TIME EVOLUTION IN A TWO-DIMENSIONAL ULTRARELATIVISTIC-LIKE ELECTRON GAS BY RECURRENCE RELATIONS METHOD

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The time evolution of the density fluctuation of a two-dimensional high-density ultrarelativistic-like electron gas is studied at the long wavelength and zero temperature limits. The model we consider is a reduced version of the relativistic Sawada model within the massless Dirac particles frame. Time correlation functions are exactly calculated through the recurrence relations method, and a dynamic equivalence between the ultrarelativistic-like and the nonrelativistic dense electron gas systems is stated by the present approach.

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1. Introduction

The study of the time evolution of perturbed systems constitutes a branch of nonequilibrium statistical mechanics in which the time correlation function (TCF) plays a central role [1]. The TCF is an important physical quantity since it can be directly related to the scattering cross section by the Van Hove relation [2]. Besides, it describes properly the nonequilibrium behavior of a system, particularly if the deviation from equilibrium is small. The velocity autocorrelation function of a Brownian particle is a classical example, while the density–density response function is a quantum analog. Generally, if $A(t)$ denotes a dynamical variable of interest, one may want to know it to obtain the TCF $\langle A(t), A(0)\rangle$, which demands to solve the Heisenberg equation $\dot{A}(t) = i\,[H, A(t)]$. A well recognized and physically appealing method that attempts to handle this problem directly is the recurrence relations method (RRM), a continued fraction based approach proposed by Lee [3] some years ago. The RRM enables one to solve the Heisenberg equation exactly, and provides a way to straightforwardly calculate TCFs.

In this work, we study the time evolution of the density fluctuation operator of a two-dimensional ultrarelativistic-like dense electron gas (2DREG), at the long wavelength and zero temperature limits. Despite the 2DREG generality, one can consider two-dimensional massless Dirac-like materials such as graphene [4] as a motivation for studying it. The apparatus we are going to employ to solve the equation of motion of the density fluctuation is the well established RRM. This approach has already been successfully applied in the study of the nonrelativistic dense electron gas (2DEG), at the long wavelength and zero temperature limits, in one [5], two [6], three [7], and D dimensions [8]. Applications of the RRM to many-body systems, in general, are also available in the literature (see, *e.g.*, reference [9]).

The rest of the paper is organized as follows. The model and the method are presented in Section 2. Section 3 is devoted to some results and discussion, and in the last section some concluding remarks are made.

2. The model and the recurrence relations method

2.1. The model

Recent progress in the field of plasmonics in graphene [10] and other massless Dirac-like systems [11] has stimulated interest in 2DREG models [12]. At the helm of the nonrelativistic electron gas theory is the Sawada model [13]. This quasiboson approach considers only electron-hole pairs interaction, and its first relativistic generalization is due to Jancovici [14]. According to his work, two kinds of excitations are allowed in relativistic dynamics, the electron–Fermi-hole pair and the electron–Dirac-hole pair, besides transverse polarized photons [14]. We will restrict our study here into a reduced version of the relativistic Sawada model that accounts only for electron–Fermi-hole pair interactions. The Hamiltonian of the 2DREG we shall deal with is

$$
\begin{aligned}
H \;=\; & \sum_{q} \epsilon_q \left[c_q^\dagger c_q - b_q^\dagger b_q \right] + \tfrac{1}{2} V_q \sum_{p\neq p'} \left[b_p c_{p+q} + c^\dagger_{-p-q} b^\dagger_{-p} \right] \\
& \times \sum_{p'} \left[b_{-p'} c_{-p'-q} + c^\dagger_{-p'-q} b^\dagger_{-p'} \right] , \qquad (1)
\end{aligned}
$$

where $V_q = 2\pi e^2/q$ is the Fourier transform of the Coulomb potential, $c_q^\dagger$ and c_q represent particle creation and annihilation operators, $b_q^\dagger$ and b_q are Fermi-hole creation and annihilation operators, $\epsilon_q = v_{\mathrm{F}}|\boldsymbol{q}|$ is the linear energy dispersion (we are considering $\hbar = 1$ from now on), and v_{F} is the Fermi velocity. In equation (1), the sum over $\boldsymbol{p}$ implies the condition $|\boldsymbol{p}| < q_{\mathrm{F}}$ and $|\boldsymbol{p}+\boldsymbol{q}| > q_{\mathrm{F}}$, where q_{F} is the Fermi momentum. Spin and pseudospin degeneracies are taken into account later.

In accordance with the linear response theory [15], the 2DREG will undergo a relaxation process if slightly perturbed by an external field of the form

$$H_{\mathrm{ext}} = \sum_q \rho_q(t) p_q e^{i\omega t}\,, \tag{2}$$

where p_q and ω are the Fourier component and the frequency of the field, respectively, while ρ_q is the electron–Fermi-hole pair density-fluctuation operator

$$\rho_q(t) = e^{iHt}\rho_q e^{-iHt} \qquad \text{and} \qquad \rho_q = \sum_p \left[b_p c_{p+q} + c^{\dagger}_{-p-q} b^{\dagger}_{-p}\right] \tag{3}$$

with $\rho_q(0) = \rho_q$. Its time evolution satisfies the Heisenberg equation of motion

$$\dot{\rho}_q(t) = i\left[H, \rho_q(t)\right]\,, \tag{4}$$

and the RRM approach will be employed in order to solve it exactly.

2.2. The recurrence relations method

Let $\mathcal{L}$ be a d-dimensional realized Hilbert space of $\rho(t)$. The variable (operator) $\rho(t)$ is a vector in this space, and its norm is a constant of motion since the system is assumed to be Hermitian. Hence, $\rho(t)$ evolves on time changing only its direction, delineating a trajectory in $\mathcal{L}$. The trajectory will be closed, and the motion periodic, if the Hilbert space dimensionality d is finite. If d is infinite, the trajectory turns out to be open, and the motion aperiodic. For some time instant $t \geq 0$, we express $\rho(t)$ by an orthogonal expansion

$$\rho(t) = \sum_{\nu=0}^{d-1} f_\nu a_\nu(t)\,, \tag{5}$$

where $\{f_\nu\}$ constitutes a set of orthogonalized basis vectors which spans $\mathcal{L}$, and $\{a_\nu(t)\}$ is a set of real time-dependent projection functions. We assume that $\mathcal{L}$ is realized by the Kubo scalar product (KSP) [16],

$$(X, Y) = \beta^{-1}\int_0^\beta d\lambda \left\langle e^{\lambda H} X e^{-\lambda H} Y^{\dagger}\right\rangle - \langle X\rangle \left\langle Y^{\dagger}\right\rangle\,, \tag{6}$$

for every X, $Y \subset \mathcal{L}$, where $\langle \dots \rangle$ represents the canonical ensemble average, dagger the Hermitian conjugation, and β the inverse temperature (we are considering $k_{\mathrm{B}} = 1$)[1]. The basis vectors $\{f_\nu\}$ must satisfy the *first recurrence relation* (RR1)

$$f_{\nu+1} = \dot{f}_\nu + \Delta_\nu f_{\nu-1}\,, \qquad 0 \le \nu \le d-1\,, \tag{7}$$

such that $\dot{f}_\nu = i[H, f_\nu]$, and

$$\Delta_\nu = \frac{(f_\nu, f_\nu)}{(f_{\nu-1}, f_{\nu-1})}\,, \qquad \nu = 0, 1, 2, \ldots, d-1\,. \tag{8}$$

Quantity (8) is a relative norm, henceforth referred to as ν-recurrant. Since there are only d basis vectors, we set $f_{-1} \equiv 0$ and $\Delta_0 \equiv 1$ to avoid ambiguities. Equation (7) represents a set of d hierarchical difference equations. Once we state the basal vector as $f_0 = \rho$, hence f_1 is calculable and then also is the Δ_1.

The trajectory of interest is governed by the equation of motion (4), and since the basis vectors $\{f_\nu\}$ satisfy it and also the RR1, the autocorrelation functions $\{a_\nu\}$ are ruled out by the *second recurrence relation* (RR2)

$$\Delta_{\nu+1} a_{\nu+1}(t) = -\dot{a}_\nu(t) + a_{\nu-1}(t)\,, \qquad 0 \le \nu \le d-1\,, \tag{9}$$

where $\dot{a}_\nu(t) = da_\nu(t)/dt$, and $a_{\nu-1} \equiv 0$. Analogously to RR1, RR2 is also a hierarchical set of d equations. The basal autocorrelation a_0 is unknown *a priori*, however it is possible to determine it through the analytical theory of continued fractions due to Mori [16]. By taking the Laplace transform of

[1] It is worth mentioning that the KSP (6) has already been used in several situations where $\beta \to \infty$ (see, *e.g.*, [6, 7]). It is a standard linear response relation whose validity at $T = 0$ was established many years ago [2, 17, 18]. To see it clearly, consider the KSP (6) in a more general form [19]

$$(X, Y) = \beta^{-1} \int_0^\beta d\lambda g(\lambda) \left\langle e^{\lambda H} X e^{-\lambda H} Y^\dagger \right\rangle - \langle X \rangle \left\langle Y^\dagger \right\rangle\,,$$

with the weight function, $g(\lambda)$, satisfying the conditions

$$g(\lambda) \ge 0\,, \qquad g(\beta - \lambda) = g(\lambda)\,, \qquad \beta^{-1} \int_0^\beta d\lambda g(\lambda) = 1\,.$$

By choosing $g(\lambda) = \frac{1}{2}\beta[\delta(\lambda) + \delta(\beta - \lambda)]$, the KSP (6) turns into

$$(X, Y) = \tfrac{1}{2} \left\langle XY^\dagger + Y^\dagger X \right\rangle - \langle X \rangle \left\langle Y^\dagger \right\rangle\,,$$

and is readily evaluated at $\beta > 0$ situations, including the important case of $T = 0$.

RR2, regarding that $a_0(t=0)=1$, and $a_\nu(t=0)=0$, $\nu \geq 1$, one has

$$1 = z\tilde{a}_0 + \Delta_1 \tilde{a}_1\,, \tag{10a}$$

$$\tilde{a}_{\nu-1} = z\tilde{a}_\nu + \Delta_{\nu+1}\tilde{a}_{\nu+1}\,. \tag{10b}$$

After some algebraic manipulation of equations (10a)–(10b), we obtain the continued fraction representation of $\tilde{a}_0$,

$$\tilde{a}_0(z) = 1/z + \Delta_1/z + \Delta_2/z + \ldots\,, \tag{11}$$

that can also be written as

$$\tilde{a}_0(z) = \frac{1}{z+\phi(z)}\,, \qquad \phi(z) = \cfrac{\Delta_1}{z + \cfrac{\Delta_2}{z+\Delta_3 \ddots}}\,. \tag{12}$$

If d is finite, the r.h.s. of equation (12) is a polynomial of finite order with a finite number of zeroes, and its Laplace transform corresponds to a periodic function. On the other hand, if d is infinite, the r.h.s. of relation (12) is an infinite continued fraction.

For certain physical models, the trajectory of the dynamical variable in the realized Hilbert space is soft, which means that the sequence of Δs converges. Hence, one can find $\tilde{a}_0(z)$ and its Laplace transform, the relaxation function $a_0(t)$. Therefore, all autocorrelation functions $a_\nu(t)$, $\nu \geq 1$, are provided by the RR2.

The RRM also enables one to construct subspaces of $\mathcal{L}$ in order to study the time evolution of other set of autocorrelation, such as memory functions, in a similar fashion [20].

3. Results and discussion

By following the RRM, we straightforwardly obtained the relative norms for the interacting 2DREG at the long wavelength limit,

$$\Delta_1 = \tfrac{1}{2}q^2 v_{\mathrm{F}}^2 + \Gamma\,, \tag{13a}$$

$$\Delta_\nu = \Delta = \tfrac{1}{4}q^2 v_{\mathrm{F}}^2\,, \qquad \nu \geq 2\,, \tag{13b}$$

where Γ has dimension of squared frequency. For the 2DREG model we have considered, the Hilbert space dimensionality is $d=\infty$, and the infinite sequence of recurrants is convergent. Hence, the relaxation function for the density fluctuation is given by

$$a_0(t) = A_{\mathrm{s}} \sum_{n=0}^{\infty} (-\alpha)^n \left[\partial/\partial(t)\right]^{2n} J_1\left(2\sqrt{\Delta}\,t\right)/2\sqrt{\Delta}\,t + A_{\mathrm{p}} \cos\left(2\sqrt{\Delta/\alpha}\,t\right)\,, \tag{14}$$

where J_1 is the Bessel function of first order, and $\alpha = 4\Delta\left(1-\Delta/\Delta_1\right)/\Delta_1$. The coefficients $A_{\rm s} = 1-(1-\alpha)^{-1/2}$ and $A_{\rm p} = \left[(1-\alpha)^{1/2} - (1-\alpha)\right]/\left(\frac{1}{2}\alpha\right)$ stand for the single particle (electron-hole pair) and the collective (plasma) excitations, respectively. In the limit $\alpha = 1$, which represents no interaction, the relaxation for the ideal system is given by a damping function, *i.e.*, $a_0(t) = J_0(2\sqrt{\Delta}\ t)$. On the other hand, the asymptotic behavior of the interacting system is primarily an oscillatory plasma excitation, *i.e.*,

$$a_0(t) \sim t^{-3/2}\cos\left(2\sqrt{\Delta}\,t - 3\pi/4\right) + A_{\rm p}\cos\left(2\sqrt{\Delta/\alpha}\,t\right)\,, \qquad t\to\infty\,. \quad (15)$$

Within the linear response approach [15], dynamical quantities such as susceptibility, structure factor, and spectral density, can be readily obtained from the relaxation function.

An important discussion we briefly carry out here is concerning the recurrants (13a)–(13b). According to the RRM, the dimensionality $d = (f_1 f_2 \ldots f_{d-1})$ and the shape $\sigma = (\Delta_1\Delta_2\ldots\Delta_{d-1})$ of the Hilbert space are the static properties that characterize the TCFs related to a dynamical variable in a given system towards a relaxational process [21]. Therefore, different systems may belong to the same *dynamical universality class* if they have the same d and σ, *i.e.*, identical relaxation functions (this subject is nicely exemplified in [22]). Some years ago, Lee and Hong [6] have applied the RRM to study the time evolution of the density fluctuation in a two-dimensional nonrelativistic electron gas (2DEG), at the long wavelength and zero temperature limits. The corresponding recurrants they have obtained for the interacting system are

$$\Delta_1 \;=\; 2q^2\epsilon_{\rm F}^2 + \Gamma\,, \quad (16a)$$
$$\Delta_\nu \;=\; \Delta = q^2\epsilon_{\rm F}^2\,, \qquad \nu\geq 2\,, \quad (16b)$$

where $\epsilon_{\rm F} = q_{\rm F}^2/2m$ is the Fermi energy, and Γ is the squared classical plasma frequency. By comparison of expressions (13a)–(13b) and (16a)–(16b), one can see that the recurrants for the corresponding 2DREG and 2DEG have the same structure, besides the Hilbert space dimensionality for both systems is $d=\infty$. Hence, these systems are dynamically equivalent and possess the same TCFs.

4. Concluding remarks

We have studied the time evolution of the density fluctuation operator of a two-dimensional ultrarelativistic-like dense electron gas (2DREG), at the long wavelength and zero temperature limits. The employed machinery was the recurrence relations method (RRM), a well established continued fraction approach that has, so far, been successfully applied to nonrelativistic many-body systems. To the best of our knowledge, this work is the first application of the RRM to relativistic problems. We verified that both the 2DREG and the 2DEG are dynamically equivalent, *i.e.*, they have identical relaxation functions.

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REFERENCES

[1] K.H. Li, *Phys. Rep.* **134**, 1 (1986).
[2] W. Marshall, R.D. Lowde, *Rep. Prog. Phys.* **31**, 705 (1968).
[3] M.H. Lee, *Phys. Rev. Lett.* **49**, 1072 (1982).
[4] K.S. Novoselov *et al.*, *Nature* **438**, 197 (2005).
[5] M.H. Lee, J. Hong, N.L. Sharma, *Phys. Rev. A* **29**, 1561 (1984).
[6] M.H. Lee, J. Hong, *Phys. Rev. Lett.* **48**, 634 (1982); *Phys. Rev. B* **26**, 2227 (1982); *B* **32**, 7734 (1985).
[7] M.H. Lee, J. Hong, *Phys. Rev. B* **30**, 6756 (1984).
[8] N.L. Sharma, *Phys. Rev. B* **45**, 3552 (1992).
[9] M.H. Lee, J. Hong, J. Florencio, *Phys. Scr. T* **19**, 498 (1987).
[10] A.N. Grigorenko, M. Polini, K.S. Novoselov, *Nature Photonics* **6**, 749 (2012).
[11] T. Stauber, *J. Phys.: Condens. Matter* **26**, 123201 (2014).
[12] D.K. Efimkin, Y.E. Lozovik, A.A. Sokolik, *Nanoscale Res. Lett.* **7**, 163 (2012).
[13] K. Sawada, *Phys. Rev.* **106**, 372 (1957); K. Sawada, N. Fukuda, K.A. Brueckner, R. Brout, *Phys. Rev.* **108**, 507 (1957); R. Brout, *Phys. Rev.* **108**, 515 (1957).
[14] B. Jancovici, *Nuovo Cim.* **25**, 428 (1962).
[15] R. Kubo, *Rep. Prog. Phys.* **29**, 255 (1966).
[16] H. Mori, *Prog. Theor. Phys.* **33**, 423 (1965); **34**, 399 (1965).
[17] H. Falk, L.W. Bruch, *Phys. Rev.* **180**, 442 (1969).
[18] F.J. Dyson, E.H. Lieb, B. Simon, *J. Stat. Phys.* **18**, 335 (1978).
[19] V.S. Viswanath, G. Müller, *The Recursion Method — Application to Many-Body Dynamics*, Lecture Notes in Physics, Springer-Verlang, Berlin 1994.
[20] U. Balucani, M.H. Lee, V. Tognetti, *Phys. Rep.* **373**, 409 (2003).
[21] M.H. Lee, J. Florencio, J. Hong, *J. Phys. A: Math. Gen.* **22**, L331 (1989).
[22] S. Sen, *Phys. Rev. B* **53**, 5104 (1996).