

## UNSTABLE FIXED POINTS AND REFORMATION

M. HOWARD LEE

Department of Physics and Astronomy, University of Georgia  
Athens, GA 30602, USA

and

Korea Institute for Advanced Study, Seoul 130-012, Korea

MHLee@uga.edu

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An unstable fixed point of a map cannot be calculated by iteration from almost any initial value. If such a map is concave, it is possible to turn it convex by reforming the map. An unstable fixed point is thereby made stable for iteration. The technique of reformation is presented with examples from the physics and math literature.

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**1. Introduction**

For chaotic maps, fixed points are of special interest [1–3]. They are obtained numerically, almost exclusively by the Fixed-Point Analysis (FPA). It is a powerful yet simple iterative method, with which one can easily and quickly calculate fixed points to any desired degree of accuracy. To calculate them by FPA, their “characters” must be stable (to be defined below). If not, FPA is powerless. A map may contain as many unstable fixed points as stable ones. It is thus desirable to find a way to make FPA applicable also to the unstable classes of fixed points, which is precisely the goal of this work. If our goal could be realized, FPA could also be used to calculate real positive roots of functions and polynomials. This is because roots may be put in the form of fixed points.

The basic idea is to change the character of a fixed point from being unstable to stable but without changing its value. This is accomplished by transformation or substitution, or re-ordering or others, collectively termed reformation. As our work will show, there are no unique ways nor universal algorithms for reformation. Like doing an integral, the reformation will depend on the nature of a map or a function, making this new approach an interesting challenge in itself.

## 2. Fixed-Point Analysis (FPA)

Here we provide a brief review of the conventional version of FPA. Let  $x$  and  $x'$  be a pair of real positive numbers in an interval such as  $(0,1)$ . If  $f$  is a map which takes  $x$  to  $x'$ , we write the process or mapping as

$$x' = f(x). \quad (1)$$

If  $f$  is a nonlinear function of  $x$ , it is possible that, for some value  $x = x^*$  say, it may map on to itself:  $f(x^*) = x^*$ . Such a special value of  $x$  is termed a fixed point of  $f$  and its existence must depend on the nature of  $f$ .

We shall term  $|df(x^*)/dx|$  the character of  $x^*$ . If it is less than 1, the character of  $x^*$  is said to be stable (sometimes attractive). If greater than 1, the character of  $x^*$  is unstable (sometimes repulsive).

If the character of a fixed point is stable, the following iteration process is well established: If  $x^{(1)}$  is an initial value in the interval then,

$$f(x^{(1)}) = x^{(2)}, f(x^{(2)}) = x^{(3)}, \dots, f(x^{(N-1)}) = x^{(N)}, \dots, f(x^{(\infty)}) = x^*. \quad (2)$$

A fixed point is reached by iteration for almost any initial value in the interval. An accurate value of  $x^*$  is easily and quickly obtained this way. For most cases of practical interest, typically  $N = 50$ – $100$  steps of iteration are sufficient.

If a map is given, there is a simple way to determine by graph whether the character of a fixed point is stable or not. Plot (1) and draw a bisector of slope 1 on it. Suppose  $f$  describes concave up with respect to the bisector and it intersects the bisector at two points  $x_1$  and  $x_2$ , ( $x_1 < x_2$ ). The two points are the fixed points of  $f$ .

Since  $x_1$  is a result of  $f$  crossing the bisector from above, its character is usually stable. Since  $x_2$  is a result of  $f$  crossing the bisector from below, its character is always unstable (since the slope must be greater than 1 to be able to intersect the bisector from below).

For  $f$  describing concave up with respect to the bisector,  $x_1$  is obtained by FPA (2), starting from almost any initial value in the interval. But  $x_2$  cannot be obtained. Almost any initial point will move away from it if an iteration is unfolded. Below, we will illustrate these issues through one simple map.

An Ising model on the square lattice [4] yields a map  $x' = f(x)$ ,  $0 < x < \infty$ , where

$$f(x) = \{(x + x^{-1})/2\}^{3/2}. \quad (3)$$

It is concave up (as supposed above) and there are two intersections or two fixed points at  $x_1 = 1$  and  $x_2 \approx 7.6$ . By graph, one can easily determine that

the character of  $x_1$  is stable but that of  $x_2$  unstable. The second fixed point, which is of physical interest, thus cannot be obtained by FPA. An iteration from almost any initial value will move away from it. If one could turn  $f$  into concave down without altering the intersecting points, one might be able to turn  $x_2$  stable making it amenable to FPA. This is what a reformation intends to do, formally described in Sec. 4.

### 3. Enmapping

By definition, the fixed points of  $f$  are the roots of  $\phi(x) = f(x) - x$ . Thus, obtaining the fixed points is formally the same as solving

$$\phi(x) = 0. \quad (4)$$

Now the l.h.s. could represent a function or a polynomial. If it were, by “enmapping” it, *i.e.*, putting it in the form of a map (also see below), one could obtain its real positive roots by FPA if their characters are stable and by FPA after reformation if their characters are unstable. In this way, one could obtain real positive roots of any function or polynomial just as easily and quickly to any desired degree of accuracy just as the fixed points of a map. Below, we illustrate this idea through one simple example.

The integral representation of  $\Gamma(s+1) = \int_0^\infty e^{\phi(x)} dx$ ,  $\text{Re } s > -1$ , where

$$\phi(x) = s \log x - x, \quad (5)$$

which gives  $d\phi(x)/dx = 0$  at  $x = s$ . One can obtain an asymptotic form of  $\Gamma(s+1)$  by Laplace’s method by expanding (5) in the neighborhoods of  $x = s$  [5, 6], which is bounded by  $x_1$  and  $x_2$ , ( $x_1 < x_2$ ), the two real positive roots of  $\phi$ . For the purpose of illustration, let  $s = 3$ . To obtain the roots, we put the r.h.s. of (5) as:  $x = 3 \log x$ , which is enmapped to read:  $x' = f(x)$ ,  $0 < x < \infty$ , where

$$f(x) = 3 \log x. \quad (6)$$

The fixed points of  $f$  are the roots of  $\phi$ . We see that  $f$  plots concave down on the bisector, so that the character of  $x_1$  is unstable but that of  $x_2$  stable. Thus the second root is readily obtained by FPA.

The character of  $x_1$  can be reformed (*i.e.* made stable) if  $f$  is made concave up. To do so, we write  $x = 3 \log x$  as  $x = e^{x/3}s$ , which is enmapped as  $x' = h(x)$ ,  $0 < x < \infty$ , where now

$$h(x) = e^{x/3}. \quad (7)$$

The fixed points of  $h$  are the same fixed points of  $f$ . That is, reforming does not alter the fixed points. But  $h$  is concave up. As a result, the character of  $x_1$  in  $h$  is stable, making it amenable to FPA.

We have illustrated two different ways a function can be enmapped. The more complex is a function, the more ways are there to enmap it.

Going from  $f$  to  $h$  may also be regarded as changing variables from  $x$  to  $t$  by  $x = e^{t/3}$  or more simply  $e^t$ . In the plane of  $x$ ,  $f$  is concave down but in the plane of  $t$ ,  $h$  is concave up. The problem posed by (5) is among the simplest and clearest. A more formal treatment of the reformation is given in Sec. 4.

#### 4. General principle of reformation

The reforming method transforms a map into another without altering the fixed points, but altering their characters. By this method, an unstable fixed point having been made stable, becomes amenable to FPA. The method applies to all fixed points greater than 0.

The basic idea is as follows: The fixed points of  $f(x)$  are the real positive roots of  $f(x) - x = 0$ . The roots are unchanged if it is replaced by *e.g.*  $(f(x))^2 - x^2 = 0$ . The roots are still unchanged if the latter is multiplied by a constant or by any power of  $x$ ,  $x \neq 0$  (both to be denoted by  $\gamma$ ). If this resulting process is expressed as:  $\gamma((f(x))^2 - x^2) = Q(x) - x$ ,  $x \neq 0$ , the roots of  $Q(x) - x$  are also the roots of  $f(x) - x$ . It is possible that the characters of the same roots may be different for  $f$  and for  $Q$ . If this occurs, a fixed point may be unstable for  $f$  but stable for  $Q$ . We shall term this process a reforming operation denoted as

$$\mathbf{R}[f(x) - x] = Q(x) - x. \quad (8)$$

##### 4.1. Procedure

The procedure for the reforming method is described below, applicable to maps and equations. (For equations, it begins immediately with Step 2.)

Step 1: Take a map  $x' = f(x)$ .

Step 2: De-map it into the form of an equation

$$f(x) - x = 0. \quad (9)$$

Since the real positive roots of (9) are the fixed points of  $f(x)$ , de-mapping does not alter the fixed points.

Step 3: Generate a function  $Q(x)$  by reforming (9) by an operation  $\mathbf{R}$ :

$$\mathbf{R}[f(x) - x] = Q(x) - x = 0. \quad (10)$$

The operation  $\mathbf{R}$  on  $f(x) - x = 0$  means any operation which transforms  $f - x = 0$  into  $Q - x = 0$  without altering the roots, with the

possible exception of the root  $x = 0$ . The  $\mathbf{R}$  operation depends on  $f$ , so that there may be a variety of ways to achieving (10), *i.e.*, one  $f$  may yield several independent  $Q$ s. See Sec. 4.2 below.

Step 4: Enmap (10) as

$$x' = Q(x). \quad (11)$$

The fixed points of  $Q$  are still the same fixed points of  $f$  with the possible exception of  $x = 0$ . By reformation, it is possible that the characters of some of the fixed points may have been changed. An unstable fixed point of  $f$  may have turned a stable fixed point of  $Q$ .

#### 4.2. Possible types of $R$ operation

Because the  $\mathbf{R}$  operation depends on the nature of  $f$  itself, there cannot be just one general “algorithm” for it. It is somewhat like doing an integral. As we know, it all depends on the nature of the integrand. Similarly, there are different possibilities for the  $\mathbf{R}$  operation. Since  $Q = x + \mathbf{R}[f - x]$ , we want to express the  $\mathbf{R}$  operation in the form

$$\mathbf{R}[f(x) - x] = -x + \dots \text{ (remainder)} \quad (12)$$

by a suitable choice of  $\gamma$ , so that  $Q$  is given by the remainder. Listed below are  $\mathbf{R}$  operations for three different general forms of  $f$  by which  $Q$ s are generated in this way:

Type-1.  $f$  raised to power

$$\mathbf{R}[f - x] = \gamma (f^k - x^k), \quad (13)$$

$k$  is an integer or rational number, where  $\gamma$  is a constant or a simple power of  $x$ , chosen to yield the r.h.s. of (12). It is called a reforming prefactor or simply a reformer. If  $\gamma$  is a power of  $x$ , the  $\mathbf{R}$  operation requires  $x \neq 0$ . If a fixed point happens to be  $x = 0$ , it cannot be reformed by this reformer.

Type-2.  $f$  in a new variable by  $x = u(y)$

$$\mathbf{R}[f(x) - x] = \gamma(f(u(y)) - u(y)) \quad (14)$$

$$= -y + \dots \text{ (remainder)}, \quad (15)$$

where the remainder is  $Q(y)$ . Here,  $\gamma$ , the reformer is a constant or a simple power of  $y$ ,  $y \neq 0$ .

Type-3.  $f(x) - x = p(x)$ ,  $p(x)$  a polynomial

$$\mathbf{R}[p(x)] = \gamma p(x), \quad (16)$$

where  $\gamma$  is a constant or a simple power of  $x$  ( $x \neq 0$ ).

In Sec. 5, the reforming operation is illustrated for different types of  $f$ . As stated above, there are several different ways to achieving the  $\mathbf{R}$  operation even for one form of  $f$ . They will be denoted  $Q_k(x)$ ,  $k = 1, 2, \dots$  to distinguish them.

## 5. Illustrations of Type-1

### 5.1. Ising map [4]

Consider  $x' = f(x)$ , where

$$f(x) = \{(x + x^{-1})/2\}^{3/2}. \quad (17)$$

There are two fixed points  $x_1 = 1$  and  $x_2 \approx 7.6$ , of which  $x_1$  is stable and  $x_2$  unstable. Thus  $x_2$ , which is of physical interest, is not amenable to FPA. Our goal is to reform the map (17) into another, in which  $x_2$  is stable.

If we choose  $k = 2/3$  (see Eq. (13)),

$$\mathbf{R}[f(x) - x] = \gamma \left( (x + x^{-1})/2 - x^{2/3} \right). \quad (18)$$

If  $\gamma = -2$ , r.h.s. of (18)  $= -x - x^{-1} + 2x^{2/3}$ . Hence,

$$Q_1(x) = 2x^{2/3} - x^{-1}. \quad (19)$$

Thereby,  $x' = f(x)$  has been re-mapped to  $x' = Q_1(x)$ . The fixed points of  $Q_1$  are  $x_1$  and  $x_2$ , the same fixed points of  $f$ , but in  $Q_1$  their characters have changed as can be verified graphically. The re-mapping has turned  $x_1$  and  $x_2$  unstable and stable, respectively. As a result, it is now possible to obtain the value of  $x_2$  accurately by FPA.

If (18) is considered Type-2, we can change  $x$  to  $y$  by  $x = y^3$ . If  $\gamma = -y^{-2}$ ,  $y \neq 0$ , r.h.s. of (18)  $= -y - y^{-5} + 2$ . Hence,

$$Q_2(y) = 2 - y^{-5}. \quad (20)$$

If the above is re-mapped as  $y' = Q_2(y)$ , there are two fixed points  $y_1$  and  $y_2$ , corresponding to  $x_1$  and  $x_2$ , respectively, by  $x = y^3$ . One can verify graphically that the character of  $y_2$  is also stable. Evidently, this map

$y' = Q_2(y)$  is simpler than the previous reformed map  $x' = Q_1(x)$ . By FPA, we obtain:

$$\begin{aligned} y_2 &= 1.965948236645490 \left( y^{(1)} = 1.1; N = 20 \right), \\ x_2 &= 7.598296491. \end{aligned}$$

A comparison of the slope at the fixed point is revealing:

$$\begin{aligned} df(x_2)/dx &= 1.448922355, \\ dQ_1(x_2)/dx &= 0.695534639, \\ dQ_2(y_2)/dy &= 0.0860391642. \end{aligned}$$

If  $\gamma = -2x^2$ ,  $x \neq 0$ , r.h.s. of (18)  $= -x - x^3 + 2x^{8/3}$ . Thus,

$$Q_3(x) = 2x^{8/3} - x^3. \quad (21)$$

Now  $x' = f(x)$  is re-mapped to  $x' = Q_3(x)$ . The fixed points are still the same. But this reforming operation does not achieve the desired goal. As one can verify graphically, both fixed points are now unstable:

$$\begin{aligned} dQ_3(x_1)/dx &= 7/3, \\ dQ_3(x_2)/dx &= -16.57803652. \end{aligned}$$

The above failure indicates that the reforming operation is not to be done routinely but with judicious care.

### 5.2. Chaotic map [7]

We next consider  $x' = f(x)$ , where

$$f(x) = \sin^2 ax, \quad 0 \leq x \leq 1, \quad a = \pi/2. \quad (22)$$

There are 3 fixed points:  $x_1 = 0$ ,  $x_2 = 1/2$ , and  $x_3 = 1$ , of which  $x_1$  and  $x_3$  are stable and  $x_2$  unstable. Since the fixed points are known, FPA is not needed. But we consider this map to further illustrate the workings of the reforming operation. If  $k = 1/2$ ,

$$R[f - x] = \gamma \left( \sin ax - x^{1/2} \right). \quad (23)$$

By expressing  $\sin ax = ax - ax + \sin ax$  and  $\gamma = -1/a$ , r.h.s. of (23)  $= -x + 1/ax^{1/2} + x - a^{-1} \sin ax$ . Hence,

$$Q_1(x) = a^{-1}x^{1/2} + x - a^{-1} \sin ax. \quad (24)$$

Evidently, the fixed points of  $Q_1(x)$  are the same fixed points of  $f(x) : x_1 = 0, x_2 = 1/2$  and  $x_3 = 1$ . By (24),

$$dQ_1(x)/dx = 1 + (1/2a)x^{-1/2} - \cos ax, \quad (25)$$

yielding:

$$\begin{aligned} dQ_1(0)/dx &= \text{infinity}, \\ dQ_1(1/2)/dx &= 0.74305133, \\ dQ_1(1)/dx &= 1.318309886. \end{aligned}$$

In  $Q_1(x)$ ,  $x_2$  has been made stable.

Another possible **R** operation is to write  $f = \sin^2 ax$  as  $1/2(1 - \cos 2ax)$ ,  $a = \pi/2$ . If  $k = 1$  now and by expressing  $\cos 2ax = 2(ax)^2 - 2(ax)^4 + \cos 2ax$

$$\mathbf{R}[f(x) - x] = \gamma (1/2 - (ax)^2 + (ax)^4 - 1/2 \cos 2ax - x). \quad (26)$$

If  $\gamma = -a^{-2}x^{-1}$ ,  $x \neq 0$ , r.h.s. of (26)  $= -x + 1/a^2 + 1/2a^2(\cos 2ax - 1)/x + x$ . Hence,

$$Q_2(x) = 1/a^2 + 1/2a^2(\cos 2ax - 1)/x + x, \quad x > 0. \quad (27)$$

The fixed points  $x_2 = 1/2$  and  $x_3 = 1$  are unaltered in  $Q_2(x)$ . But observe that  $Q_2(0) = 1/a^2 \neq 0$ , i.e.,  $x_1 = 0$  is not a fixed point of  $Q_2(x)$ . A particular form for  $\gamma$  excludes  $x = 0$ . As noted in Sec. 4, the reforming method keeps all the fixed points invariant, fixed point zero possibly excepted.

By (27),

$$dQ_2(x)/dx = 1 - 1/a^2(\cos 2ax - 1)/x^2 - 1/a(\sin 2ax)/x, \quad (28)$$

yielding:

$$\begin{aligned} dQ_2(1/2)/dx &= 0.537329924, \\ dQ_2(1)/dx &= 1.405284735. \end{aligned}$$

The second reforming operation has turned the fixed point  $x_2 = 1/2$  also stable.

## 6. Illustration of Type-2

For this type, we return to (5) and (6) with  $s = 3$ . Of the two fixed points,  $x_2$  is stable,  $f$  being concave down. Hence by FPA

$$\begin{aligned} x_2 &= 4.536403653 \left( x^{(1)} = 5; N = 51 \right), \\ df(x_1)/dx &= 0.661316811. \end{aligned} \quad (29)$$



To obtain  $x_1$ , we turn to the reforming method by substitution  $x = u(t)$ . If  $u(t) = e^t$ ,

$$\mathbf{R}[f(x) - x] = \gamma(f(u(t)) - u(t)) = \gamma(3t - e^t). \quad (30)$$

If  $\gamma = -1/3$ , the r.h.s. is  $-t + e^t/3$ . Thus,

$$Q(t) = e^t/3. \quad (31)$$

There are two fixed points  $t_1$  and  $t_2$  corresponding to  $x_1$  and  $x_2$ , related by  $t = \log x$ . The reformed function  $Q(t)$  is concave up in  $t$  with respect to the bisector and now  $t_1$  is stable. By FPA on  $Q(t)$ ,

$$\begin{aligned} t_1 &= 0.619061287 \left( x^{(1)} = 1; N = 45 \right), \\ dQ(t_1)/dt &= 0.619061287 \text{ (since } dQ/dt = Q \text{)}. \end{aligned} \quad (32)$$

By  $x_1 = e^{t_1}$ ,

$$\begin{aligned} x_1 &= 1.857183961, \\ df(x_1)/dx &= 1.61534895. \end{aligned} \quad (33)$$

We have thus obtained the root  $x_1$  by applying the reforming method to the original function  $f$  for which  $x_1$  was an unstable fixed point.

## 7. Illustrations of Type-3

If  $p(x) = 0$ , where  $p(x)$  is a polynomial in  $x$ , real positive roots can be obtained by FPA following (16):

$$Q(x) = x + \gamma p(x). \quad (34)$$

We shall illustrate the analysis by considering a polynomial from the logistic map  $x' = f(x)$ ,  $f(x) = ax(1 - x)$ ,  $x = (0, 1)$  and  $1 < a \leq 4$ , where  $a$  is a parameter [1–3].

A superstable point in the logistics map occurs at  $x = 1/2$  [8]. One is to determine the value(s) of  $a$  where it occurs. One defines 4-cycle by  $f^4(x) - x = 0$ , where  $f^4(x) = f(f(f(f(x))))$ . If  $x = 1/2$  for  $f$  defined by the logistic map, it results in the following polynomial:

$$P(t) = t^6 - 18t^5 + 123t^4 - 524t^3 + 1511t^2 - 1858t + 4861, \quad (35)$$

where  $t = (a - 1)^2$ . There are 6 roots. The roots must be complex conjugate pairs or real positive or some combinations like 2 pairs of complex conjugates and two real positive roots. Evidently,  $t = 0$  is not a root.

Following (34) but in variable  $t$ , if  $\gamma = -t^{-5}$ ,  $t \neq 0$ ,

$$Q_1(t) = 18 - 123t^{-1} + 524t^{-2} - 1511t^{-3} + 1858t^{-4} - 4861t^{-5}. \quad (36)$$

The map  $t' = Q_1(t)$  intersects points at  $t_1$  and  $t_2$  ( $t_1 < t_2$ ). Evidently, 4 other roots are two pairs of complex conjugates. As  $Q_1$  is concave down,  $t_1$  is unstable and  $t_2$  stable. By FPA, we obtain:

$$\begin{aligned} t_2 &= 8.76319922611 \left( t^{(1)} = 12; N = 49 \right), \\ a_2 &= 3.960270127. \end{aligned}$$

To obtain  $t_1$ , we reform it by taking  $\gamma = (1/18)t^4$  in (34):

$$Q_2(t) = 1/18 [t^2 + 123 - 524t^{-1} + 1511t^{-2} - 1858t^{-3} + 4861t^{-4}]. \quad (37)$$

The map  $t' = Q_2(t)$  is concave up, where there are observed the same two fixed points  $t_1$  and  $t_2$ . But now  $t_1$  is stable and  $t_2$  unstable. The new map has successfully reformed  $t_1$ . By FPA,

$$\begin{aligned} t_1 &= 6.2428105653 \left( t^{(1)} = 2.8; N = 74 \right), \\ a_1 &= 3.4985616999. \end{aligned}$$

The above  $a_1$  value is probably the most accurately determined parametric value for the superstable 4-cycle in the bifurcation domain. The  $a_2$  value comes as a surprise as it places another superstable 4-cycle in the thick of the chaotic domain. According to the theorems due to Sharkowskii [9] and Li and Yorke [10], the domain of chaos for 1d unimodular maps begins where a 3-cycle exists. For the logistic map, the onset value of 3-cycle is  $a = 1 + \sqrt{8} = 3.8284271 \dots$  [11].

## 8. Concluding remarks

We have demonstrated that unstable fixed points of a map can also be calculated by iteration if reformed. A geometric interpretation of reformation is to transform a map which if concave to convex. A formal process for achieving it is presented. The underlying principle holds that the fixed point is fundamental but not its character, hence reformable. We have shown that the idea may be used to obtain real positive roots of any function or polynomial.

There are other numerical techniques for obtaining roots. Perhaps the best known is Newton's method [12, 13]. A root of  $\phi(x)$  is successively approximated as if iterated, the  $n^{\text{th}}$  approximation being given as

$$x_{n+1} = x_n - \phi(x_n)/(d\phi(x_n)/dx), \quad n = 1, 2, \dots \quad (38)$$

An iteration by Newton's method depends on the derivative, thus one more set of calculations than FPA. In addition, it requires that, in the vicinity of a root, the slope not vanish, its sign not change. There are no inflection points. FPA is not burdened by such constraints. For these reasons, stable fixed points have almost always been calculated by FPA.

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