COMPLETE SYNCHRONIZATION OF COUPLED MULTIPLE-TIME-DELAY COMPLEX CHAOTIC SYSTEM WITH APPLICATIONS TO SECURE COMMUNICATION

Fangfang Zhang

School of Electrical Engineering and Automation, Qilu University of Technology
Jinan 250353, China

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Considering that time-delay is frequently encountered in a variety of practical chaotic systems, we investigate the complete synchronization (CS) of coupled multiple-time-delay complex chaotic systems and design the control law by error feedback, which is simple in principle and easy to implement in engineering. A communication scheme is further designed according to chaotic masking. We take coupled multiple-time-delay complex Lorenz system as an example, make simulations and verify the effect of the controllers. The error feedback is extended to complex chaotic systems with multiple-time-delay. The CS of real chaotic systems and complex chaotic systems without time-delay are its special cases.

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1. Introduction

Since Fowler *et al.* [1] introduced the complex Lorenz equations, complex systems have played an important role in many branches of physics, especially for chaos-based secure communication, where the complex variables (doubling the number of variables) increase the contents and security of the transmitted information [2]. The synchronization of complex chaotic systems has attracted great attention in the last decades, such as phase synchronization (PHS) and anti-phase synchronization (APHS) [3], complete synchronization (CS) [4], projective synchronization (PS) and modified projective synchronization (MPS) [5], anti-synchronization (AS) [6, 7], modified function projective synchronization (MFPS) [8, 9], lag synchronization (LS) [10], anti-lag synchronization [11], full state hybrid projective synchronization (FSHPS) [12], modified projective phase synchronization (MPPS) [13], hybrid modified function projective synchronization (HMFPS) [14], modified projective synchronization with complex scaling factors (CMPS) [15, 16], complex function projective synchronization (CFPS) [17], modified function projective lag synchronization [18], combination synchronization [19], combination–combination synchronization [20], *etc.*

Time delay is frequently encountered in a variety of practical chaotic systems. Due to the effect of time delay, complex chaotic systems may possess unexpected characteristics and the synchronization performance of these systems is hardly assured. However, the above papers [1–20] did not consider time delay, and there are a very few literature on the time-delay complex chaotic system. To the authors knowledge, there is no other relevant literature except our recent works [21, 22]. In our paper [21], we have discussed the characteristics of one-time-delay complex chaotic system and designed self-time-delay synchronization (STDS) controller; then, we discussed self-time-delay synchronization of time-delay coupled complex chaotic system and its applications to communication in our paper [22]. From the point of application, the papers [21, 22] refer only to STDS, and do not consider CS, which is the simplest and most widely used form in real chaos-communication techniques. In this paper, we will discuss CS of the coupled multiple-time-delay complex chaotic system and its applications to communication, and design the control laws only by error feedback. The controller in [22] contains nonlinear part and error feedback, and it works based, in essence, on the offset of the nonlinear part. Considering the difficulty in the implementing of nonlinear part in engineering, the controller in this paper is much simpler than the controller in [22].

Furthermore, there are a number of papers [23–26] about communication based on the real chaotic signal, while the communication techniques based on complex chaotic systems have rarely been studied [22, 27]. Especially, there is almost no paper about the chaos-communication techniques considering time delay or time lag based on complex chaotic systems except our paper [22], where we consider the time lag of transmission, and describe the transmitter as original chaotic system and the receiver as its one-time-delay system. In this paper, we describe both the transmitter and the receiver as multiple-time-delay systems because multiple-time-delay is inevitable in the real world. Moreover, the signals generated by time-delay complex chaotic systems are more complicated and difficult to decipher.

Finally, the synchronization of coupled chaotic systems has been intensively investigated since its introduction in 1990 by Pecora and Carrol [28] due to its potential applications in physiology [29], chemical kinetics [30], physics [31] and secure communications [32–35]. Therefore, inspired by the above discussion, aiming at coupled multiple-time-delay complex chaotic system, in this paper we study the complete synchronization and its applications to secure communication for better applications.

The rest of this paper is organized as follows. We design the controllers of coupled multiple-time-delay complex chaotic systems in Section 2. Section 3 is devoted to the communication scheme. We make simulations in Section 4. Finally, some conclusions are given in Section 5.

2. The synchronization of coupled multiple-time-delay complex chaotic systems

As the synchronization of coupled chaotic systems is intensively investigated in communication, we consider the general coupled multiple-time-delay complex chaotic system described by n dimensional differential equations

$$\begin{cases} \dot{\boldsymbol{y}} &= \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z}) + \boldsymbol{A}_1\boldsymbol{y}(\boldsymbol{t}-\boldsymbol{\tau}) + \boldsymbol{A}_2\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r})\,, \\ \dot{\boldsymbol{z}} &= \boldsymbol{g}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}(\boldsymbol{t}-\boldsymbol{\tau}),\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r}))\,, \\ \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z}) + \boldsymbol{A}_1\boldsymbol{x}(\boldsymbol{t}-\boldsymbol{\tau}) + \boldsymbol{A}_2\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r}) + \boldsymbol{v}\,, \end{cases} \tag{1}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_q)^{\mathrm{T}}$ and $\boldsymbol{r} = (r_1, r_2, \ldots, r_q)^{\mathrm{T}}$ are constant time-delay vectors. The state vector of drive system is broken into two parts $(\boldsymbol{y},\boldsymbol{z})$, and $\boldsymbol{y} = (y_1, y_2, \ldots, y_q)^{\mathrm{T}}$ and $\boldsymbol{z} = (z_1, z_2, \ldots, z_{n-q})^{\mathrm{T}}$. $\boldsymbol{A}_1$ is the $q\times q$ real constant matrix and $\boldsymbol{A}_2$ the $q\times(n-q)$ real constant matrix. The state vector $\boldsymbol{z}$ is treated as the coupling vector. $\boldsymbol{x} = (x_1, x_2, \ldots, x_q)^{\mathrm{T}}$ is a complex state vector of response system. The vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ are $q\times 1$ and $(n-q)\times 1$ complex function vectors, respectively. The designed controller is $\boldsymbol{v} = \boldsymbol{v}^{\mathrm{r}} + j\boldsymbol{v}^{\mathrm{i}}$, where $\boldsymbol{v}^{\mathrm{r}} = (v_1, v_3, \ldots, v_{2q-1})^{\mathrm{T}}$, $\boldsymbol{v}^{\mathrm{i}} = (v_2, v_4, \ldots, v_{2q})^{\mathrm{T}}$. Superscripts 'r' and 'i' stand for the real and imaginary parts of the complex vector $\boldsymbol{v}$, respectively. The objective is to design the controller $\boldsymbol{v}$ such that $\boldsymbol{x}(\boldsymbol{t})$ synchronizes $\boldsymbol{y}(\boldsymbol{t})$ asymptotically.

Lemma 1 [36] If $n\times n$ real matrix is the row (column) diagonally dominant matrix where all of the diagonal elements are negative, then all eigenvalues of this matrix have negative real parts.

Assumption 1 The complex function vector $\boldsymbol{f}$ satisfies local Lipchitz condition *i.e.* for any compact set S, there exist positive constants $l_{\mathrm{rmax}}(S)$ and $l_{\mathrm{imax}}(S)$ such that

$$\begin{aligned} \|\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{y})\| &= \left\|\boldsymbol{f}(\boldsymbol{x})^{\mathrm{r}}-\boldsymbol{f}(\boldsymbol{y})^{\mathrm{r}}+j\left(\boldsymbol{f}(\boldsymbol{x})^{\mathrm{i}}-\boldsymbol{f}(\boldsymbol{y})^{\mathrm{i}}\right)\right\| \\ &\le \|\boldsymbol{f}(\boldsymbol{x})^{\mathrm{r}}-\boldsymbol{f}(\boldsymbol{y})^{\mathrm{r}}\| + \|\boldsymbol{f}(\boldsymbol{x})^{\mathrm{i}}-\boldsymbol{f}(\boldsymbol{y})^{\mathrm{i}}\| \\ &\le [l_{\mathrm{rmax}}(S)+l_{\mathrm{imax}}(S)]\|\boldsymbol{x}-\boldsymbol{y}\|\,, \qquad \forall x,y\in S\,, \end{aligned} \tag{2}$$

where $\|\cdot\|$ denotes the Euclidean norm. The existence of $l_{\mathrm{rmax}}(S)$ and $l_{\mathrm{imax}}(S)$ are related to the choice of the compact set S.

Theorem 1. *For any initial conditions* $\boldsymbol{y}(0), \boldsymbol{z}(0), \boldsymbol{x}(0)$ *of coupled multiple-time-delay system (1) under Assumption 1, the complete synchronization between* $\boldsymbol{x(t)}$ *and* $\boldsymbol{y(t)}$ *will occur by the following control law (3)*

$$\boldsymbol{v} \;=\; -\boldsymbol{K}(\boldsymbol{x}-\boldsymbol{y})\,, \tag{3}$$

where $\boldsymbol{K} = diag(k_1,\, k_2,\, \ldots,\, k_q)$ *is the real positive constant control strength matrix.*

Proof. Set $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t)$, then $\boldsymbol{e}(t-\tau) = \boldsymbol{x}(t-\tau) - \boldsymbol{y}(t-\tau)$. For simplicity, we rewrite $\boldsymbol{e}(t), \boldsymbol{x}(t), \boldsymbol{y}(t)$ as $\boldsymbol{e}, \boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{e}(t-\tau), \boldsymbol{x}(t-\tau), \boldsymbol{y}(t-\tau)$ as $\boldsymbol{e}_\tau, \boldsymbol{x}_\tau, \boldsymbol{y}_\tau$. We get the error dynamical system as

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}} - \dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z}) - \boldsymbol{K}\boldsymbol{e} + \boldsymbol{A}_1\boldsymbol{e}_\tau\,. \tag{4}$$

Therefore, we have

$$\begin{cases} \dot{\boldsymbol{e}}^{\mathrm{r}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z})^{\mathrm{r}} - \boldsymbol{K}\boldsymbol{e}^{\mathrm{r}} - \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z})^{\mathrm{r}} + \boldsymbol{A}_1\boldsymbol{e}^{\mathrm{r}}{}_\tau\,, \\ \dot{\boldsymbol{e}}^{\mathrm{i}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z})^{\mathrm{i}} - \boldsymbol{K}\boldsymbol{e}^{\mathrm{i}} - \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z})^{\mathrm{i}} + \boldsymbol{A}_1\boldsymbol{e}^{\mathrm{i}}{}_\tau\,. \end{cases} \tag{5}$$

The control strength matrix $\boldsymbol{K}$ is used to adjust the convergence rate of synchronization.

According to local Lipchitz condition, we set

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z}) &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z})^{\mathrm{r}} - \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z})^{\mathrm{r}} + j\left[\boldsymbol{f}(\boldsymbol{x},\boldsymbol{z})^{\mathrm{i}} - \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z})^{\mathrm{i}}\right] \\ &= l_{\mathrm{r}}(t)\boldsymbol{e} + jl_{\mathrm{i}}(t)\boldsymbol{e} \\ &= \left[l_{\mathrm{r}}(t)\boldsymbol{e}^{\mathrm{r}} - l_{\mathrm{i}}(t)\boldsymbol{e}^{\mathrm{i}}\right] + j\left[l_{\mathrm{r}}(t)\boldsymbol{e}^{\mathrm{i}} + l_{\mathrm{i}}(t)\boldsymbol{e}^{\mathrm{r}}\right] \\ &\le \left[l_{\mathrm{rmax}}(S) + l_{\mathrm{imax}}(S)\right]\|\boldsymbol{x}-\boldsymbol{y}\|\,, \end{aligned} \tag{6}$$

where $l_{\mathrm{r}}(t) = \mathrm{diag}\{l_{\mathrm{r}1}, l_{\mathrm{r}2}, \ldots, l_{\mathrm{r}q}\}$ and $l_{\mathrm{i}}(t) = \mathrm{diag}\{l_{\mathrm{i}1}, l_{\mathrm{i}2}, \ldots, l_{\mathrm{i}q}\}$.

Then Eq. (5) becomes

$$\begin{cases} \dot{\boldsymbol{e}}^{\mathrm{r}} = \left[l_{\mathrm{r}}(t)\boldsymbol{e}^{\mathrm{r}} - l_{\mathrm{i}}(t)\boldsymbol{e}^{\mathrm{i}}\right] - \boldsymbol{K}\boldsymbol{e}^{\mathrm{r}} + \boldsymbol{A}_1\boldsymbol{e}^{\mathrm{r}}{}_\tau\,, \\ \dot{\boldsymbol{e}}^{\mathrm{i}} = \left[l_{\mathrm{r}}(t)\boldsymbol{e}^{\mathrm{i}} + l_{\mathrm{i}}(t)\boldsymbol{e}^{\mathrm{r}}\right] - \boldsymbol{K}\boldsymbol{e}^{\mathrm{i}} + \boldsymbol{A}_1\boldsymbol{e}^{\mathrm{i}}{}_\tau\,. \end{cases} \tag{7}$$

For positive definite matrixes $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{M}_1, \boldsymbol{M}_2$, we adopt Lyapunov–Krasovskii function $V : C \to R$ as

$$\begin{aligned} V\left(\boldsymbol{e}^{\mathrm{r}}, \boldsymbol{e}^{\mathrm{i}}\right) \;=\; & \boldsymbol{e}^{\mathrm{r}^{\mathrm{T}}}(t)\boldsymbol{P}_1\boldsymbol{e}^{\mathrm{r}}(t) + \boldsymbol{e}^{\mathrm{i}^{\mathrm{T}}}(t)\boldsymbol{P}_2\boldsymbol{e}^{\mathrm{i}}(t) + \int_{-\tau}^{0} \boldsymbol{e}^{\mathrm{r}^{\mathrm{T}}}(t+\xi)\boldsymbol{M}_1\boldsymbol{e}^{\mathrm{r}}(t+\xi)d\xi \\ & + \int_{-\tau}^{0} \boldsymbol{e}^{\mathrm{i}^{\mathrm{T}}}(t+\xi)\boldsymbol{M}_2\boldsymbol{e}^{\mathrm{i}}(t+\xi)d\xi\,. \end{aligned} \tag{8}$$

If the initial condition is $(0, \boldsymbol{\phi}^{\mathrm{r}}(0), \boldsymbol{\phi}^{\mathrm{i}}(0))$, then

$$\begin{aligned} V(\phi) &= \boldsymbol{\phi}^{\mathrm{r}\mathrm{T}}(0)\boldsymbol{P}_1\boldsymbol{\phi}^{\mathrm{r}}(0) + \int_{-\tau}^{0} \boldsymbol{\phi}^{\mathrm{r}\mathrm{T}}(\xi)\boldsymbol{M}_1\boldsymbol{\phi}^{\mathrm{r}}(\xi)d\xi + {\boldsymbol{\phi}^{\mathrm{i}}}^{\mathrm{T}}(0)\boldsymbol{P}_2\boldsymbol{\phi}^{\mathrm{i}}(0) \\ &\quad + \int_{-\tau}^{0} {\boldsymbol{\phi}^{\mathrm{i}}}^{\mathrm{T}}(\xi)\boldsymbol{M}_2\boldsymbol{\phi}^{\mathrm{i}}(\xi)d\xi \\ &\le \mu_{\mathrm{r}}\boldsymbol{\phi}^{\mathrm{r}\mathrm{T}}(0)\boldsymbol{\phi}^{\mathrm{r}}(0) + \int_{-\tau}^{0} \lambda_{\mathrm{r}}\boldsymbol{\phi}^{\mathrm{r}\mathrm{T}}(\xi)\boldsymbol{\phi}^{\mathrm{r}}(\xi)d\xi + \mu_{\mathrm{i}}{\boldsymbol{\phi}^{\mathrm{i}}}^{\mathrm{T}}(0)\boldsymbol{\phi}^{\mathrm{i}}(0) \\ &\quad + \int_{-\tau}^{0} \lambda_{\mathrm{i}}{\boldsymbol{\phi}^{\mathrm{i}}}^{\mathrm{T}}(\xi)\boldsymbol{\phi}^{\mathrm{i}}(\xi)d\xi \\ &\le (\mu_{\mathrm{r}} + \lambda_{\mathrm{r}}\tau)\boldsymbol{\phi}^{\mathrm{r}\mathrm{T}}(0)\boldsymbol{\phi}^{\mathrm{r}}(0) + (\mu_{\mathrm{i}} + \lambda_{\mathrm{i}}\tau){\boldsymbol{\phi}^{\mathrm{i}}}^{\mathrm{T}}(0)\boldsymbol{\phi}^{\mathrm{i}}(0)\,, \end{aligned} \tag{9}$$

where $\mu_{\mathrm{r}}, \lambda_{\mathrm{r}}, \mu_{\mathrm{i}}, \lambda_{\mathrm{i}}$ is the maximum eigenvalue of $\boldsymbol{P}_1, \boldsymbol{M}_1, \boldsymbol{P}_2, \boldsymbol{M}_2$, respectively.

Its derivative along $\dot{\boldsymbol{e}}$ is

$$\begin{aligned} \dot{V}\left(\boldsymbol{e}^{\mathrm{r}}, \boldsymbol{e}^{\mathrm{i}}\right) &= \dot{\boldsymbol{e}}^{\mathrm{r}\mathrm{T}}(t)\boldsymbol{P}_1\boldsymbol{e}^{\mathrm{r}}(t) + \boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t)\boldsymbol{P}_1\dot{\boldsymbol{e}}^{\mathrm{r}}(t) + {\dot{\boldsymbol{e}}^{\mathrm{i}}}^{\mathrm{T}}(t)\boldsymbol{P}_2\boldsymbol{e}^{\mathrm{i}}(t) + {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t)\boldsymbol{P}_2\dot{\boldsymbol{e}}^{\mathrm{i}}(t) \\ &\quad + \left[\boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t)\boldsymbol{M}_1\boldsymbol{e}^{\mathrm{r}} - \boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t-\tau)\boldsymbol{M}_1\boldsymbol{e}^{\mathrm{r}}(t-\tau)\right] \\ &\quad + \left[{\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t)\boldsymbol{M}_2\boldsymbol{e}^{\mathrm{i}} - {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t-\tau)\boldsymbol{M}_2\boldsymbol{e}^{\mathrm{i}}(t-\tau)\right]. \end{aligned} \tag{10}$$

Substituting (7) into (10), we have

$$\begin{aligned} \dot{V}\left(\boldsymbol{e}^{\mathrm{r}}, \boldsymbol{e}^{\mathrm{i}}\right) &= \boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t)\left({l_{\mathrm{r}}}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{P}_1 l_{\mathrm{r}} - \boldsymbol{K}\boldsymbol{P}_1 - \boldsymbol{P}_1\boldsymbol{K} + \boldsymbol{M}_1\right)\boldsymbol{e}^{\mathrm{r}}(t) \\ &\quad + {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t)\left({l_{\mathrm{r}}}^{\mathrm{T}}\boldsymbol{P}_2 + \boldsymbol{P}_2 l_{\mathrm{r}} - \boldsymbol{K}\boldsymbol{P}_2 - \boldsymbol{P}_2\boldsymbol{K} + \boldsymbol{M}_2\right)\boldsymbol{e}^{\mathrm{i}}(t) \\ &\quad + \boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t)\left({l_{\mathrm{i}}}^{\mathrm{T}}\boldsymbol{P}_2 - \boldsymbol{P}_1 l_{\mathrm{i}}\right)\boldsymbol{e}^{\mathrm{i}}(t) + {v^{\mathrm{i}}}^{\mathrm{T}}(t)\left(\boldsymbol{P}_2 l_{\mathrm{i}} - {l_{\mathrm{i}}}^{\mathrm{T}}\boldsymbol{P}_1\right)\boldsymbol{e}^{\mathrm{r}}(t) \\ &\quad + \boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t-\tau){\boldsymbol{A}_1}^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{e}^{\mathrm{r}}(t) + {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t-\tau){\boldsymbol{A}_1}^{\mathrm{T}}\boldsymbol{P}_2\boldsymbol{e}^{\mathrm{i}}(t) \\ &\quad + \boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t)\boldsymbol{P}_1\boldsymbol{A}_1\boldsymbol{e}^{\mathrm{r}}(t-\tau) + {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t)\boldsymbol{P}_2\boldsymbol{A}_1\boldsymbol{e}^{\mathrm{i}}(t-\tau) \\ &\quad - \boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t-\tau)\boldsymbol{M}_1\boldsymbol{e}^{\mathrm{r}}(t-\tau) - {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t-\tau)\boldsymbol{M}_2\boldsymbol{e}^{\mathrm{i}}(t-\tau) \\ &= \boldsymbol{\theta}^{\mathrm{T}}(t)\boldsymbol{L}(t)\boldsymbol{\theta}(t)\,, \end{aligned} \tag{11}$$

where $\boldsymbol{\theta}^{\mathrm{T}} = (\boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t), {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t), \boldsymbol{e}^{\mathrm{r}\mathrm{T}}(t-\tau), {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t-\tau))$, and $\boldsymbol{L}(t) =$

$$\begin{pmatrix} {l_{\mathrm{r}}}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{P}_1 l_{\mathrm{r}} - \boldsymbol{K}\boldsymbol{P}_1 - \boldsymbol{P}_1\boldsymbol{K} + \boldsymbol{M}_1 & \frac{{l_{\mathrm{i}}}^{\mathrm{T}}\boldsymbol{P}_2 - \boldsymbol{P}_1 l_{\mathrm{i}} + \boldsymbol{P}_2 l_{\mathrm{i}} - {l_{\mathrm{i}}}^{\mathrm{T}}\boldsymbol{P}_1}{2} & \frac{{\boldsymbol{A}_1}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{A}_1}{2} & 0 \\ \frac{{l_{\mathrm{i}}}^{\mathrm{T}}\boldsymbol{P}_2 - \boldsymbol{P}_1 l_{\mathrm{i}} + \boldsymbol{P}_2 l_{\mathrm{i}} - {l_{\mathrm{i}}}^{\mathrm{T}}\boldsymbol{P}_1}{2} & {l_{\mathrm{r}}}^{\mathrm{T}}\boldsymbol{P}_2 + \boldsymbol{P}_2 l_{\mathrm{r}} - \boldsymbol{K}\boldsymbol{P}_2 - \boldsymbol{P}_2\boldsymbol{K} + \boldsymbol{M}_2 & 0 & \frac{{\boldsymbol{A}_1}^{\mathrm{T}}\boldsymbol{P}_2 + \boldsymbol{P}_2\boldsymbol{A}_1}{2} \\ \frac{{\boldsymbol{A}_1}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{A}_1}{2} & 0 & -\boldsymbol{M}_1 & 0 \\ 0 & \frac{{\boldsymbol{A}_1}^{\mathrm{T}}\boldsymbol{P}_2 + \boldsymbol{P}_2\boldsymbol{A}_1}{2} & 0 & -\boldsymbol{M}_2 \end{pmatrix}. \tag{12}$$

According to local Lipchitz condition, all elements of $l_{\mathrm{r}}(t)$ and $l_{\mathrm{i}}(t)$ are bounded (*i.e.* $-l_{\mathrm{rmax}} < |l_{\mathrm{r}s}(t)| < l_{\mathrm{rmax}}$ and $-l_{\mathrm{imax}} < |l_{\mathrm{r}s}(t)| < l_{\mathrm{imax}}$, where $s = 1, 2, \ldots, q$). If we select $\boldsymbol{P}_1 = \boldsymbol{P}_2 = \boldsymbol{I}, \boldsymbol{M}_1 = N_1\boldsymbol{I}, \boldsymbol{M}_2 = N_2\boldsymbol{I}$, where N_1 and N_2 are positive integers, then $\boldsymbol{L}(t)$ becomes

$$\begin{pmatrix} 2{l_{\mathrm{r}}}^{\mathrm{T}} - 2\boldsymbol{K} + N_1 I & 0 & \frac{{\boldsymbol{A}_1}^{\mathrm{T}} + \boldsymbol{A}_1}{2} & 0 \\ 0 & 2{l_{\mathrm{r}}}^{\mathrm{T}} - 2\boldsymbol{K} + N_2 I & 0 & \frac{{\boldsymbol{A}_1}^{\mathrm{T}} + \boldsymbol{A}_1}{2} \\ \frac{{\boldsymbol{A}_1}^{\mathrm{T}} + \boldsymbol{A}_1}{2} & 0 & -\boldsymbol{M}_1 & 0 \\ 0 & \frac{{\boldsymbol{A}_1}^{\mathrm{T}} + \boldsymbol{A}_1}{2} & 0 & -\boldsymbol{M}_2 \end{pmatrix}. \tag{13}$$

We can select suitable $\boldsymbol{K}, N_1, N_2$ to make the real symmetric matrix $\boldsymbol{L}(t)$ be a row diagonally dominant matrix where all of the diagonal elements are negative. According to Lemma 1, all eigenvalues of real symmetric matrix $\boldsymbol{L}(t)$ are negative.

Set $-w(w > 0)$ as the maximum eigenvalue of the matrix $\boldsymbol{L}$, then

$$\begin{aligned} \dot{V} &\leq -w\left(\| \boldsymbol{e}^{\mathrm{rT}}(t) \|^2 + \| {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t) \|^2 + \| \boldsymbol{e}^{\mathrm{rT}}(t-\tau) \|^2 + \| {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t-\tau) \|^2\right) \\ &\leq -w\left(\| \boldsymbol{e}^{\mathrm{rT}}(t) \|^2 + \| {\boldsymbol{e}^{\mathrm{i}}}^{\mathrm{T}}(t) \|^2\right). \end{aligned} \tag{14}$$

According to the theorems in [36], $\boldsymbol{e}(t) = 0$ is the globally equilibrium point with exponential asymptotic stability of system (1). The proof is completed. □

Remark 1 If $\boldsymbol{\tau} = \boldsymbol{r} = 0$, then system (1) can indicate most chaotic complex systems satisfying Assumption 1 without time delay, such as complex Lorenz system, complex Chen system, complex Lü system, complex Van der Pol oscillator, complex Duffing system and so on. If $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are real state variables, the problem becomes the CS of real chaotic systems. The CS of real chaotic systems and complex chaotic systems without time delay are its special cases.

Remark 2 The convergence rate of the error system can be flexibly regulated by choosing the matrix $\boldsymbol{K}$. In a certain range, the larger the absolute values of $\boldsymbol{K}$ is, the faster the convergence speed of errors is, *i.e.* the speed of realizing synchronization of systems (1) increases.

3. A communication scheme of coupled multiple-time-delay complex chaotic system

In this section, the application of coupled multiple-time-delay complex chaotic system to secure communication is investigated in theory. We adopt $L1$ as the transmitter and $L2$ as the receiver by the following systems:

$$L1: \begin{cases} \dot{\boldsymbol{y}} & = & \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z}) + \boldsymbol{A}_1\boldsymbol{y}(\boldsymbol{t}-\boldsymbol{\tau}) + \boldsymbol{A}_2\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r}) + \boldsymbol{B}\boldsymbol{h}\,, \\ \dot{\boldsymbol{z}} & = & \boldsymbol{g}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}(\boldsymbol{t}-\boldsymbol{\tau}),\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r}))\,, \\ \boldsymbol{s} & = & \boldsymbol{p}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}(\boldsymbol{t}-\boldsymbol{\tau}),\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r})) + \boldsymbol{B}\boldsymbol{h}\,, \end{cases} \tag{15}$$

$$L2: \begin{cases} \dot{\boldsymbol{x}} & = & \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z}) + \boldsymbol{A}_1\boldsymbol{x}(\boldsymbol{t}-\boldsymbol{\tau}) + \boldsymbol{A}_2\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r}) + \boldsymbol{s} - \boldsymbol{s'}\,, \\ \boldsymbol{s'} & = & \boldsymbol{p}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{x}(\boldsymbol{t}-\boldsymbol{\tau}),\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r}))\,, \end{cases} \tag{16}$$

where $\boldsymbol{h} = (h_1, h_2, \ldots, h_q)^{\mathrm{T}}$ is the information signal vector, and $\boldsymbol{B} = \mathrm{diag}\{b_1, b_2, \ldots, b_q\}$ is its parameter matrix. The transmitted signal (*i.e.* the output of the transmitter) is $\boldsymbol{s}$ and the output of the receiver is $\boldsymbol{s'}$. $\boldsymbol{p}$ is the $q \times 1$ complex function vector. As the synchronization between $\boldsymbol{x}$ and $\boldsymbol{y}$, the recovered signal is $\boldsymbol{h}_g = \boldsymbol{B}^{-1}(\boldsymbol{s} - \boldsymbol{s'})$. Next, we need to deduce the special form of $\boldsymbol{s}$ and $\boldsymbol{s'}$.

From (15) and (16), we get the error system as

$$\begin{aligned} \dot{\boldsymbol{e}} &= \dot{\boldsymbol{x}} - \dot{\boldsymbol{y}} \\ &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{y},\boldsymbol{z}) + \boldsymbol{A}_1\boldsymbol{e}(\boldsymbol{t}-\boldsymbol{\tau}) - \boldsymbol{B}\boldsymbol{h} + \boldsymbol{s} - \boldsymbol{s'}\,. \end{aligned} \tag{17}$$

Based on the controller (3), we choose

$$\begin{cases} \boldsymbol{s} & = & \boldsymbol{K}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{h}\,, \\ \boldsymbol{s'} & = & \boldsymbol{K}\boldsymbol{x}\,. \end{cases} \tag{18}$$

Therefore, $\boldsymbol{p}(\boldsymbol{y},\boldsymbol{z},\boldsymbol{y}(\boldsymbol{t}-\boldsymbol{\tau}),\boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r})) = \boldsymbol{K}\boldsymbol{y}$ and $\boldsymbol{p}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{x}(\boldsymbol{t}-\boldsymbol{\tau}), \boldsymbol{z}(\boldsymbol{t}-\boldsymbol{r})) = \boldsymbol{K}\boldsymbol{x}$.

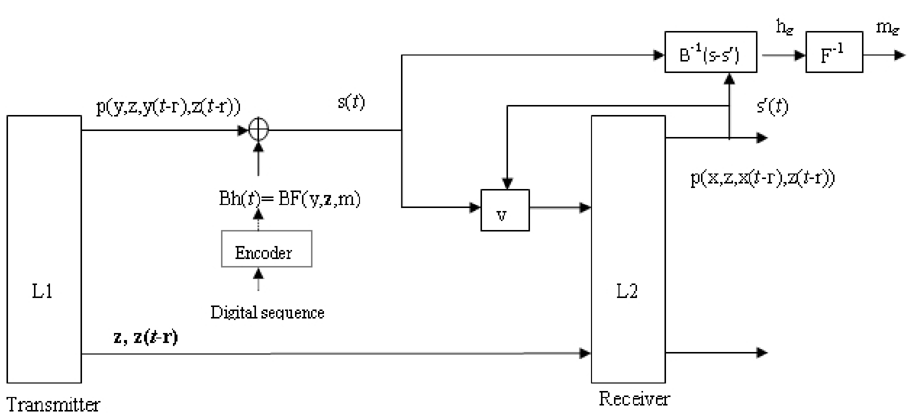

Fig. 1. The block diagram of our communication scheme based on coupled multiple-time-delay complex chaotic system.

Figure 1 shows the block diagram of our communication scheme. In practice, to decrease the complexity of controller, we always set $q = 1$. These will be illustrated in Fig. 1 and numerical simulation. The transmitted signal is denoted as $s = p(y, z, y(t-\tau), z(t-r)) + Bh$. The other state vectors $\boldsymbol{z}$ and $\boldsymbol{z(t-r)}$ are transmitted to the receiver. At the receiver end, the controller is $v = s - s'$. As x approaches y, the recovered signal $h_g \to h$. We can also employ $m(t)$ as the information signal, and $h(t) = F(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{m})$ is a function of information signal in order to increase the security of communication scheme.

4. Numerical simulation and discussion

We adopt the following coupled time-delay complex Lorenz system to describe $L1$ and $L2$:

$$L1: \begin{cases} \dot{y} &= -a_1 y + a_1 z_1(t - r_1) + Bh\,, \\ \dot{z}_1 &= a_2 y - y z_2 - z_1\,, \\ \dot{z}_2 &= -a_3 z_2(t - r_2) + (1/2)(\bar{y} z_1 + y \bar{z}_1)\,, \end{cases} \tag{19}$$

$$L2: \dot{x} = -a_1 x + a_1 z_1(t - r_1) + (s - s')\,, \tag{20}$$

where $y = u_1 + ju_2$, $z_1 = u_3 + ju_4$ are complex state variables of drive system $L1$, and $x = u_1' + ju_2'$ is the complex state variable of response system $L2$. The coupling variable is z_1 and the real variable $z_2 = u_5$.

We choose the initial condition $y(0) = -1 - 2j$, $z_1(0) = -3 - 4j$, $z_2(0) = 1$, $x(0) = 1 + 2j$, $r_1 = r_2 = 0.088s$, and $a_1 = 14, a_2 = 35, a_3 = 3.7$. The chaotic attractor projections and states of system (19) are shown in Fig. 2. The values of five Lyapunov exponents are $LE_1 = 1.235$, $LE_2 = 0.818$, $LE_3 = 0.367$, $LE_4 = 0.147$, $LE_5 = -0.765$. The number of positive Lyapunov exponent, which indicates the complexity of signals, is more than one. It indicates that the complex chaotic signals generated by time-delay complex chaotic systems are more complicated and difficult to predict. This problem had been discussed in Ref. [21] and is omitted here.

According to Fig. 1 and (18), the controller is designed as

$$\begin{aligned} v &= s - s' \\ &= k_1 y + Bh - k_1 x \\ &= -k_1 e_1 + Bh\,, \end{aligned} \tag{21}$$

where $e_1 = x - y$.

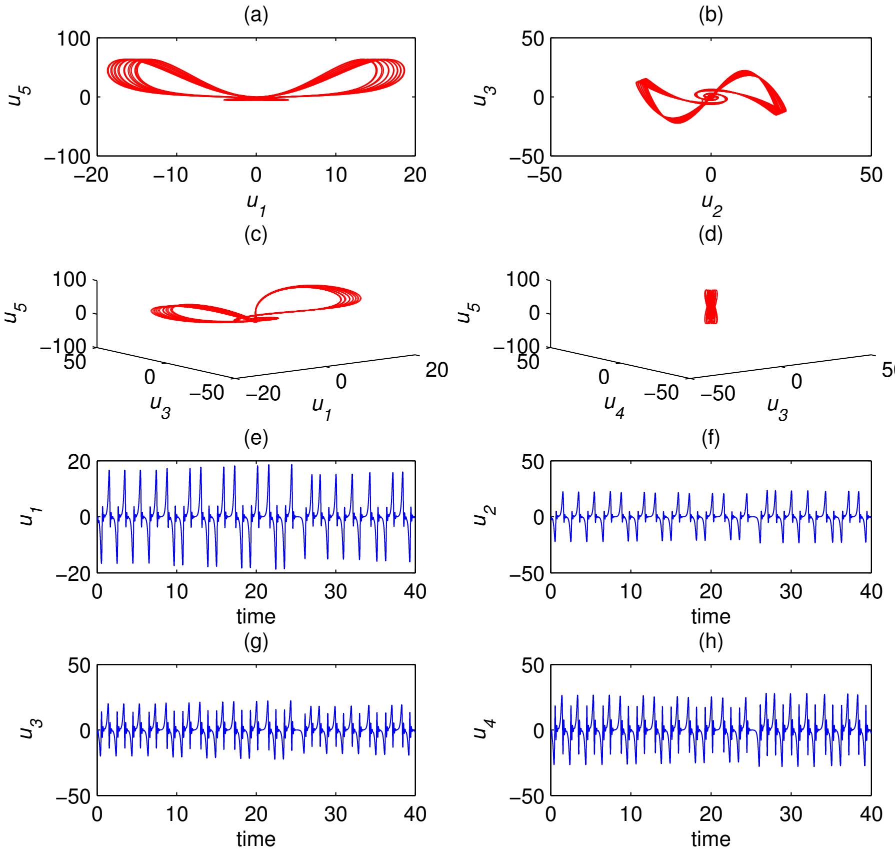

Fig. 2. The different chaotic attractor projections and states of two time-delay complex Lorenz system ($r_1 = r_2 = 0.088s$, $u_1(0) = -1$, $u_2(0) = -2$, $u_3(0) = -3$, $u_4(0) = -4$, $u_5(0) = 1$) (a) $u_1 - u_5$; (b) $u_2 - u_3$; (c) $u_1 - u_3 - u_5$; (d) $u_3 - u_4 - u_5$; (e) $u_1(t)$; (f) $u_2(t)$; (g) $u_3(t)$; (h) $u_4(t)$.

4.1. Analog signals

We transmit the famous melody *To Alice.wav*, written by Beethoven, with $B = 1$. It contains two components. These components $h^{\mathrm{r}}(t), h^{\mathrm{i}}(t)$ are read by `Matlab` software and transmitted by the real part and imaginary part of $s(t)$. We obtain that the transmitted signal $s(t)$ covers up completely the information signal shown in Fig. 3. The error vector of CS is depicted in Fig. 4, where the error vector converges quickly to zero as time increases. These results show that CS takes place with the melody. The information signal $h(t)$ and the recovered sinal $h_g(t)$ are depicted in Fig. 5. It is easy to find that the information signal $h(t)$ is recovered accurately.

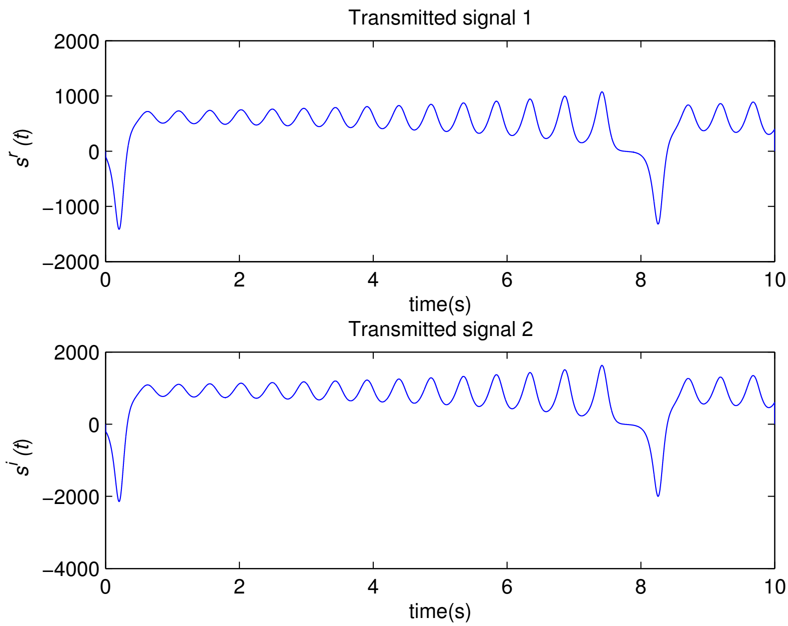

Fig. 3. The transmitted signal $s(t)$ for the melody *To Alice.wav*.

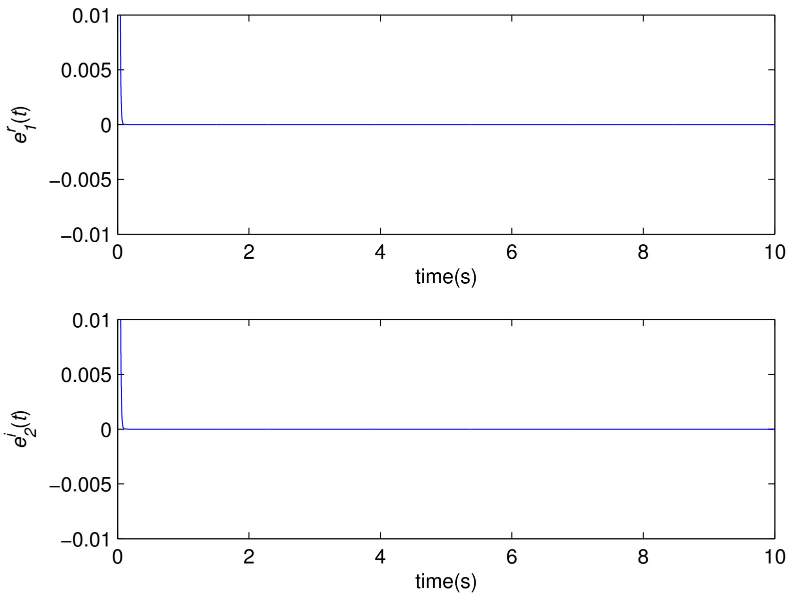

Fig. 4. The error vector of CS for the melody *To Alice.wav* which converges quickly to zero.

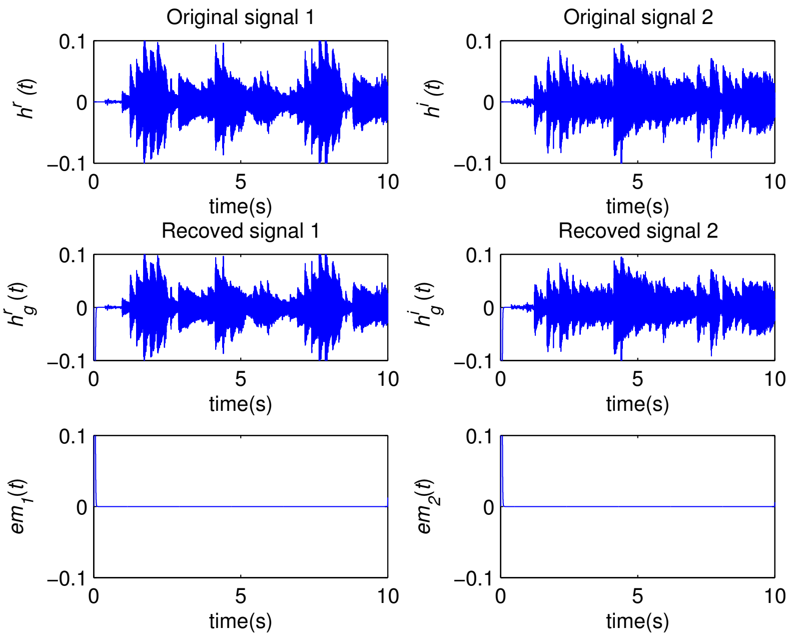

Fig. 5. The analog signal $h(t)$ extracted from the melody *To Alice.wav*, its recovered signal $h_g(t)$ and their error.

4.2. Digital signals

In this subsection, we transmit binary digital sequence which is produced by random function with `Matlab` software. From Fig. 1, M binary bits must first be converted into corresponding scaling factors by 2^M-ary. Here, the assignment of scaling factors to digital symbols is carried out by 16-ary, *i.e.*, $M = 4$. The symbol duration used here is equal to 4 time units. We adopt $B = 0.01$. The error vector of CS is depicted in Fig. 6, where the error vector converges quickly to zero as time increases. These results show that CS takes place with this binary digital sequence. The corresponding transmitted signal $s(t)$ is shown in Fig. 7, where $s(t)$ is an analogous signal and completely covers up the binary digital sequence. The digital sequence $h(kT)$ and the recovered sinal $h_g(kT)$ are also depicted in Fig. 7. It is easy to find that the digital sequence $h(kT)$ is recovered accurately.

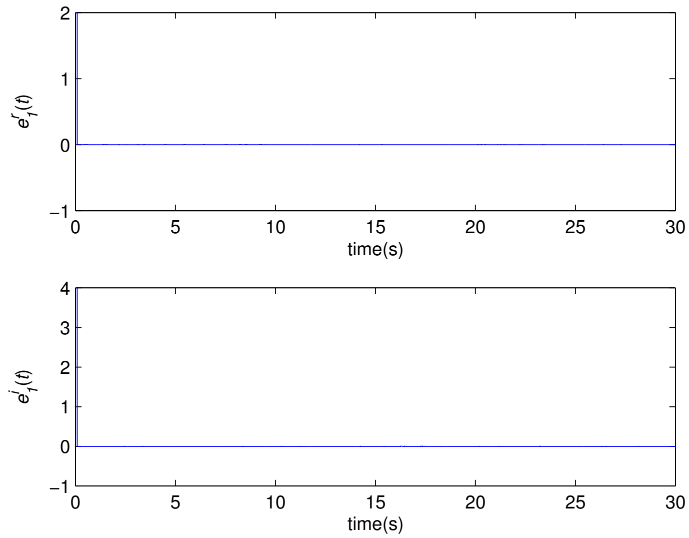

Fig. 6. The error vector of CS for binary digital sequence which converges quickly to zero.

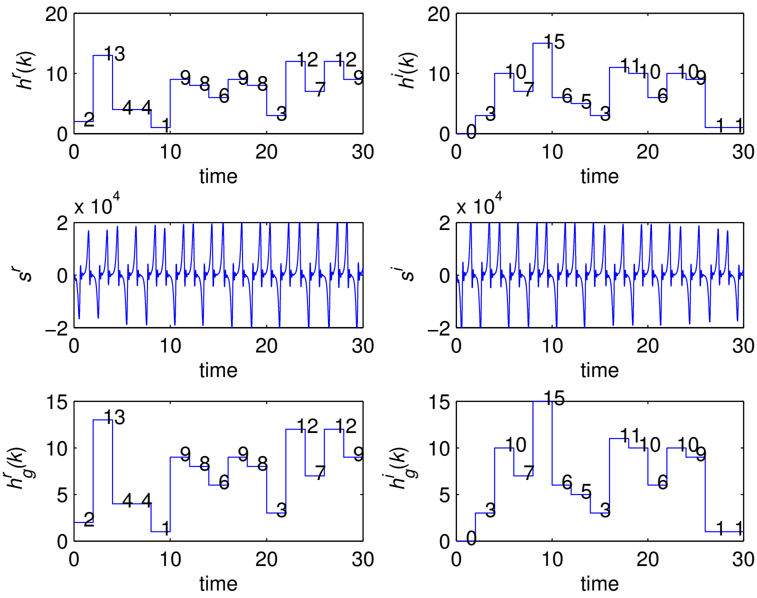

Fig. 7. The digital signals $h(k)$ of 2^4-ary and corresponding transmitted signals and its recovered signal $h_g(k)$.

5. Conclusion

Time delay is frequently encountered in a variety of practical chaotic systems, and it is inevitable at most times. The signals of complex chaotic systems with time delay are more complicated and difficult to decipher. Therefore, it is more suitable to describe actual chaotic systems as time-delay systems, especially in chaos communication. The paper investigates the complete synchronization of coupled multiple-time-delay complex chaotic systems and its application to chaos communication. The error feedback controller is extended to complex chaotic systems with time delay. The CS of real chaotic systems and complex chaotic systems without time delay are its special cases.

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REFERENCES

[1] A.C. Fowler, J.D. Gibbon, *Physica D* **4**, 139 (1982).

[2] G.M. Mahmoud, T. Bountis, E.E. Mahmoud, *Int. J. Bifurcat. Chaos* **17**, 4295 (2007).

[3] G.M. Mahmoud, E.E. Mahmoud, *Nonlinear Dyn.* **61**, 141 (2010).

[4] G.M. Mahmoud, E.E. Mahmoud, *Nonlinear Dyn.* **62**, 875 (2010).

[5] G.M. Mahmoud, E.E. Mahmoud, *Math. Comput. Simulat.* **80**, 2286 (2010).

[6] S.T. Liu, P. Liu, *Nonlinear Anal. RWA* **12**, 3046 (2011).

[7] P. Liu, S.T. Liu, *Phys. Scr.* **83**, 065006 (2011).

[8] H. Zhu, Adaptive modified function projective synchronization of a new chaotic complex system with uncertain parameters ICCRD, $3^{\rm rd}$rd Int. Conf. on Computer Research Development 4, Shanghai, China, March 11–13, 2011, pp. 451–455.

[9] P. Liu, S.T. Liu *et al.*, *Phys. Scr.* **85**, 035005 (2012).

[10] G.M. Mahmoud, E.E. Mahmoud, *Nonlinear Dyn.* **67**, 1613 (2012).

[11] E.E. Mahmoud, *J. Franklin Inst.* **349**, 1247 (2012).

[12] P. Liu, S.T. Liu, *Nonlinear Dyn.* **70**, 585 (2012).

[13] E.E. Mahmoud, *Math. Comput. Simulat.* **89**, 69 (2013).

[14] C. Luo, X.Y. Wang, *J. Franklin Inst.* **350**, 2646 (2013).

[15] F.F. Zhang, S.T. Liu, W.Y. Yu, *Chin. Phys. B* **22**, 120505 (2013).

[16] G.M. Mahmoud, E.E. Mahmoud, *Nonlinear Dyn.* **73**, 2231 (2013).

[17] S.T. Liu, F.F. Zhang, *Nonlinear Dyn.* **76**, 1087 (2014).

[18] C. Luo, X.Y. Wang, *J. Vib. Control.* **20**, 1831 (2014).

[19] J.W. Sun, G.Z. Cui, Y.F. Wang, Y. Shen, *Nonlinear Dyn.* **79**, 953 (2015).

[20] X.B. Zhou, L.L. Xiong, X.M. Cai, *Abstr. Appl. Anal.* **2014**, 953265 (2014).

[21] F.F. Zhang, S.T. Liu, W.Y. Yu, *Acta Phys. Sinica* **62**, 220505 (2013).

[22] F.F. Zhang, S.T. Liu, *Int. J. Mod. Phys. C* **25**, 1350102 (2014).

[23] O.I. Moskalenko, A.A. Koronovskii, A.E. Hramov, *Phys. Lett. A* **374**, 2925 (2010).

[24] H.J. Liu, X.Y. Wang, Q.L. Zhu, *Phys. Lett. A* **375**, 2828 (2011).

[25] C.C. Cheng, Y.S. Lin, S.W. Wu, *J. Franklin Inst.* **349**, 2626 (2012).

[26] M. Eisencraft *et al.*, *Commun. Nonlinear Sci. Numer. Simulat.* **17**, 4707 (2012).

[27] G.M. Mahmoud, E.E. Mahmoud, A.A. Arafa, *Phys. Scr.* **87**, 055002 (2013).

[28] L.M. Pecora, T.L. Carroll, *Phys. Rev. Lett.* **64**, 821 (1990).

[29] L. Glass, *Nature* **410**, 277 (2001).

[30] Y.N. Li, L. Chen, Z.S. Cai, X.Z. Zhao, *Chaos Soliton Fract.* **22**, 767 (2004).

[31] M. Lakshmanan, K. Murali, *Chaos in Nonlinear Oscillators: Controlling and Synchronization*, World Scientific, Singapore 1996.

[32] C. Hua, B. Yang, G.X. Ouyang, X.P. Guan, *Phys. Lett. A* **342**, 305 (2005).

[33] X.Y. Wang, M.J. Wang, *Commun. Nonlinear Sci. Numer. Simulat.* **14**, 1502 (2009).

[34] J.S. Lin, C.F. Huang, T.L. Liao, J.J. Yan, *Digit. Signal Process* **20**, 229 (2010).

[35] X.J. Wu, H. Wang, H.T. Lu, *Nonlinear Anal. RWA* **12**, 1288 (2011).

[36] N.W. Fang, *J. Anhui University Natural Science* **1**, 18 (1995).

[37] J. Hale, *Theory of Functional Differential Equations*, Springer, Berlin 1977.