condition is that imposed on the geometric mean, $\langle \ln x\rangle = c + \langle\gamma\rangle t$, which results in a power law distribution [30].

4. Tsallis entropy vs. Tsallis distributions ($q_1 + q_2 = 2$)

One has to start with some explanatory remarks. The Tsallis distribution can be also obtained via the MaxEnt procedure from Tsallis entropy,

$$S_q = -\frac{1}{1-q}\sum\left(1 - p_i^q\right) = -\langle \ln_q p_i\rangle_q = -\langle \ln_{2-q} p_i\rangle_{q=1}\,, \tag{37}$$

where $\ln_q x = \left(x^{1-q} - 1\right)/(1-q)$ and $\langle x\rangle_q = \sum p_i^q x_i$. Now, depending on the condition imposed, one gets from S_q

$$\text{either} \qquad f(x) = q[1+(1-q)x]^{-\frac{1}{1-q}} \qquad \text{for} \qquad \langle x\rangle_1 \tag{38}$$

$$\text{or} \qquad f(x) = (2-q)[1+(q-1)x]^{\frac{1}{1-q}} \qquad \text{for} \qquad \langle x\rangle_q\,. \tag{39}$$

However, after replacement of q by $q_1 = 2 - q$, the distribution (38) becomes the usual Tsallis distribution (39). Therefore, one encounters an apparent puzzle, namely the q_1 of Tsallis distribution does not coincides with the q_2 of the corresponding Tsallis entropy, instead they are connected by relation $q_1 + q_2 = 2$. The natural question therefore arises: is such a relation seen in the data? As shown in [31] that seems really be the case, at least quantitatively. This is seen when comparing $q = q_1$ obtained from the data on p_T *distributions* (*cf.* Fig. 2 (a)) to $q = q_2$ obtained from the data on *multiplicities* in p–A collisions assuming that entropy is proportional to the number of particles produced (*cf.* Fig. 2 (b)). Whereas q_1 is deduced from a

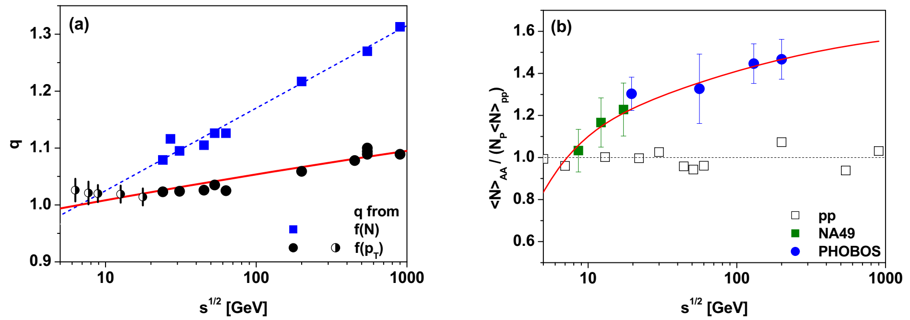

Fig. 2. (Color online) (a) Energy dependencies of the parameters q obtained from, respectively: multiplicity distributions $P(N)$ [33] (squares), from different analysis of transverse momenta distributions $f(p_T)$ in $p+p$ data ([34] — circles, full symbols) and from data on $f(p_T)$ from Pb+Pb collisions ([35] — half filled circles). (b) Energy dependence of the charged multiplicity for nucleus–nucleus collisions divided by the superposition of multiplicities from proton–proton collisions fitted to data on multiplicity taken from [35] (NA49) and from compilation [36].

Tsallis distribution taken in one of the forms discussed above, q_2 is deduced directly from the corresponding entropy S_q of the p–A collision. Assume that such collision can be adequately described by a superposition model in which the main ingredients are ν nucleons which have interacted at least once [32]. Assume further that they are identical and independent and produce n_i secondaries of each other. As a result, $N = \sum_{i=1}^{\nu} n_i$ are produced in one collision and the mean multiplicity is $\langle N\rangle = \langle\nu\rangle\langle n_i\rangle$, where $\langle\nu\rangle$ is the mean number of nucleons participating in the collision and $\langle n_i\rangle$ the mean multiplicity in i^{th} elementary collision. The corresponding entropy $S_q^{(\nu)}$ of such process will then be q-sum of ν entropies $S_q^{(1)}$ of individual collisions and is given by

$$S_q^{(\nu)} = \sum_{k=1}^{\nu} \frac{\nu!}{(\nu-k)!k!}(1-q)^{k-1}\left[S_q^{(1)}\right]^k = \frac{\left[1+(1-q)S_q^{(1)}\right]^{\nu}-1}{1-q}. \tag{40}$$

Notice that $\ln[1+(1-q)S_q^{(\nu)}] = \nu\ln[1+(1-q)S_q^{(1)}]$ and $S_q^{(\nu)} \stackrel{q\to 1}{\longrightarrow} \nu\, S_1^{(1)}$. For $q<1$, entropy $S_q^{(\nu)}$ is nonextensive because $S_q^{(\nu)}/\nu \stackrel{\nu\to\infty}{\longrightarrow} \infty$. For $q>1$, one has $S_q^{(\nu)} \geq 0$ only for $q < 1 + 1/S_q^{(1)}$ and $S_q^{(\nu)}/\nu \stackrel{\nu\to\infty}{\longrightarrow} 0$, *i.e.*, entropy is extensive, $0 \leq S_q^{(\nu)}/\nu \leq S_q^{(1)}$.

As $\langle N_{AA}\rangle > N_P\langle N_{pp}\rangle = \nu\langle N_{pp}\rangle$, the nonextensivity parameter obtained from the corresponding entropies must be smaller than unity, $q_2 < 1$. On the other hand, all estimates of the nonextensivity parameter from Tsallis distributions lead to $q_1 > 1$.

5. Dressed Tsallis distributions (log-periodic oscillations)

The pure power-like distributions are known to be in many cases decorated by specific log-periodic oscillations (*i.e.*, multiplied by some dressing factor R) [37]. They suggest some hierarchical fine structure existing in the system under consideration and are usually regarded as possibly indicating some kind of multifractality in the system. Closer inspection of recent data from the LHC [1–3, 38, 39] reveals that, for large transverse momenta p_{T}, one observes a similar effect, *cf.* Fig. 3. So far, the prevailing opinion is that this is just an apparatus induced artifact with no meaning. However, its persistence in the type of experiment considered, energy or type of collision process (provided that the range of p_{T} covered is large enough) calls for some explanation. Such an explanation was considered in [40, 41]. In [37], the only possibility investigated was to attribute such oscillations to complex values of the power index n in Eq. (1). As shown in [40] and also below, it also works in the case of quasi-power-like Tsallis distributions. However, because one now also has a scale parameter T and a constant term, one can offer another explanation: real n but log-periodically oscillating scale parameter T. This was discussed in detail in [41]. We shall present both possibilities here.

5.1. Complex nonextensivity parameter q

For simple power laws, one has some function $O(x)$ which is scale invariant, $O(\lambda x) = \mu O(x)$ and $O(x) = Cx^{-m}$ with $m = -\ln\mu/\ln\lambda$. However, this can be written as $\mu\lambda^m = 1 = e^{i2\pi k}$, where k is an arbitrary integer. We then have not a single power m but rather a whole family of complex powers, m_k, with $m_k = -\ln\mu/\ln\lambda + i2\pi k/\ln\lambda$. Their imaginary part signals a hierarchy of scales leading to log-periodic oscillations. This means that, in fact, $O(x) = \sum_{k=0} w_k \mathrm{Re}\left(x^{-m_k}\right) = x^{-\mathrm{Re}(m_k)}\sum_{k=0} w_k \cos\left[\mathrm{Im}\left(m_k\right)\ln(x)\right]$ (where w_k are coefficients of the expansion). This is the origin of the usual dressing factor appearing in [37] and used to describe data

$$R(E) = a + b\cos\left[c\ln(E+d) + f\right] \tag{41}$$

(only w_1 and w_2 terms are kept).

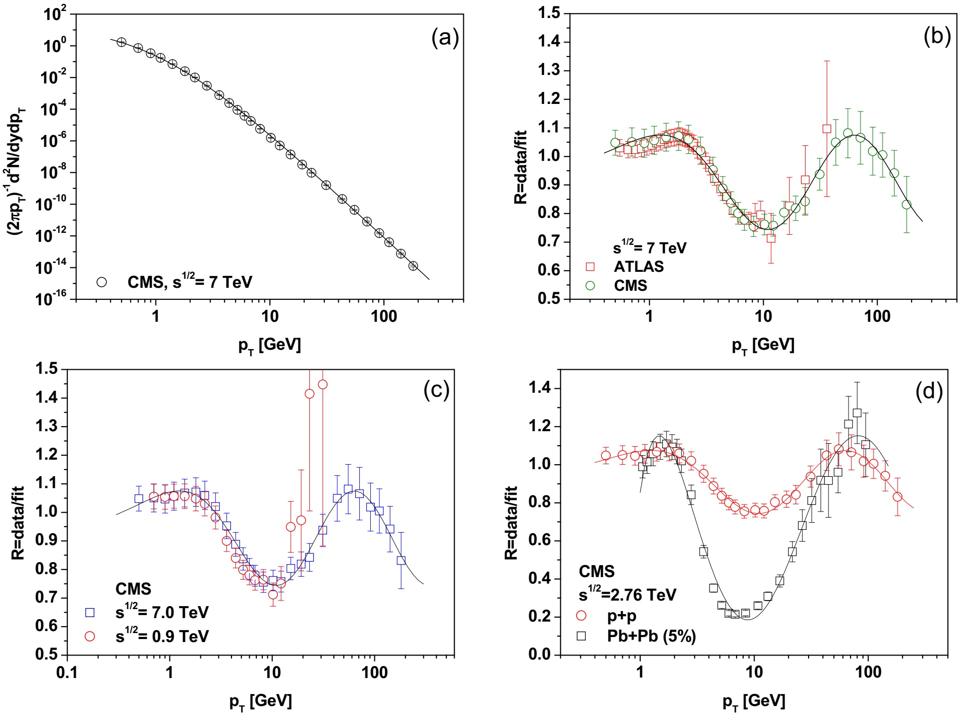

Fig. 3. (Color online) Examples of log-periodic oscillations. (a) $dN/dp_{\rm T}$ for the highest energy 7 TeV, the Tsallis behavior is evident. Only CMS data are shown [1], others behave essentially in the identical manner. (b) Log-periodic oscillations showing up in different experimental data like [1] or ATLAS [3] taken at 7 TeV. (c) Results from CMS [1] for different energies. (d) Results for different systems ($p+p$ collisions compared with Pb+Pb taken for 5% centrality [38]). Results from ALICE are very similar [39].

It turns out that a similar scaling solution can also be obtained in the case of a Tsallis quasi-power-slike distribution. To this end, one must start from the stochastic network approach, Section 2.2 and Eq. (5), in which the Tsallis distribution is obtained by introducing a scale parameter depending on the variable considered. In our case, it is $df(E)/dE = -f(E)/T(E)$ resulting in

$$f(E) = \frac{n-1}{nT_0}\left(1+\frac{E}{nT_0}\right)^{-n} \qquad \text{for} \qquad T(E) = T_0 + \frac{E}{n}\,. \tag{42}$$

In final difference form (with change in notation: T_0 replaced by T)

$$\frac{df(E)}{dE} = -\frac{f(E)}{T(E)} \quad \Longrightarrow \quad f(E+\delta E) = \frac{-n\delta E + nT + E}{nT+E} f(E)\,. \tag{43}$$

We consider a situation in which $\delta E = \alpha n T(E) = \alpha(nT + E)$. It depends now on the new scale parameter α ($\alpha < 1/n$ in order to keep changes of δE to be of the order of T) and can be very small but always remains finite. It can now be shown that $f[E + \alpha(nT + E)] = (1 - \alpha n)f(E)$ which, when expressed in the new variable $x = 1 + E/(nT)$, corresponds formally to the following scale invariant relation:

$$g[(1+\alpha)x] = (1-\alpha n)g(x)\,. \tag{44}$$

Following the same procedure used to obtain dressed solutions of scale invariant functions discussed at the beginning of this section, one arrives at the dressed Tsallis distribution (we keep, as before, only the two lowest terms, $k = 0$ and $k = 1$)[5]

$$g(E) \simeq \left(1 + \frac{E}{nT}\right)^{-m_0} \left\{ w_0 + w_1 \cos\left[\frac{2\pi}{\ln(1+\alpha)} \ln\left(1 + \frac{E}{nT}\right)\right]\right\} \tag{45}$$

with $m_0 = -\ln(1-\alpha n)/\ln(1+\alpha) \stackrel{\alpha\to 0}{\longrightarrow} n$. In addition to the scale parameter α, one has two more parameters occurring in the dressing factor R, w_0 and w_1. The other parameters occurring in Eq. (41) are expressed by the original parameters in the following way: $a/b = w_0/w_1$, $c = 2\pi/\ln(1+\alpha)$, $d = nT$ and $f = -2\pi \ln(nT)/\ln(1+\alpha) = -c \ln d$. One can, however, consider a more involved evolution process, with κ sequential cascades; in this case, the additional parameter κ changes parameter c in (41), $c \to c' = c/\kappa$. It does not affect the slope parameter m_0 but changes the frequency of oscillations which now decrease as $1/\kappa$. Comparison with data requires $\kappa \sim 22$ (*cf.* [40] for details).

5.2. *Log-periodically oscillating T*

As mentioned before, one can translate a dressed Tsallis distribution into a normal one but with a log-periodically oscillating in p_{T} scale factor T, *cf.* Fig. 4 (a). The formula used there to fit the obtained results resembles that for dressing factor (41)

$$T = \bar{a} + \bar{b}\sin\left[\bar{c}\left(\ln(E + \bar{d}\right) + \bar{f}\,\right]. \tag{46}$$

In fit shown in Fig. 4 (a), parameters (generally energy-dependent) are $\bar{a} = 0.143$, $\bar{b} = 0.0045$, $\bar{c} = 2.0$, $\bar{d} = 2.0$, $\bar{f} = -0.4$.

[5] Notice that in Eq. (45) $n \neq m_0$. However, n and T are both unknown *a priori* parameters. Therefore, for fitting purposes, where we have to use two (and not three) parameter Tsallis distribution, we use $[1 + E/(m_0 T')]^{-m_0}$ with fitting parameters m_0 and T'. In terms of T, n and α we have $T' = nT/m_0 = -nT\ln(1+\alpha)/\ln(1-\alpha n)$.

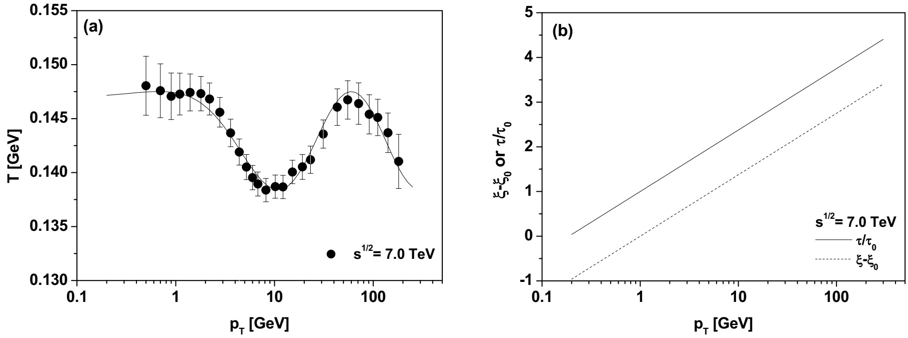

Fig. 4. (Color online) (a) Oscillations of scale parameter T leading to identical dressed Tsallis distribution as shown in Fig. 3 (b), obtained for CMS data at 7 TeV and fitted using Eq. (46). (b) Dependencies of τ/τ_0 from Eq. (54) and $\xi - \xi_0$ from Eq. (51) resulting in oscillations of T shown in panel (a).

To explain Eq. (46), one uses a stochastic equation for the temperature evolution [42] written in the Langevin formulation with energy-dependent noise, $\xi(t,E)$, and allowing for time-dependent $E = E(t)$ [6]

$$\frac{dT}{dE}\frac{dE}{dt} + \frac{1}{\tau}T + \xi(t,E)T = \Phi\,. \tag{47}$$

Assuming now a scenario of *preferential attachment* (*cf.* Section 2.2 above) known from the growth of networks [16], one has

$$\frac{dE}{dt} = \frac{E}{n} + T\,, \tag{48}$$

and Eq. (47) has now the form

$$\left(\frac{E}{n} + T\right)\frac{dT}{dE} + \frac{1}{\tau}T + \xi(t,E)T = \Phi\,. \tag{49}$$

After straightforward manipulations (*cf.* [40] for details), one gets for large E (*i.e.*, neglecting terms $\propto 1/E$)

$$\frac{1}{n}\frac{d^2T}{d(\ln E)^2} + \left[\frac{1}{\tau} + \xi(t,E)\right]\frac{dT}{d(\ln E)} + T\frac{d\xi(t,E)}{d(\ln E)} = 0\,. \tag{50}$$

[6] Notice the change of notation, we discuss formulas for energy E but results are for transverse momenta p_{T} here. However, they are taken at the midrapidity, *i.e.*, for $y \simeq 0$ and for large transverse momenta, $p_{\mathrm{T}} > M$, and in this region, one has $E = \sqrt{M^2 + p_{\mathrm{T}}^2}\cosh(y) \simeq p_{\mathrm{T}}$.

Let us now assume that noise $\xi(t,E)$ increases logarithmically with energy,

$$\xi(t,E) = \xi_0(t) + \frac{\omega^2}{n}\ln E\,. \tag{51}$$

In this case, Eq. (50) becomes an equation for the damped hadronic oscillator with solution in the form of log-periodic oscillation of temperature with frequency ω and depending on initial conditions phase shift parameter ϕ

$$T = C\exp\left\{-n\left[\frac{1}{2\tau} + \frac{\xi(t,E)}{2}\right]\ln E\right\}\sin(\omega\ln E + \phi)\,. \tag{52}$$

Averaging the noise fluctuations over time t and taking into account that the noise term cannot on average change the temperature, $1/\tau + \langle\xi(t,E)\rangle = 0$, one arrives at

$$T = \bar{a} + \frac{b'}{n}\sin(\omega\ln E + \phi)\,. \tag{53}$$

This should now be compared with the parametrization of $T(E)$ given by Eq. (46) and used to fit data in Fig. 4 [7].

We close with the remark that, instead of using energy-dependent noise $\xi(t,E)$ given by Eq. (51) and keeping the relaxation time τ constant, we could equivalently keep the energy-independent white noise, $\xi(t,E) = \xi_0(t)$, but allow for the energy-dependent relaxation time, for example, in the form of

$$\tau = \tau(E) = \frac{n\tau_0}{n + \omega^2\ln E}\,. \tag{54}$$

In this case, the temperature evolution has the form

$$T(t) = \langle T\rangle + [T(t=0) - \langle T\rangle]E^{-t\omega^2/n}\exp\left(-\frac{t}{\tau_0}\right)\,, \tag{55}$$

and T gradually approaches its equilibrium value $\langle T\rangle$. Actually, for $\tau = \tau(E)$, as in our case, this approach towards equilibrium is faster for large E. This is because, in addition to the usual exponential relaxation characteristic for the $\tau = \text{const}$ case, we have an additional factor $\sim E^{-t\omega^2/n}$.

6. Summary and conclusions

We presented examples of possible mechanisms resulting in quasi-power distributions exemplified by the Tsallis distribution, Eq. (2). Our presentation had to be limited, therefore we did not touch thermodynamic connections of this distribution [8, 9] or the possible connection of Tsallis distributions with QCD calculations discussed recently [43, 44].

[7] Notice that only a small amount of T, of the order of $\bar{b}/\bar{a} \sim 3\%$, emerges from the stochastic process with energy-dependent noise; the main contribution comes from the usual energy-independent Gaussian white noise.

The main results presented here can be summarized in the following points:

— Statistical physics consideration, as well as "induced partition process", results in Eq. (15), *i.e.*, in a Tsallis distribution with $q=(N-3)/(N-2) < 1$. Fluctuations of the multiplicity N modify the parameter q which is now equal to $q = 1 + \mathrm{Var}(N)/\langle N\rangle^2 - 1/\langle N\rangle$, *cf.* Eq. (30). Notice that conditional probability for the BG distribution again results in Eq. (15).

— Fluctuations of the multiplicity N are equivalent to results of an application of superstatistics, where the convolution

$$f(E) = \int g(T) \exp\left(-\frac{E}{T}\right) dT \tag{56}$$

becomes a Tsallis distribution, Eq. (2), for

$$g(T) = \frac{1}{\Gamma(n)T}\left(\frac{nT_0}{T}\right)^n \exp\left(-\frac{nT_0}{T}\right) . \tag{57}$$

— Differentiating Eq. (56), one gets

$$\frac{df(E)}{dE} = -\frac{1}{T(E)} f(E) , \qquad \text{where} \qquad T(E) = T_0 + \frac{E}{n} . \tag{58}$$

This is nothing else than a "preferential attachment" case, again resulting in a Tsallis distribution which for $T(E) = T_0$ becomes a BG distribution, *cf.* Eq. (5).

— Replacing in Eq. (58) differentials by finite differences, *cf.* Eq. (43), one gets for $\delta E = \alpha n T(E)$ the scale invariant relation, Eq. (44), which results in log-periodic oscillations in Tsallis distributions[8].

In addition to this line of reasoning, we have also brought in the problem of the apparent duality between the nonextensive parameters obtained from the whole phase space measurements of multiplicity and more local measurements of transverse momenta. This point deserves an experimental and phenomenological scrutiny.

[8] Among numerous other explanations, we can therefore say that we have demonstrated that a Tsallis distribution, which can be regarded as generalization to real power n of such well known distributions as the Snedecor distribution (with $n = (\nu + 2)/2$ with integer ν, for $\nu \to \infty$ it becomes an exponential distribution), can be extended to complex nonextensivity parameter.

Finally, we tentatively suggested that, by choosing the right constraints, which account for additive or multiplicative processes considered, one can also get a Tsallis distribution directly from the Shannon information entropy.

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