REMARKS ON MATHEMATICAL FOUNDATIONS OF QUANTUM MECHANICS*

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We show that the set-theoretic forcing is the essential part of the continuous measurement of a suitably rich Boolean algebra of quantum observables. The Boolean algebra structure of quantum observables enables us to give a classical and geometric meaning to the results of measurements of the observables. The measurement takes place in the semiclassical state of the system which is the generic filter added by a forcing to the ZFC model based on the Borel measure algebra. The analogue of the semiclassical state (the pseudoclassical state) was described by Wesep in 2006 in his studies of the local hidden variables program in quantum mechanics.

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1. Introduction

It is widely accepted paradigm that most of the mathematics required by theories of physics is based on the formal theory of sets, *i.e.* Zermelo–Fraenkel set theory with the axiom of choice (ZFC). Such statement is motivated by the fact that most of mathematics itself can be developed on basis of ZFC. More careful studies of this phenomenon in the particular context of quantum mechanics (QM) were performed by Benioff [1] several years after inventing the method of forcing in set theory (Cohen [2]). The remarkable result obtained by Benioff is that neither any single model of set theory nor any of its generic extensions by forcing, can serve as formal carrier for mathematics of QM along with the statistical predictions of QM. More recently, one of the present authors [3–5] proposed the approach where forcing becomes an important ingredient of the QM formalism and its geometric continuum limit. Moreover, Wesep [6] showed the exceptional role of forcing in the local hidden variables (LHV) program. One of general aims of this paper is to show and explain the following:

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Proposition 1. *Set-theoretic forcing is the essential building block of the formalism of QM.*

Especially, forcing becomes non-trivial for Boolean local contexts where continuum-valued observables (like position of a particle) are considered. For such observables, the geometric classical limit can be naturally approached. Thus, another aim of the paper, which supports Prop. 1, is to show the following:

Proposition 2. *Set-theoretic forcing is the essential ingredient of the local continuum Boolean geometric limit of QM.*

While thinking of the eventual modification of QM formalism towards the agreeing it with (quantum) dynamics of (pseudo)Riemannian geometry in dimension 4, as appearing in general relativity, forcing can be the helpful tool. Facing no sign of supersymmetry nor extra dimensions in up-to-date accelerator experiments, perhaps we need careful reconsideration of foundations of QM also from such unconventional perspectives. In the next section, we will discuss basic features of the method of forcing and present formal and conceptual reasons supporting its applicability in QM. This serves as the partial validation of the Props. 1 and 2. We demonstrate also some formal results relating forcing and QM. We close the paper with the brief summary and discussion of the results.

2. Why forcing in QM?

For the purposes of this paper, forcing is the deriving property of some (generic) ultrafilter on the partial order or Boolean algebra. Closely related to this is the genericity of a propositional algebra (PA). The last was considered by Wesep in the case of LHV in QM [6] and is also important for the results in this paper.

Let $\mathbb{P}$ be a partially ordered set, *i.e.* $(\mathbb{P},<)$ is a partial order (PO). Then $p \in \mathbb{P}$ is called a forcing condition and a condition p is stronger than $q \in \mathbb{P}$ if $p < q$ [7]. A set of conditions $D \subset \mathbb{P}$ is dense in $\mathbb{P}$ whenever, for all $p \in \mathbb{P}$, there exists $d \in D$ such that $d \leqslant p$. $G \subset \mathbb{P}$ is a filter in $\mathbb{P}$ if the following conditions hold true:

(1) $(p \in G \wedge p \leqslant q) \Rightarrow q \in G$,

(2) $(p \in G \wedge q \in G) \Rightarrow (\exists_{r\in G}\colon r \leqslant p \wedge r \leqslant q)$.

We say that a set $G \subset \mathbb{P}$ is *generic* if G is a filter in $\mathbb{P}$ and $G \cap D \neq \emptyset$ for every dense set $D \subset \mathbb{P}$. We say that partially ordered set $\mathbb{P}$ is *separative* if for all $p,q \in \mathbb{P}$ one has $p \nleqslant q \Rightarrow \exists_{r\leqslant p}\colon r\perp q$. Here, $r\perp q$ means incompatibility of the conditions p and q, where *compatibility* $p|q$ means that there exists such v that $v \leqslant p$ and $v \leqslant q$. Given the family K of subsets of $\mathbb{P}$ such that $K \cap \mathcal{P}(\mathbb{P}) \neq \emptyset$, we say that some filter G is K-*generic* if $G \cap D_K \neq \emptyset$ for every D_K that is a dense subset of $\mathbb{P}$ in K.

Lemma 1 (*e.g.* [7], Thm. 1.5)**.** $(\mathbb{P}, <)$ *is separative if and only if it can be embedded densely in a unique complete Boolean algebra* $\mathfrak{B}(\mathbb{P})$.

For an arbitrary partial order $(\mathbb{P}, <)$, one can determine the complete Boolean algebra $\mathfrak{B}(\tilde{\mathbb{P}})$ such that $\tilde{\mathbb{P}}$ is separative. $\tilde{\mathbb{P}}$ is the quotient $\mathbb{P}/_\sim$, where $\sim$ is the equivalence relation on $\mathbb{P}$ defined by $p \sim q$ iff $\forall_{r\in\mathbb{P}}\colon (r|p \iff r|q)$.

Let $\mathfrak{B}$ be some maximal complete Boolean algebra of projections chosen from the lattice of projections $\mathbb{L}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. We say that a self-adjoint (s-a) operator A is in $\mathfrak{B}$ ($A \in (\mathfrak{B})$) whenever $A = \int \lambda dP_\lambda$ for some projective measure dP_λ with values in $\mathfrak{B}$.

Lemma 2 (Takeuti, 1978 [8], Lemma 1.1)**.** *For every family* $\{A_\alpha\}$ *of pairwise commuting s-a operators, there exists a Boolean algebra of projections* $\mathfrak{B}$ *such that* $A_\alpha \in (\mathfrak{B})$ *for every* α.

The maximal Boolean algebra $(\mathfrak{B})$ as above is called the *Boolean algebra of quantum observables* on $\mathcal{H}$. For such $\mathcal{H}$ and $\{A_\alpha\}$, one formulates the spectral theorem as:

Theorem 1 (Spectral theorem [8], p. 68, Thm. 6.1)**.** *There exists a measure space* (X, μ) *and a unitary map* $U\colon \mathcal{H} \to \mathcal{L}^2(X, \mu)$ *such that the operator* $UA_\alpha U^{-1}$ *is the multiplication by a real measurable function* f_α *on* X. *Conversely, let* f *be any real measurable function on* X *and* F *the corresponding multiplication by* f *operator, then* $U^{-1}FU$ *is a s-a operator on* $\mathcal{H}$ *which commutes with each* A_α.

This theorem, in general, gives the correspondence between the algebra of s-a operators which are in $\mathfrak{B}$ and the measure algebra defined on Borel subsets of X. In fact, there is the isomorphism between the maximal projection algebra $(\mathfrak{B})$ generating the family $\{A_\alpha\}$ and the measure algebra of the space (X, μ), given by:

$$P \mapsto S_{\mathrm{P}}\,, \tag{1}$$

where S_{P} is the characteristic function of the subset S of X. The multiplication by S_{P} is the projection operator on $\mathcal{L}^2(X, \mu)$ as in the spectral theorem. The measure algebra of (X, μ) is, in general, non-atomic.

Given finitely many s-a commuting operators $A_1, A_2, \ldots, A_k$ generated by finite Boolean algebras of projections, one can diagonalize the family simultaneously and any measurement projects the state of the system on some common eigenstate with the probability given by the Born rule. Due to the Boolean nature of the algebra, there preexist measured values of the observables. The corresponding Boolean algebra $(\mathfrak{B})$ is atomic.

We are interested in the following question: Given the family of commuting s-a operators on $\mathcal{H}$ (hence simultaneously measurable), is there any intrinsic algebraic feature assigned to such Boolean family which would indicate quantumness of the observables? To grasp this problem, we consider the *classical geometric limit* of QM by taking infinite projection algebras and continuous observables based on them, like the measurement of the position of a particle in spacetime. In this way, we have both the classical spacetime and the (quantum) measurement of the spacetime coordinate which have to be related. We are going to argue that the set-theoretic forcing is the essence of the quantumness in such geometric set-up.

The Kochen–Specker theorem shows the impossibility to build the consistent $\{0,1\}$ valuation on every propositional system made of projections on closed subspaces of a given Hilbert space $\mathcal{H}$, provided $\dim(\mathcal{H}) \geqslant 3$. If such valuation exists, quantum observables would have preassigned, before the measurement, values in some state. Then the system would allow classical-like description. The existence of any (semi)classical state $\psi_{\rm cl}$ giving such valuation is possible for Boolean contexts only, since for the noncommuting algebras of observables and for the Hilbert spaces of dimension at least 3 the Kochen–Specker theorem forbids the existence of $\psi_{\rm cl}$. In such case, for every $P \in \mathfrak{B} \subset \mathbb{L}(\mathcal{H})$, there is the definite value 0 or 1 for each *yes–no* elementary proposition (see Appendix A): $[\![\psi \in \varDelta^P]\!] \in \{0,1\}$, where $P(\mathcal{H}) = \varDelta^P \subset \mathcal{H}$.

For every Boolean algebra of projections $\mathfrak{B} \subset \mathbb{L}(\mathcal{H})$ let us define [6] the *semiclassical state* $\psi_{\rm cl} \in \mathcal{H}^{\mathfrak{B}}$ as such state, that it determines the Boolean $\{0,1\}$ valuation on $\mathfrak{B}$, *i.e.* $\forall_{P\in\mathfrak{B}}\ [\![\psi_{\rm cl} \in \varDelta^P]\!] \in \{0,1\}$. Moreover, supposing that the Boolean algebra $\mathfrak{B}$ is determined by a partially ordered set of projections, the following compatibility conditions have to be fulfilled for all $P, Q \in \mathfrak{B}$:

(i) if $P(\psi_{\rm cl}) = 1$ (*i.e.* $[\![\psi_{\rm cl} \in \varDelta^P]\!] = 1$) and $P \leqslant Q$, then $Q(\psi_{\rm cl}) = 1$;

(ii) if $P(\psi_{\rm cl}) = 1$ and $Q(\psi_{\rm cl}) = 1$, then $(P \wedge Q)(\psi_{\rm cl}) = 1$.

For an atomless Boolean algebra of quantum observables $\big(\mathfrak{B}\big)$, like for the measure algebra on $\mathbb{R}$, there exists a probability measure which follows from the Maharam theorem [9]. Moreover, this measure is the uncountable product of 'coin-tossing' measures $\{0,1\}^c$.

Then, for the Boolean algebra of quantum observables $\big(\mathfrak{B}\big)$, *i.e.* such that the observables A_α are in $\big(\mathfrak{B}\big)$, one has for all $A_\alpha \in \big(\mathfrak{B}\big)$

$$\exists_{r_\alpha \in \mathbb{R}} \colon A_\alpha(\psi_{\rm cl}) \xmapsto{m} r_\alpha\,, \tag{2}$$

where m-arrow is the measurement of A_α in the state $\psi_{\rm cl}$. The measured value r_α can be preassigned to the observable before the experiment and the state $\psi_{\rm cl}$ is compatible with all the observables A_α, *i.e.* each observable has a definite value in the state $\psi_{\rm cl}$ even before the measurement [6]. Note that for a classical system, one simply has $A_\alpha(\psi) = r_\alpha \in \mathbb{R}$ for all α.

Given the atomless (Boolean) algebra, one has a tool for grasping the difference between quantum and classical observables (both in the Boolean context), since $\psi_{\rm cl}$ has to be carried by a very special algebraic structure in the quantum case:

Theorem 2. *For any complete atomless Boolean algebra $\big(\mathfrak{B}\big)$ of quantum observables, any semiclassical state $\psi_{\rm cl}$ is well defined generic filter G in $\big(\mathfrak{B}\big)$. Such a classical state is added by the non-trivial forcing to the ground Boolean model $V^{(\mathfrak{B})}$ of ZFC, i.e. $G \in V^{(\mathfrak{B})}[G]$ and $G \notin V$ (G is the generic ultrafilter on $\big(\mathfrak{B}\big)$ in V).*

Proof. The second part of the theorem follows from the following lemma:

Lemma 3 ([7], Prop. 2.1)**.** *For every ultrafilter G that is generic over a Boolean valued model $V^{(\mathfrak{B})}$ of ZFC, one has $G \notin V^{(\mathfrak{B})}$ (and $G \in V^{(\mathfrak{B})}[G]$).*

This is equivalent to [7]: $\big(\mathfrak{B}\big)$ *is atomless if and only if the generic ultrafilter over $V^{(\mathfrak{B})}$ is not in $V^{(\mathfrak{B})}$*. We use the canonical interpretation of V in $V^{(\mathfrak{B})}$ (see [10] and Appendix B). Let us suppose that $\big(\mathfrak{B}\big)$ is atomless and F is the generic ultrafilter on $\big(\mathfrak{B}\big)$ in the model. Then $\big(\mathfrak{B}\big) \setminus F$ is dense in $V^{(\mathfrak{B})}$ — otherwise there would be atoms. Hence $(\big(\mathfrak{B}\big) \setminus F) \cap F \neq \emptyset$, so $F \notin V^{(\mathfrak{B})}$.

Thus indeed, if $\psi_{\rm cl}$ is a generic filter in an atomless $\big(\mathfrak{B}\big)$, then the state $\psi_{\rm cl}$ is added by the forcing as in the theorem.

Now, let $\big(\mathfrak{B}\big)$ be an atomless Boolean algebra of observables made of the projections from $\mathbb{L}(\mathcal{H})$. To prove the genericity of G, we need just a slight reformulation of the results in [6]. Let us observe that the conditions *(i)* and *(ii)* define some filter $F^{\psi_{\rm cl}}$ on $\big(\mathfrak{B}\big)$ composed of those P which are true on $\psi_{\rm cl}$, *i.e.* $F^{\psi_{\rm cl}} = \{P \in \big(\mathfrak{B}\big) \colon \psi_{\rm cl}(P) = 1\}$. This is an ultrafilter. We need to show it is generic and can be represented by some vector state in the Hilbert state space $\mathcal{H}^{(\mathfrak{B})}$. To fix the latter statement, let us consider as the set $\mathcal{H}^{(\mathfrak{B})} = \{F^{\psi_{\rm cl}} \colon \psi_{\rm cl} \text{ is semiclassical}\}$. Then $P \in F^{\psi_{\rm cl}} \iff F^{\psi_{\rm cl}}(P) = 1$. It is important to characterize $F^{\psi_{\rm cl}}$ via sets $S \subset \mathcal{H}^{(\mathfrak{B})}$ of the ultrafilters on $\big(\mathfrak{B}\big)$. First, the family of subsets of ultrafilters $Z = \{S \colon S \subset \mathcal{H}^{(\mathfrak{B})}\}$ carries the structure of the Boolean algebra $\big(\mathfrak{B}\big)$. Every $P \in \big(\mathfrak{B}\big)$ is uniquely determined by the subset $S \in Z$ such that $P \in F \iff F \in S$ for all $F \in \mathcal{H}^{(\mathfrak{B})}$. As the Boolean value of the subset S we take $[\![S]\!] = P \in \big(\mathfrak{B}\big)$. Then one shows:

Lemma 4 ([6], Inference 2, 3)**.** *If $S, S' \in Z$, then $\sim S, S \cup S', S \cap S' \in Z$, and $[\![\sim S]\!] = \sim[\![S]\!]$, and $[\![S \cup S']\!] = [\![S]\!] \vee [\![S']\!]$, and $[\![S \cap S']\!] = [\![S]\!] \wedge [\![S']\!]$,* *and* $(S \subset S' \Rightarrow [\![S]\!] \leqslant [\![S']\!])$*. Moreover, supposing* $Z' \subset Z$ *and* $\bigcup Z' \in Z$*, one has* $[\![\bigcup Z']\!] = \bigvee\{[\![S]\!] \colon S \in Z'\}$*.*

Now, we are expressing the genericity on $\big(\mathfrak{B}\big)$ via the genericity on Z:

Lemma 5 ([6], Inference 4)**.** *Consider the family* K *of subsets* $b \subset \big(\mathfrak{B}\big)$ *of the form* $b = \{[\![S]\!] \colon S \in Z'\}$*, for all* $Z' \subset Z$ *such that* $\bigcup Z' \in Z$*. Then every* $F \in \mathcal{H}^{(\mathfrak{B})}$ *is* K*-generic.*

Finally, since K contains every $P \in \big(\mathfrak{B}\big)$ and is closed under Boolean operations as in Lemma 4, every dense subset in the PO of the algebra $\big(\mathfrak{B}\big)$ is also in K. Then every filter G which is K-generic is generic in $\big(\mathfrak{B}\big)$. □

We say that a value r of an observable A in $\big(\mathfrak{B}\big)$, measured in the state G, is *added by the set-theoretic forcing* $V^{(\mathfrak{B})} \to V^{(\mathfrak{B})}[G]$ if G is a generic filter on an atomless Boolean algebra $\big(\mathfrak{B}\big)$.

Corollary 1. *The value of any observable measured in the classical state* G*, as in Thm. 2, is added by the forcing* $V^{(\mathfrak{B})} \to V^{(\mathfrak{B})}[G]$*.*

Even though the values assigned to the observables from the Boolean algebra preexist, their actual measurability is the matter of set-theoretic forcing performed over the algebra. In the classical and continuous case, the Boolean algebra of observables assigned to the system gives preassigned values in the experiments which is unconditional on any forcing performed. Let us formulate yet another consequence:

Corollary 2. *If there were local hidden variables for a quantum system on* $\mathcal{H}$*, there would exist a semiclassical state* $\psi_{\mathrm{cl}}^{\mathrm{LHV}}$ *which would coincide with* ψ_{cl} *for every atomless Boolean algebra of continuous observables chosen from the lattice of projections* $\mathbb{L}(\mathcal{H})$*.*

This is the direct consequence of the analysis of the LHV program in [6]. Note that the LHV requires the agreement of the local contextual Boolean $\psi_{\mathrm{cl}}^{(\mathfrak{B})s}$ such that there exists a single $\psi_{\mathrm{cl}}^{\mathrm{LHV}}$ projecting down to every $\psi_{\mathrm{cl}}^{(\mathfrak{B})}$. It is known from the Kochen–Specker theorem that this is not possible for $\dim(\mathcal{H}) \geqslant 3$. However, the existence of $\psi_{\mathrm{cl}}^{(\mathfrak{B})s}$ is still allowed, in particular for an atomless $\big(\mathfrak{B}\big)$ corresponding to the continuous observables. The set-theoretic forcing is then non-trivial in such contexts. This last we consider as an important formal ingredient of the shift:

$$\textit{micro-scale (quantum)} \longrightarrow \textit{macro-scale (classical)}.$$

The results will be presented elsewhere.

3. Discussion

As shown by Benioff [1], neither any model of ZFC nor its generic extension by forcing are sufficient for grasping the mathematics of QM along with the statistical predictions of QM. However, it is still not excluded that forcing becomes the essential player of QM formalism — this is the dynamics and change of models of ZFC rather than any fixed single model that would matter in QM. One way of looking at the changes of the models is via forcing relation.

Indeed, the results of the previous section indicate very special role played by set-theoretic forcing in the QM formalism. The results of the continuous measurement, *e.g.* the position, in the *Boolean* contexts are also classically valid. Moreover, when approaching the classical geometric regime (large scales) as in general relativity, the spacetime continuum, locally modeled by $\mathbb{R}^4$, refers to the real line $\mathbb{R}$. We showed that the structure of the real line encodes the shift from quantum, though Boolean, observables to classical positions represented by the number parameters from $\mathbb{R}$. To see it more clearly, let us consider again the measure algebra, *i.e.* Borel subsets of $\mathbb{R}$ modulo the ideal of subsets of measure zero. This is the Boolean algebra which we considered in the context of spectral theorem in Sec. 2. Equivalently, one can take the algebra of Lebesgue measurable subsets modulo those of measure zero. They are atomless Boolean algebras and due to Lemma 3, they carry a non-trivial forcing in the Boolean valued model $Sh((\mathfrak{B})) \simeq V^{(\mathfrak{B})}$. Now, it is known that this *random* forcing adds a single random real number which cannot be rational and is not a member of any, definable in $V^{(\mathfrak{B})}$, set of Lebesgue measure zero. Indeed, the structure of the real line matters when passing from quantum to classical macro-description (or conversely). This structure is known in mathematics to be extremely rich on some formal level and deals necessary with forcing constructions [11]. Note that this structure of the real line was used by one of the authors as the tool to approach the non-standard (exotic) smooth geometries on $\mathbb{R}^4$ [5, 12]. Such non-flat Riemannian geometries recently appeared as being important and new player in cosmology and in the QM–GR incompatibility issue (see *e.g.* [13, 14]). However, we defer the analysis of this important relation between gravitational physics, based on $\mathbb{R}$-structure where only dense linear order matters, and deep forcing-like structure of $\mathbb{R}$ (relevant in quantum mechanics), for the forthcoming work.

From the point of view of intuitionism or topos theory, forcing is written in the formal structure of Grothendieck toposes. The reason is that in such toposes the logic of forcing is naturally expanded over stages of sieves as on the stages of a partial order in set theory (see *e.g.* Appendix A in [15]). Here, we avoided using toposes and chose more set theory based approach which is different, and give other results and perspective than the one based on toposes. This certainly sheds new light on the still mysterious link between QM and GR, in a way which is independent of the topos theory and some other traditional approaches (like quantum logics or categories). However, the use of such set-theoretic and forcing methods is related to the results of the other approaches in a rather non-trivial way which is worth further exploring.

Appendix A

Propositional algebra and the lattice of projections

Let $\mathcal{H}$ be an arbitrary Hilbert space. By $L(\mathcal{H})$ we denote the set of all projections on $\mathcal{H}$. For any projection P its range is a unique closed subspace of $\mathcal{H}$. According to this fact, we sometimes refer to such spaces also as projections, depending on the context. For arbitrary projections P_1 and P_2, one can define the operations of *meet* $P_1 \wedge P_2 = P_1 \cap P_2$ and *join* $P_1 \vee P_2 = \mathrm{span}(P_1 \cup P_2)$, where span is the usual linear span. We define a PO on the set $L(\mathcal{H})$ by the condition $P_1 \leqslant P_2 \iff \mathrm{ran}(P_1) \subset \mathrm{ran}(P_2)$. Thus, using the definitions of meet and join as above, the partial order $(L(\mathcal{H}), \leqslant)$ carries the structure of an atomic complete non-distributive lattice, called a *lattice of projections*. It is denoted by $\mathbb{L}(\mathcal{H})$.

We call any orthogonal projection $P\colon \mathcal{H} \to \mathcal{H}$ a *proposition*. Note that if states of some physical system are represented in $\mathcal{H}$, then proposition P corresponds to the measurement with two possible outcomes — 0 and 1. One can think of such measurement in terms of the question '*is P equalto1?*' with possible answers '*yes*' and '*no*'. We say that two propositions P_1 and P_2 are *orthogonal* when all vectors in P_1 are orthogonal to all vectors in P_2, *i.e.* $\mathrm{ran}(P_1) \perp \mathrm{ran}(P_2)$. Note that commuting propositions correspond to simultaneously answerable questions about a physical system.

For every proposition P, we define its *complement* by the condition $\sim P = \{x \in \mathcal{H}\colon x \perp y \text{ for all } y \in P\}$. Let C be an arbitrary set of commuting propositions closed under the operations of complementation and meet. We call an algebraic structure $\mathfrak{C} = (C, \vee, \wedge, \sim, \hat{0}, \hat{1})$ a *propositional algebra* (PA). Clearly, by definition, every PA is a Boolean algebra.

Appendix B

Elements of forcing and models of ZFC

Forcing in set theory was invented as a tool to prove various independence results, like the independence of the axiom of choice (AC) and the continuum hypothesis (CH) from the axioms of ZF and ZFC correspondingly. The idea is to construct models of ZFC, where AC (or CH) hold true and some other models of ZF (and ZFC) where they do not. If such models exist, which indeed is true, then AC and CH are independent of ZF and ZFC respectively.

The forcing procedure, soon after its invention, was reformulated as the property of ultrafilters on partial orders and on Boolean algebras containing these POs. In the 1960s of the 20th century, Scott and Solovay, and independently Vopěnka and Hajék, gave the unifying description of forcing in terms of Boolean valued models of ZFC.

For every forcing to be non-trivial, the crucial is the existence of a generic filter G on a partial order $(\mathbb{P}, <)$ or on the Boolean algebra $\mathfrak{B}(\tilde{\mathbb{P}})$, such that G would be the generic ultrafilter in the model V. The appearance of the generic filters in $\mathfrak{B}$, more than logical consequences of the forcing, is the particularly important feature of the results in this paper. However, a possible development of the approach such that the geometric and gravitational aspects are included, should additionally deal with the logical and set-theoretic content of the forcing (*e.g.* random reals). Here, we explain only how it happens that a generic filter can be included in generic extensions of some ZFC models. This serves as yet another (and, in fact, complementary) reasoning, than one based on the nonexistence of atoms in Boolean algebras (*cf.* Lemma 3).

Lemma 6 ([10], Lemma 14.4)**.** *If $\mathcal{D}$ is a countable family of dense subsets of partially ordered set $\mathbb{P}$, then for every $p \in \mathbb{P}$ there exists a generic filter G on $\mathbb{P}$ such that $p \in G$.*

Then one can prove the following:

Theorem 3 (Generic model theorem [10], Thm. 14.5)**.** *Let $\mathbb{P}$ be a PO in a transitive ZFC model V, and $G \subset \mathbb{P}$ a generic filter on $\mathbb{P}$. Then there exists a transitive ZFC model $V[G]$ (a generic extension of V) such that:*

(1) $V \subset V[G]$ and $G \in V[G]$;

(2) The ordinal numbers in V are the same as in $V[G]$.

Every $x \in V$ has the canonical name in $V^{\mathfrak{B}}$, denoted by $\overline{x} \in V^{\mathfrak{B}}$, such that it is the function $\overline{x}\colon \{\overline{y}\colon y \in x\} \to \mathfrak{B}$ for which $\overline{x}(\overline{y}) = 1$ for all $y \in x$. An ultrafilter G on $\mathfrak{B}$ over V has the canonical name:

$$\overline{G}\colon \{\overline{u}\colon u \in \mathfrak{B}\} \to \mathfrak{B} \text{ such that } \overline{G}(\overline{u}) = u\,. \tag{B.1}$$

For a transitive ZFC model V and a complete Boolean algebra $\mathfrak{B}$ in V, let G be a V-generic ultrafilter on $\mathfrak{B}$. Then the interpretation by G of any $x \in V^{\mathfrak{B}}$ (recall that $x \in V^{\mathfrak{B}}_{\alpha+1}$ is a function on $V^{\mathfrak{B}}_{\alpha}$ with values in $\mathfrak{B}$, where α is a nonlimit ordinal), denoted by x^G, is defined inductively according to:

(1) $\emptyset^G = \emptyset$;

(2) $x^G = \{y^G\colon x(y) \in G\}$.

Now, let $V[G]$ be the transitive model of ZFC built of the interpretations by G, that is:

$$V[G] = \left\{ x^G \colon x \in V^{\mathfrak{B}} \right\} . \tag{B.2}$$

Let $\overline{G}$ be the canonical name of a generic ultrafilter G. Then, from (B.1) its interpretation is $\overline{G}^G = G$. Hence, from (B.2), it follows that $G \in V[G]$.

Finally, note that the model $V[G]$ is isomorphic to the quotient $V^{\mathfrak{B}}/_G$ ([10], p. 224, Ex. 14.15) which is the generic extension $V^{\mathfrak{B}}[G]$ used in the Thm. 2.

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