ON THE "FORCE-FREE SURFACE" OF THE MAGNETIZED CELESTIAL BODIES

V. Epp[a,b], M.A. Masterova[a,c]

[a]Tomsk State Pedagogical University, Kievskaya, 60, 634061 Tomsk, Russia
[b]Tomsk State University, pr. Lenina, 36, 634050 Tomsk, Russia
[c]Tomsk Polytechnic University, pr. Lenina, 30, 634050 Tomsk, Russia

(Received February 10, 2015; revised version received May 7, 2015)

The field of a uniformly magnetized rotating sphere is studied with special attention to the surface, where the electric and magnetic fields are orthogonal to each other. The equation of this surface, valid at arbitrary distances from the rotating magnetized sphere, is obtained. Inside the light cylinder, this surface can be considered as a force-free surface, *i.e.* as a place where the particles with strong radiation damping can be trapped due to their energy loss. Outside the light cylinder, this surface makes just a geometric locus which moves with a superlight velocity around the axis of rotation. The 2- and 3-dimensional plots of the force-free surface are constructed. Estimation of influence of the centrifugal force on the particle dynamics is made. It is shown that in the case of strong magnetic field, the centrifugal force is negligible small everywhere except a narrow neighbourhood of the light cylinder.

DOI:10.5506/APhysPolB.46.1593
PACS numbers: 94.30.–d, 97.60.–s, 41.20.–q

1. Introduction

Investigations of the magnetosphere of magnetized celestial bodies is an important physical problem. The structure of the electromagnetic field, and dynamics of the charged particles in this field are the basic problems in understanding of the nature of pulsars radiation (*cf.* [1]).

The model of rotating neutron star with strong magnetic field was originally constructed by Goldreich and Julian [2]. They suggested that the charged particles are moving along magnetic field lines and are accelerated by the electric field component parallel to the magnetic field.

It was shown in Ref. [3] that over the polar caps of rotating magnetized body with aligned magnetic and rotational axes, there is a surface defined by equation $\boldsymbol{E} \cdot \boldsymbol{H} = 0$. The electric field does not accelerate the charged particles along the magnetic field lines in this region. By this reason, this surface is referred to as a force-free surface. It was shown that the particle potential energy is minimal at the force-free surface. It was suggested that the particles which undergo the radiative damping force, collect around this surface. If the axis of rotation coincides with the magnetic axis, the force-free surface forms a dome above the polar cap which rests on the surface of the star.

The force-free surface in the vicinity of an inclined magnetic rotator was studied numerically in Refs. [4–6]. The neutron star was considered as a homogeneously magnetized sphere rotating around an axes inclined with respect to the magnetic axes. It was shown by numerical integration of the particle trajectories that a relativistic particle is captured at the $\boldsymbol{E}\cdot\boldsymbol{H}=0$ surface if the damping force is higher than some definite critical value [4, 5]. The perspective views of the force-free surfaces for an oblique rotator were plotted using the vacuum fields of Deutsch [7]. The force-free surface for a charged magnetized sphere was studied by Biltzinger and Thielheim [6].

The analytical expression for the force-free surface was obtained by Istomin and Sobyanin [8] for the near-field region. It was shown that the radiation of charge particles causes the trapping of these particles at the force-free surface. Further motion of the particles consists of relativistic oscillations near the surface and of regular drift along the surface. Some part of the charged particles captured by the force-free surface can move along the surface away from the centre of the field. As the azimuthal motion under the Coriolis force is restricted by the dipole magnetic field, the particles are moving with approximately constant angular velocity co-rotating with the neutron star magnetosphere. The kinetic energy of the particles increases due to the growth of the azimuthal velocity with respect to an inertial reference frame. Thus accelerated charged particles leave the force-free surface in the vicinity of the light cylinder, maintaining the pulsar wind [9, 10]. In order to study this process, we have to know the geometry of the force-free surface not only in the vicinity of the neutron star, but throughout the entire space inside of the light cylinder.

In the following, the force-free surface is considered at the distances up to the light cylinder and some remarks are made about the surface beyond the cylinder. In Section 2, we obtain the equations of the force-free surface for the electromagnetic field of the nonrelativistic inclined magnetized sphere. The geometry of the force-free surface is studied in Section 3. We summarize our results and discuss the further possible developments in Section 4.

A few words about notations. Greek letters are reserved to label space-time indices and run the set of values $0,1,2,3$, while Latin letters correspond to only spatial indices $1,2,3$. Throughout the work, summation over repeated indices is implied.

2. Equation of the force-free surface

The electromagnetic field of a neutron star can be approximated by the field of inclined rotating magnetized sphere [1, 7, 11]. Taking into account that a solid sphere is incompatible with the theory of relativity (see related discussion in [12, 13] and references therein), we consider the electromagnetic field of a nonrelativistic rotating body. This field is described by the 4-vector potential A^ν, which in the coordinate system with temporal t and spherical r, θ, φ coordinates, has the following components [13] (axis Z is directed along the vector of angular velocity)

$$\begin{aligned} A^0 &= -\frac{\omega r_0^2 \mu}{6cr^3}\left[3C\sin 2\theta \sin\alpha + \cos\alpha(3\cos 2\theta + 1)\right] , \\ A^1 &= 0\,, \\ A^2 &= -\frac{\mu}{r^3} S \sin\alpha\,, \\ A^3 &= \frac{\mu}{r^3 \sin\theta}(\cos\alpha \sin\theta - C\sin\alpha\cos\theta)\,, \end{aligned} \tag{1}$$

where $S = \sin\lambda - \rho\cos\lambda$, $C = \cos\lambda + \rho\sin\lambda$, $\lambda = \varphi - \omega t + \rho = r\omega/c$, μ is the module of the dipole moment vector of the magnetized sphere rotating with angular frequency ω and forming an angle α between the dipole vector and axis Z, r_0 is the radius of the sphere, and c is the speed of light. The field described by the potential (1) coincides with the field of Deutsch [7] in approximation of a nonrelativistic sphere ($r_0\omega/c \ll 1$).

Let us find the equation of the surface specified by condition $\boldsymbol{E}\cdot\boldsymbol{H} = 0$. Note that time and the azimuthal angle φ are represented in the formula for potential only in combination $\phi - \omega t$. That means that the equation of the force-free surface in the co-rotating reference frame does not depend on time. On the other hand, the equation $\boldsymbol{E}\cdot\boldsymbol{H} = 0$ can be written in a covariant form $F^{\mu\nu}F_{\mu\nu} = 0$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. Hence, the shape of the surface does not depend on choice of the reference frame. The derivation of the force-free surface equation in the rotating reference frame is slightly more complicated and involves the issues of definition of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ in the noninertial reference frame. We shall develop this equation in the co-rotating reference frame, just to show how it works. Besides, this reference frame is more convenient for numerical calculations and graphical illustrations which we present in Sections 3 and 4.

We define the coordinates in the co-rotating reference frame as follows: $x^{0'} = ct, x^{1'} = r, x^{2'} = \theta, x^{3'} = \psi = \varphi - \omega t$. The vector potential (1) in this frame reads:

$$
\begin{aligned}
A_{0'} &= \frac{\mu\omega}{cr}\left[C\frac{\sin 2\theta}{2}\sin\alpha\left(1-\frac{r_0^2}{r^2}\right)-\sin^2\theta\cos\alpha\left(1-\frac{r_0^2}{r^2}\right)-\frac{2}{3}\cos\alpha\frac{r_0^2}{r^2}\right],\\
A_{1'} &= 0\,,\\
A_{2'} &= \frac{\mu}{r}S\sin\alpha\,,\\
A_{3'} &= \frac{\mu\sin\theta}{r}(C\sin\alpha\cos\theta-\sin\theta\cos\alpha)\,. \qquad (2)
\end{aligned}
$$

This gives the following components of electromagnetic field tensor $F_{\mu'\nu'} = \partial_{\mu'}A_{\nu'} - \partial_{\nu'}A_{\mu'}$

$$
\begin{aligned}
F_{0'1'} &= -\frac{3\omega r_0^2\mu}{2cr^3}\left[\sin 2\theta\sin\alpha\left(C-\rho^2\frac{\cos\lambda}{3}\right)+\cos\alpha(3\cos 2\theta+1)\right]\\
&\quad -\frac{\mu\omega}{cr^2}\left[\sin^2\theta\cos\alpha-\frac{\sin 2\theta}{2}\sin\alpha\left(C-\rho^2\cos\lambda\right)\right],\\
F_{0'2'} &= \frac{\mu\omega}{cr}\left[\frac{r_0^2}{r^2}\left(C\cos 2\theta\sin\alpha-\sin 2\theta\cos\alpha\right)-C\cos 2\theta\sin\alpha+\sin 2\theta\cos\alpha\right],\\
F_{0'3'} &= \frac{\mu\omega}{2cr}S\sin 2\theta\sin\alpha\left(1-\frac{r_0^2}{r^2}\right),\\
F_{1'3'} &= \frac{\mu}{r^2}\sin\theta\left[\sin\theta\cos\alpha-\cos\theta\sin\alpha\left(C-\rho^2\cos\lambda\right)\right],\\
F_{1'2'} &= -\frac{\mu}{r^2}\sin\alpha\left(S-\rho^2\sin^2\lambda\right),\\
F_{2'3'} &= -\frac{2\mu}{r}\sin\theta(C\sin\theta\sin\alpha+\cos\theta\cos\alpha)\,. \qquad (3)
\end{aligned}
$$

The 3 vectors of the electric field E_i and of the magnetic field H_i in arbitrary curved spacetime are defined by expressions [14, 15] (further on, the primes are omitted)

$$
E_i = F_{0\,i}\,, \qquad B^i = -\frac{1}{2\sqrt{\gamma}}\varepsilon^{ijk}F_{jk}\,, \qquad (4)
$$

where ε_{ijk} is the completely antisymmetric Levi-Civita tensor, γ is the determinant of the space metric tensor

$$
\gamma_{ij} = -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}}\,.
$$

The force-free surface is defined by equation $E_i B^i = 0$. Substituting the components $F_{\mu'\nu'}$ from (3), we obtain

$$4\cos\theta\Big\{\cos\theta\cos\alpha+\sin\theta\sin\alpha\big[\cos(\psi+\rho)+\rho\sin(\psi+\rho)\big]\Big\}^2$$
$$+\sin\alpha\Big\{\big[\cos(\psi+\rho)+\rho\sin(\psi+\rho)\big]\cos\alpha\sin\theta-\sin\alpha\cos\theta\Big\}\left(\frac{\rho^2}{a^2}-1\right)=0\,, \tag{5}$$

where the notation $a = \omega r_0/c$ is introduced. Equation (5) defines the force-free surface in the rotating reference frame for the inner area of the light cylinder, *i.e.* for $\rho < 1$.

In the case of $\rho \ll 1$, this yields

$$\rho^2 = a^2\left(1-4\frac{\cos\theta[\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos\psi]^2}{\sin\alpha[\cos\psi\cos\alpha\sin\theta-\sin\alpha\cos\theta]}\right). \tag{6}$$

This expression was originally obtained in paper [8]. As one can see, the geometry of the force-free surface does not depend on a in this approximation. The radius of the sphere plays a role of a scale factor. However, if ρ is comparable with unity, the force-free surface has a more complicated form. In particular, it is wound up around the Z-axis due to the argument $\psi+\rho$ of the trigonometric functions in equation (5).

One can easily check that the equation of the force-free surface in the inertial reference frame is defined by the same equation (5) except the substitution $\psi = \varphi - \omega t$. However, in contrast to equation (5), the equation in the rest reference frame holds both in the inner space of the light cylinder and beyond the light cylinder[1].

3. Geometry of the force-free surface

The value of a in Eq. (5) depends sufficiently on the angular velocity of a celestial body. The greatest values of a belong presumably to pulsars. For example, $a \approx 0.2$ for the pulsar PSR J1748-2446ad. We have plotted the views of the force-free surface for different values of radius a and the inclination angle α using equation (5). In the case of the aligned rotator ($\alpha = 0$), the equation of the force-free surface reads: $\cos\theta = 0$. This equation holds in the total equatorial plane $\theta = \pi/2$.

[1] In this connection, a question arises — how the force-free surface behaves in the far-field zone? As is well-known, the vector $\boldsymbol{E}$ is orthogonal to the vector $\boldsymbol{H}$ everywhere in this zone and, consequently, the force-free surface has to fill all the space. Can the two-dimensional surface transform into the three-dimensional space? Answers to these questions are given in Appendix.

The structure of the force-free surface for $a^2 = 0.1$ and $a^2 = 0.001$, and for angle $\alpha = \pi/3$ is shown in Fig. 1. The sectional views of the surface $\boldsymbol{E} \cdot \boldsymbol{H} = 0$ for $a^2 = 0.1$ and different values of angle α are represented in Fig. 2. The plots in the first row look similar to those drawn in Refs. [4, 8] for small ρ, but figures in the second row differ essentially, especially for $\alpha = \pi/6$. This difference is caused by the fact that in the case of ρ comparable with the unity, one has to take into account the variable ρ in argument of the trigonometric functions in Eq. (5). Due to this argument, the force-free

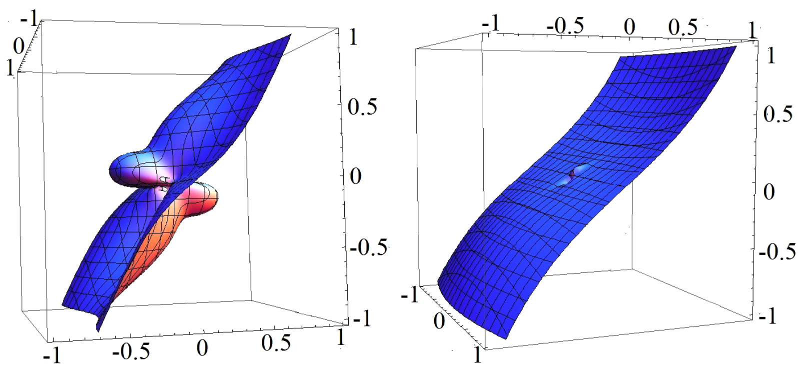

Fig. 1. The force-free surface for $\alpha = \frac{\pi}{3}$; $a^2 = 0.1$ (left) and $a^2 = 0.001$ (right).

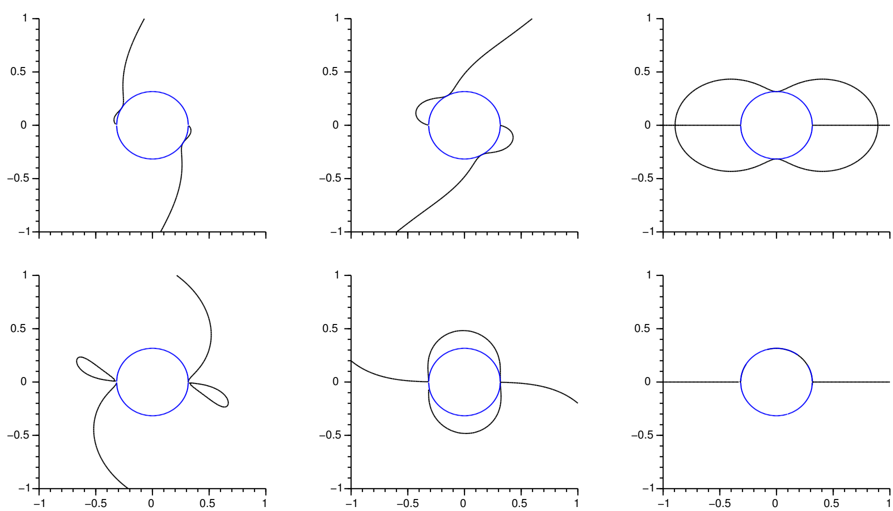

Fig. 2. Sectional views of the force-free surface for $a^2 = 0.1$ in the plane $\psi = 0, \pi$ (first row), and in the plane $\psi = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ (second row); $\alpha = \dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{\pi}{2}$, with α increasing from left to right.

surface becomes twisted around the Z-axis. This is clearly seen in the equatorial sectional view in Fig. 3. The curve in this cutting section is determined by the equation

$$\left[\cos(\varphi - \omega t + \rho) + \rho \sin(\varphi - \omega t + \rho)\right] \left(\frac{\rho^2}{a^2} - 1\right) = 0$$

for a fixed instant of t. Its shape does not depend on the inclination angle α. The surface $\boldsymbol{E} \cdot \boldsymbol{H} = 0$ is represented by the circle of radius a and the curve outside this circle. The dependence of the force-free surface on a is graphically represented in Fig. 4 as sectional views for $0.02 < a < 0.3$.

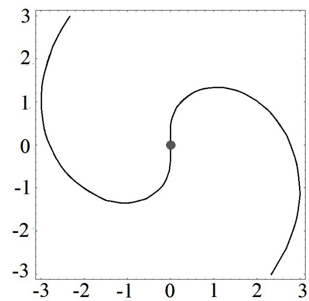

Fig. 3. Sectional view of the force-free surface in the plane $\theta = \pi/2$.

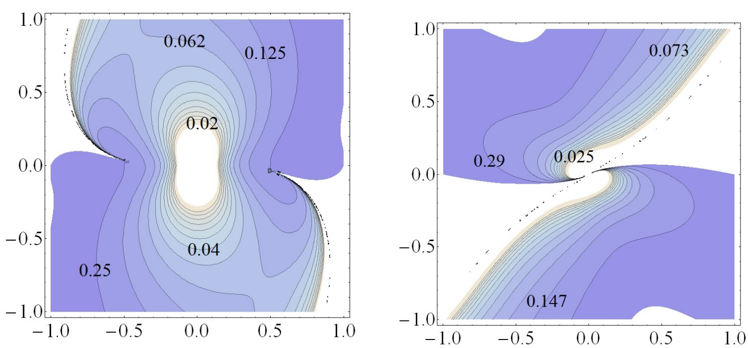

Fig. 4. Sectional view of the force-free surface for $\alpha = \dfrac{\pi}{6}$ in the plane $\psi = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ (left), and in the plane $\psi = 0, \pi$ (right) for different values of the radius of the sphere a. The values of a are marked in the figures.

4. The centrifugal force

The force-free surface is of interest because the particles under strong radiation damping can be trapped in regions where $\boldsymbol{E} \cdot \boldsymbol{H} = 0$. Making this statement we do not take into account that the particles, co-rotating with the star, undergo the centrifugal force. In the near region of an oblique rotator and in the case of strong magnetic field, the centrifugal force can be neglected. However, in the regions near the light cylinder, the centrifugal force affects sufficiently the particle dynamics. Hence, the surface $\boldsymbol{E}\cdot\boldsymbol{H}=0$ is no longer a force-free one.

Let us estimate the distance at which the centrifugal force should be taken into account. It follows from the relativistic Hamiltonian function in the rotating reference frame that the potential of the centrifugal force is equal to $U=mc^2\sqrt{1-R^2}$, where $R=\rho\sin\theta$ is the distance from the axis of rotation in units of the light cylinder radius. The module of the centrifugal force in dimensionless representation reads

$$F_{\rm c}=\frac{R}{\sqrt{1-R^2}}\,. \tag{7}$$

The longitudinal with respect to $\boldsymbol{H}$ component of the electric field acts on the particle by force

$$F_{\rm e}=eE_H=e\frac{(\boldsymbol{E}\boldsymbol{H})}{H}\,, \tag{8}$$

which, in the same dimensionless representation, has the form

$$F_{\rm e}\sim\frac{N}{\rho}\,,\qquad N=\frac{e\mu\omega^2}{mc^4}\,. \tag{9}$$

Then, the centrifugal force and the force of electromagnetic interaction are in the ratio

$$\frac{F_{\rm c}}{F_{\rm e}}\sim\frac{R\rho}{N\sqrt{1-R^2}}\,. \tag{10}$$

It is readily seen that in the case of small N, the centrifugal force can be neglected if $R\ll N$. But in the case of large N (which is typical for the neutron stars), this force can be neglected everywhere with the exception of the small vicinity of the light cylinder: $1-R\sim N^{-1}$.

5. Conclusions

We have obtained in the preceding sections the equation for the surface $\boldsymbol{E}\cdot\boldsymbol{H}=0$ valid at arbitrary distances from the rotating magnetized sphere. This surface co-rotates with the magnetized sphere as a unit. Inside the light cylinder, this surface can be considered as a force-free surface, *i.e.* as a place where the particles with strong radiation damping can be trapped due to their energy loss as described in Refs. [4, 6, 8]. Outside the light cylinder, the surface $\boldsymbol{E}\cdot\boldsymbol{H}=0$ is just a geometric locus which moves with superlight velocity (see Appendix).

The main purpose of this paper is to show that in the case of highly magnetized sphere ($N \gg 1$, with N defined by Eq. (9)), the surface $\boldsymbol{E}\cdot\boldsymbol{H} = 0$ can be used in the study of the particles dynamics in the entire space within the light cylinder, not only in the region $\rho \ll 1$. In this case, one has to take into account that the force-free surface is twisted around the rotational axis, if we consider the distances $\rho \sim 1$. The only exception is the thin belt at the inner side of the light cylinder ($1 - N^{-1} \lesssim R < 1$) where the centrifugal force is comparable with or greater than the force of electromagnetic interaction. In the case of weak electromagnetic field ($N \ll 1$), the surface $\boldsymbol{E}\cdot\boldsymbol{H} = 0$ can be considered as a force-free one only in the region of $R \ll N$. Otherwise, the centrifugal force should be taken into account.

The particles captured by the force-free surface inside the light cylinder are leaving this surface in the vicinity of the light cylinder, forming an outward flow of relativistic particles. Since the force-free surface co-rotates with the neutron star, this flow of particles has the shape of a sequence of expanding quasi-spherical surfaces having the equatorial section shown in Fig. 3. This flow of particles resembles the pulsar wind, described in details in [9]. Of course, the results of this paper are not applicable to magnetosphere filled with dense plasma.

The work was supported by the Ministry of Education and Science of the Russian Federation, project No. 867.

Appendix

Although the surface $\boldsymbol{E}\cdot\boldsymbol{H} = 0$ makes no sense in study of the particle dynamics in the far-field region, it is interesting from the academic point of view to investigate the asymptotic $\rho \to \infty$.

The scalar product $\boldsymbol{E}\cdot\boldsymbol{H}$ obtained from the Deutsch field [7] in approximation $\rho \gg 1$ is, evidently, equal to zero because this field corresponds to radiation. In order to find the equation of a force-free surface when $\rho \gg 1$, we consider equation (5) in an inertial reference frame. If we take into account only the components which are proportional to the highest power of ρ, then we obtain the following equation

$$\cos\alpha \sin\theta \sin\alpha \sin(\varphi - \omega t + \rho) = 0\,.$$

If $\alpha \neq 0, \pi/2, \pi$ and $\theta \neq 0, \pi$, we have the equation of a spiral which rotates with the angular velocity ω around the Z-axis

$$\rho = \omega t - \varphi\,. \tag{11}$$

The pitch of the spiral is equal to $\Delta\rho = 2\pi$ or, in dimensional units, $\Delta r = 2\pi c/\omega$ which is equal to the wave length of dipole radiation. As we see, the force-free surface has a form of a surface which wounds on the Z-axis in such a way that the line of its intersection with an arbitrary cone $\theta = \text{const}$ is a spiral described by equation (11). Since equation (11) does not depend on the angle θ, each turn of this surface is close to a sphere. Rotation of such quasi-spherical spiral surface produces a picture of expanding set of concentric spheres, each of them can be considered as a wave-front of a radiation field.

Hence, the "force-free" surface does not fill all the space in the wave zone. Strictly speaking, the electric field is not orthogonal to magnetic one in between the adjacent windings of the surface $\boldsymbol{E}\cdot\boldsymbol{H} = 0$ even in the far-field region. Claiming that $\boldsymbol{E}\cdot\boldsymbol{H} = 0$ in the wave zone, we are neglecting the terms in electromagnetic field equations, which decrease faster than $1/r$. If we take into account these terms, we shall see that there is a component of $\boldsymbol{E}$ parallel to $\boldsymbol{H}$, but it is of the order of λ/r, where λ is the wave length of the dipole radiation.

REFERENCES

[1] F.C. Michel, *Theory of Neutron Star Magnetospheres*, The University of Chicago Press: Chicago and London 1991.

[2] P. Goldreich, W.H. Julian, *Astrophys. J.* **157**, 869 (1969).

[3] E.A. Jackson, *Astrophys. J.* **206**, 831 (1976).

[4] B. Finkbeiner, H. Herold, T. Ertl, H. Ruder, *Astron. Astrophys.* **225**, 479 (1989).

[5] K.O. Thielheim, H. Wolfsteller, *Astrophys. J. Suppl. Ser.* **71**, 583 (1989).

[6] P. Biltzinger, K.O. Thielheim, `arXiv:astro-ph/0011306`.

[7] A.J. Deutsch, *Ann. d'Astrophys.* **18**, 1 (1955).

[8] Ya.N. Istomin, D.N. Sobyanin, *J. Exp. Theor. Phys.* **109**, 393 (2009).

[9] J.G. Kirk, Y. Lyubarsky, J. Petri, in: *Neutron Stars and Pulsars*, *Astrophys. Space Sci. Lib.* **357**, 421 (2009).

[10] M. Kutschera *et al.*, *Acta Phys. Pol. B* **35**, 1837 (2004).

[11] H.W. Babcock, T.G. Cowling, *Mon. Not. R. Astron. Soc.* **113**, 356 (1953).

[12] V. Epp, M.A. Masterova, *Astrophys. Space Sci.* **345**, 315 (2013).

[13] V. Epp, M.A. Masterova, *Astrophys. Space Sci.* **353**, 473 (2014).

[14] L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, Pergamon, N.Y. 1975.

[15] C. Møller, *The Theory of Relativity*, Oxford University Press, Oxford 1976.