ON THE EXISTENCE OF AN UPPER CRITICAL DIMENSION FOR SYSTEMS WITHIN THE KPZ UNIVERSALITY CLASS

Evandro A. Rodrigues, Fernando A. Oliveira†
Bernardo A. Mello

Institute of Physics, University of Brasilia
Caixa Postal 04513, 70919-970 Brasilia-DF, Brazil

(Received March 18, 2015; revised version received May 29, 2015)

In this work, we extend the etching model [B.A. Mello, A. Chaves, F.A. Oliveira, *Phys. Rev. E* **63**, 041113 (2001)] to $d+1$ dimensions. This permits us to investigate its exponents behaviour on higher dimensions, to try to verify the existence of an upper critical dimension for the KPZ equations. Our results show that $d=4$ is not an upper critical dimension for the etching model, and suggest that if an upper critical dimension exists it must larger than six.

DOI:10.5506/APhysPolB.46.1231
PACS numbers: 81.15.Aa, 05.10.–a, 05.40.–a, 68.35.Ja

1. Introduction

Among many equations describing the behaviour of moving surfaces, the KPZ equation [1] is of special interest, partially motivated by the fact that it describes many physical phenomena, such as flame front propagation [2, 3] and deposition of thin films [4]. Although its exponents in one dimension have been long known [5, 6], the renormalization group technique used for obtaining its values is not usable for higher dimensions. As such, either numerical simulations or approximate methods are used to obtain exponents for the KPZ in higher dimensions and even then, the numerical determination of exponents can be tricky [7].

Such methods give no final answer to the values of these exponents, which combined with the absence of exact solutions leads to the much debated possibility of an upper critical dimension (UCD) d_c for the dynamic exponents [8]. For a concise review, see [9].

† faooliveira@gmail.com

One approach consisted of trying to find a general expression for the exponents depending on the substrate dimension d, *i.e.* $\alpha = \alpha(d)$, $\beta = \beta(d)$ and $z = z(d)$. Notable examples are those for the RSOS model, by Kim and Kosterlitz [10], for the Eden model [11] by Kertész and Wolf, the heuristic approach to the strong-coupling regime by Stepanow [12], a tentative method based on quantization of the exponents by Lässig [13], and a perturbation expansion of the KPZ equation by Bouchaud and Cates [14]. Unfortunately, further numerical results have shown these results to be incorrect [15–18].

Analytical methods such as mapping of the directed polymer [19], perturbation expansion [20] and mode-coupling techniques [21] among others [22–25] observe $d_c = 4$. An asymptotic weak noise approach by Fogedby [26] suggests $d_c < 4$. Otherwise, some numerical studies found no such limit [11, 27, 28], as well as the numerical and theoretical results by Scharwartz and Perlsman [9].

We present in this work a version of the etching model by Mello *et al.* [29, 30] extended to work with $d + 1$ spatial dimensions [31]. The one dimensional version of the etching model is known to be compatible with the KPZ equation, and as such, is classified in the KPZ universality class.

Using this version of the model, we determine exponents for surfaces with $1 \leq d \leq 6$, reaching the conclusion that if there is a UCD, it must be such that $d_c > 6$.

2. The etching model in d dimensions

Proposed by Mello *et al.* [29] in 2001, the etching model simulates an one-dimensional crystalline solid submerged in a solvent liquid. Its scaling exponents are very close to those of the KPZ equation, namely $\alpha = 0.4961 \pm 0.0003$ and $\beta = 0.330 \pm 0.001$, and as such, it is believed to belong to the KPZ universality class. This model was object of extensive research in recent years [32–37].

We extend the model to $d+1$ dimensions, considering the solid a square lattice exposed to the solvent liquid, with a removal probability proportional to the exposed area. The algorithm can be described as:

1. at discrete instant T, one horizontal site $i = 1, 2 \ldots, L$ is randomly chosen;
2. $h_i(T + 1) = h_i(T) + 1$;
3. if $h_{i+\delta}(T) < h_i(T)$, do $h_{i+\delta}(T+1) = h_i(T)$, where $\delta = \pm 1$ are the first neighbours.

In the multidimensional version, i and δ are vectors and δ runs over the 2^d first neighbours of the hypercube. We consider L to be the substrate length

in each direction, with the total number of sites L^d. The normalized time t defines the time unity as L^d cellular automata iterations, *i.e.*, $t = T/L^d$. We use periodic boundary conditions on the surface to reduce unwanted finite size length effects. Albeit the model is not a direct mapping of the KPZ equation, it generally mimics its dynamics and reproduces its exponents for $1+1$ as well as the general case $d+1$.

We simulate several substrate lengths L for each dimension d, with each experiment being repeated several times. This ensemble average is necessary to reduce noise, producing higher accuracy in the resulting exponents.

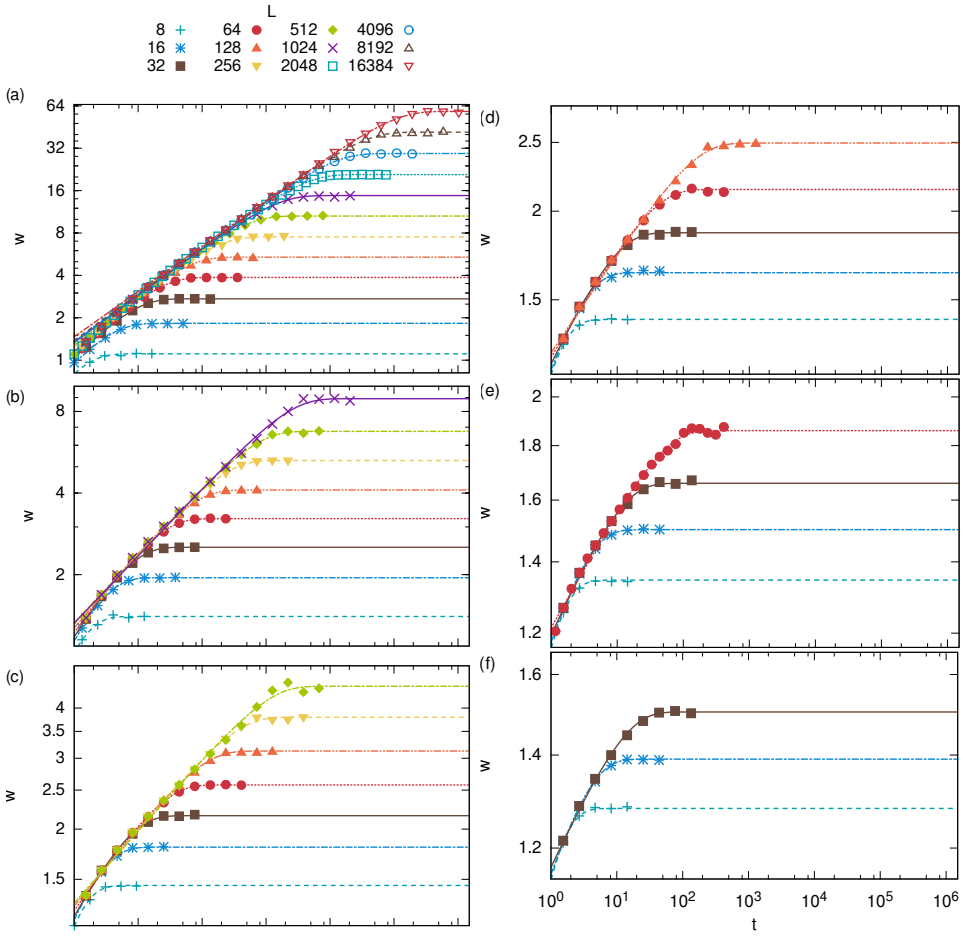


Fig. 1. Roughness $w(L,t)$ from the etching model as a function of time, for surfaces with (a) $d=1$, (b) $d=2$, (c) $d=3$, (d) $d=4$, (e) $d=5$, and (f) $d=6$, showing only data points for $t < 15t_\times$ for clarity. The lines are guides for the eyes.

In Fig. 1, we show our results for simulating roughness dynamics on various substrate lengths for dimensions from $1+1$ to $6+1$, on log–log scale. In all simulated dimensions, the expected Family–Vicsek (FV) scaling [38] is visible.

The Family–Vicsek scaling is a relation that can be used to model surface roughness dynamics by considering it composed of two different regimes: one in which it grows in a power-law-like function of time, and another one where it saturates after a saturation time $t_\times$. The values of the saturation roughness is related to the substrate length, with

$$w_\mathrm{s} \propto L^\alpha \,. \tag{1}$$

These properties are expressed in the FV relation

$$w(t,L) = w_\mathrm{s} f(t/t_\times, \beta) = \begin{cases} w_y t^\beta & \text{if} \quad t \ll t_\times \\ w_\mathrm{s} & \text{if} \quad t \gg t_\times \end{cases} . \tag{2}$$

3. Determination of exponents value

The etching model presents a transient behaviour such that times $t \lesssim 1$ are discarded from the fitting. Using the FV relation, we obtain our exponents by fitting our data to a set of power laws. We fit values of w_y, β_L, and w_s by using the two expressions of (2) at $t \ll t_\times$ and $t \gg t_\times$. Determination of $t_\times$ is made by analysing the intersection between the functions of the aforementioned regimes. This implies in a β that is not independent of L. For this reason, the parameter obtained from the fitting is called β_L and a correction is made in the form

$$\beta_L = \beta \left(1 + \frac{A_0}{L^\gamma}\right) , \tag{3}$$

where $\gamma \approx 1$. This correction considers the real value of β to be the asymptotic of β_L, eliminating finite size effects.

The values of β_L, w_s and $t_\times$ for each value of d and L were obtained from Fig. 1 and plotted in Fig. 2.

4. Dynamic exponents and the UCD

Using the fitting from the data shown in Fig. 2, we obtain the values of each exponent for dimensions ranging from $1+1$ to $6+1$. It allows us to observe how these exponents behave on higher dimensions.

In Table I, we show our results. It is simple to observe that there is no UCD $d_\mathrm{c} = 4$. *I.e.* all exponents continues to change after $d = 4$. This suggests that there is not an UCD $d_\mathrm{c} \leq 6$, in agreement with previous results [9, 11, 27, 28, 39] obtained through other models.

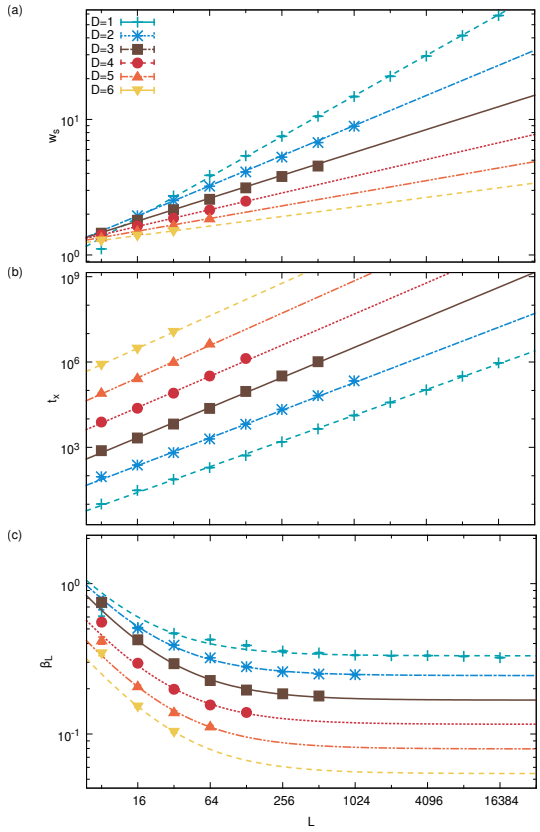


Fig. 2. Parameters of (2) plotted as functions of L with $d = 1 \ldots 6$. (a) $w_{\mathrm{s}}(L)$, (b) $t_{\times}(L)$ multiplied by 10^d for better visualization, and (c) $\beta_L(L)$. In all simulations $L = 2^n$, with n integer.

TABLE I

Dynamic exponents obtained from the fittings of Fig. 2. An evidence of the precision of these exponents is the value of $\alpha + z$, which should be 2.

d	α	β	z	$\alpha + z$
1	0.497(5)	0.331(3)	1.50(8)	2.00(1)
2	0.369(8)	0.244(2)	1.61(5)	1.98(2)
3	0.280(7)	0.168(1)	1.75(9)	2.03(2)
4	0.205(3)	0.116(3)	1.81(3)	2.02(1)
5	0.154(2)	0.079(3)	1.88(6)	2.04(1)
6	0.117(1)	0.054(1)	1.90(6)	2.01(1)

It is important to note that although the one-dimensional etching model is on the KPZ universality class, it is hard to classify it, or for that matter, any multidimensional model, on the KPZ universality class, as there are no known solutions for those cases. We can, however, compare our results with others [31] showing great concordance.

5. Conclusion

We have made a generalized version of the etching model, capable of simulating surfaces on $d+1$ dimensions. Using this version of the model, we were capable of obtaining exponents for systems up to $6+1$ dimensions. Through the obtained exponents, we have shown that the etching model does not show an upper critical dimension at $d_c = 4$. It is not possible, however, to assert that the same thing is true to the KPZ equation, as we still do not have a formal mapping of our model to the KPZ equation. However, comparison with current literature points to the same general direction of both the etching model belonging to the KPZ universality class and the absence of this UCD on the KPZ equation. We expect as well that import results obtained in stochastic process [40–46] could be used to give a more complete solution to this problem.

This work was supported by the Conselho Nacional de Desenvolvimento Cientifico e Tecnologico (CNPQ), the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), the Fundação de Apoio a Pesquisa do Distrito Federal (FAPDF), and the Companhia Nacional de Abastecimento (CONAB).

REFERENCES

[1] M. Kardar, G. Parisi, Y.C. Zhang, *Phys. Rev. Lett.* **56**, 889 (1986).
[2] N. Provatas, T. Ala-Nissila, M. Grant, K.R. Elder, *Phys. Rev. E* **51**, 4232 (1995).
[3] A.T. Campos, T.M. da Rocha Filho, *Physica A* **392**, 3903 (2013).
[4] A.E. Lita, J.E. Sanchez Jr., *Phys. Rev. B* **61**, 7692 (2000).
[5] T. Hwa, E. Frey, *Phys. Rev. A* **44**, 7873 (1991).
[6] E. Frey, U.C. Täuber, T. Hwa, *Phys. Rev. E* **53**, 4424 (1996).
[7] H.S. Wio, R.R. Deza, J.A. Revelli, C. Escudero, *Acta Phys. Pol. B* **44**, 889 (2013).
[8] E. Katzav, M. Schwartz, *Physica A* **309**, 69 (2002).
[9] M. Schwartz, E. Perlsman, *Phys. Rev. E* **85**, 050103 (2012).
[10] J.M. Kim, J.M. Kosterlitz, T. Ala-Nissila, *J. Phys. A* **24**, 5569 (1991).
[11] K. Moser, J. Kertész, D.E. Wolf, *Physica A* **178**, 215 (1991).
[12] S. Stepanow, *Phys. Rev. E* **55**, R4853 (1997).

[13] M. Lässig, *Phys. Rev. Lett.* **80**, 2366 (1998).
[14] J.P. Bouchaud, M.E. Cates, *Phys. Rev. E* **47**, 1455 (1993).
[15] B.M. Forrest, L.H. Tang, *Phys. Rev. Lett.* **64**, 1405 (1990).
[16] L.H. Tang, B.M. Forrest, D.E. Wolf, *Phys. Rev. A* **45**, 7162 (1992).
[17] T. Ala-Nissila, T. Hjelt, J.M. Kosterlitz, O. Venäläinen, *J. Stat. Phys.* **72**, 207 (1993).
[18] T. Ala-Nissila, *Phys. Rev. Lett.* **80**, 887 (1998).
[19] T. Halpin-Healy, *Phys. Rev. Lett.* **62**, 442 (1989).
[20] K.J. Wiese, *J. Stat. Phys.* **93**, 143 (1998).
[21] J.K. Bhattacharjee, *J. Phys. A* **31**, L93 (1998).
[22] M. Lässig, *Nucl. Phys. B* **448**, 559 (1995).
[23] M. Lässig, H. Kinzelbach, *Phys. Rev. Lett.* **78**, 903 (1997).
[24] A.J. McKane, T. Blum, *Phys. Rev. E* **52**, 4741 (1995).
[25] M.A. Moore, T. Blum, J.P. Doherty, M. Marsili, *Phys. Rev. Lett.* **74**, 4257 (1995).
[26] H.C. Fogedby, *Pramana* **71**, 253 (2008).
[27] E. Marinari, A. Pagnani, G. Parisi, *J. Phys. A* **33**, 8181 (2000).
[28] E. Marinari, A. Pagnani, G. Parisi, Z. Rácz, *Phys. Rev. E* **65**, 026136 (2002).
[29] B.A. Mello, A. Chaves, F.A. Oliveira, *Phys. Rev. E* **63**, 041113 (2001).
[30] B.A. Mello, *Physica A* **419**, 762 (2015).
[31] E.A. Rodrigues, B.A. Mello, F.A. Oliveira, *J. Phys. A* **48**, 035001 (2015).
[32] F.D.A. Aarão Reis, *Physica A* **364**, 190 (2006).
[33] T. Paiva, F.D.A. Aarão Reis, *Surf. Sci.* **601**, 419 (2007).
[34] S. Kimiagar, G.R. Jafari, M.R.R. Tabar, *J. Stat. Mech.* **2008**, P02010 (2008).
[35] T.J. Oliveira, F.D.A. Aarão Reis, *Phys. Rev. E* **77**, 041605 (2008).
[36] G. Tang *et al.*, *Physica A* **389**, 4552 (2010).
[37] Z. Xun *et al.*, *J. Stat. Mech.* **2012**, P10014 (2012).
[38] T. Family, F. Vicsek, *J. Phys. A* **18**, L75 (1985).
[39] S.G. Alves, T.J. Oliveira, S.C. Ferreira, *Phys. Rev. E* **90**, 020103 (2014).
[40] L.C. Lapas, I.V.L. Costa, M.H. Vainstein, F.A. Oliveira, *Europhys. Lett.* **77**, 37004 (2007).
[41] R. Morgado, M. Ciesla, L. Longa, F.A. Oliveira, *Europhys. Lett.* **79**, 10002 (2007).
[42] F.L. Forgerini, W. Figueiredo, *Phys. Rev. E* **79**, 041602 (2009).
[43] C.C.Y. Dorea, J.A. Guerra, R. Morgado, A.G.C. Pereira, *Numer. Func. Anal. Opt.* **31**, 164 (2010).
[44] R.M.S. Ferreira *et al.*, *Phys. Rev. E* **86**, 021121 (2012).
[45] R.M.S. Ferreira, L.C. Lapas, F.A. Oliveira, *Acta Phys. Pol. B* **44**, 1085 (2013).
[46] A. Fuliński, *Acta Phys. Pol. B* **44**, 1137 (2013).