CONVERGENCE OF A CLASS OF HANKEL-TYPE MATRICES*

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Let H_n be the $n \times n$ symmetric Hankel-type matrix whose $(i,j)^{\rm th}$ element on the $k^{\rm th}$ anti-diagonal (where $k = 0$ denotes the main anti-diagonal) is defined as: $H_{k,n}(i,j) = g_k(\frac{i-[\frac{n+k+1}{2}]}{n})$ if $i+j = n+1+k$ and 0 otherwise. Under suitable symmetry and summability conditions on $\{g_k\}$, we show that as $n \to \infty$, the limiting spectral distribution of $\{H_n\}$ exists and is given by $\sum_{k=-\infty}^{\infty} g_k(U)a_k$, where U is uniformly distributed on $[-1/2,\ 1/2]$ and is tensor-independent of the non-commutating variables $\{a_k\}$ which are certain symmetric pair-wise free but not completely free Bernoulli variables.

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1. Introduction

Let $\{A_{k,n}\}$, $k = 1, 2, \ldots, K$ be K sequences of $n \times n$ matrices. Then, as elements of the non-commutative probability space of $n \times n$ complex matrices with the state as average trace, they are said to converge jointly (as $n \to \infty$), if for every polynomial $P(A_{k,n},\ A^*_{k,n},\ k \leq K)$, the average trace converges. Here, A^* denotes the complex conjugate of A. The limit non-commutative (polynomial) $*$-algebra is defined by the non-commutative indeterminates (limit variables) $\{a_k\}$, where the state ϕ satisfies $\phi(P(a_k,\ a^*_k,\ k \leq K)) = \lim \frac{1}{n} \operatorname{Tr}(P(A_{k,n},\ A^*_{k,n},\ k \leq K))$ for all polynomials P. The limit non-commutative joint distribution of $\{a_k\}$ is defined as the collection of all the joint moments $\phi(a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \ldots a_{i_n}^{\epsilon_n})$ for all $1 \leq i_1, i_2 \ldots, i_n \leq K, n \geq 1$ and $\epsilon_i \in \{1, *\}$.

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When we have only one sequence of matrices, say $\{A_n\}$ (which are, for simplicity, real symmetric), then there is a related notion of convergence. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A_n. Then, the *Empirical Spectral Distribution Function* (ESD) of A_n equals

$$F_{A_n}(x) = n^{-1} \sum_{i=1}^{n} \mathbb{I}\{\lambda_i \leq x\} .$$

As $n \to \infty$, the *Limiting Spectral Distribution* (LSD) of $\{A_n\}$ is defined as the weak limit F of $\{F_{A_n}\}$, if it exists. We identify F with any random variable X whose distribution is F. This definition extends to non-symmetric matrices with complex entries in the obvious way.

It is easy to construct examples of real symmetric matrices $\{A_n\}$ where the LSD exists but there is no convergence in the non-commutative sense (that is, $\lim \frac{1}{n} \operatorname{Tr}(A_n^k)$ does not exist for some k). On the other hand, by using the moment-trace formula, it is also easy to see that if the real symmetric $\{A_n\}$ converges in the non-commutative sense (that is, $\lim \frac{1}{n} \operatorname{Tr}(A_n^k) = \mu_k$ exists for all positive integers k), and if $\{\mu_k\}$ is the moment sequence of a unique probability distribution F, then the LSD of A_n equals F.

Let U_1 and U_2 be i.i.d. random variables, uniformly distributed on the interval $(0,\ 1)$. The famous Szegö's theorem implies that if $T_n := ((t_{|i-j|}))_{1 \leq i,j \leq n}$ is the Toeplitz matrix and $\{t_k\}$ is square summable, then the LSD of T_n equals $t_0 + 2\sum_{k=1}^{\infty} t_k \cos(2\pi k U_2)$. This result was extended to the Toeplitz-type matrix $T_{n,g}$ say, where the elements of the k^{th} upper and lower diagonals, instead of being the constant t_k, are of the form of $g_k(i/n)$ in the i^{th} row for some suitable functions g_k (see [1, 2]). The limit in this case equals $g_0(2\pi U_1) + 2\sum_{k=1}^{\infty} g_k(2\pi U_1) \cos(2\pi k U_2)$.

The related Hankel matrix $H_n = ((h_{i+j}))$ and the corresponding Hankel operator has been extensively treated in the literature. See [3–6] for detailed information. Note that the elements on each *anti-diagonal* of H_n are identical. However, while in T_n the constant on the main diagonal does not change with n, the main anti-diagonal in H_n is h_{n+1}. We take a cue from this observation and the matrix $T_{n,g}$, to consider the following class of Hankel-type matrices.

In our convention of labelling the anti-diagonals, $k = 0$ refers to the main anti-diagonal and $k = 1, 2, \ldots$ denote the successive anti-diagonals below the main anti-diagonal and, similarly, the negative integers label the upper anti-diagonals. For each k, first consider the Hankel matrix $D_{k,n}$ whose k^{th} anti-diagonal elements equal one and the rest of the elements are zero. These matrices converge jointly. The non-commutative joint distribution of the limit variables $\{a_k\}$ can be described in terms of the non-commutative moments as

$$\phi\left(a_{i_1}\ldots a_{i_k}\right)=\begin{cases}\mathbb{I}_{\{i_1+i_3+\cdots+i_{2m-1}=i_2+i_4+\cdots+i_{2m}\}} & \text{if } k=2m \text{ for some } m\geq 1\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{1.1}$$

Interestingly, the above $\{a_k\}$ are symmetric Bernoulli and are pair-wise free but not completely free. This is easy to check by using the above description.

Now generalise $D_{k,n}$ as follows. Let $g_k : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ be continuous and symmetric about 0; let $H_{k,n}$ be the $n\times n$ Hankel-type matrix whose $(i,j)^{\text{th}}$ element is defined as

$$H_{k,n}(i,j)=\begin{cases} g_k\left(\frac{i-\left[\frac{n+k+1}{2}\right]}{n}\right) & \text{if } \; i+j=n+1+k\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{1.2}$$

Note that unlike the Hankel matrices considered usually in the literature, for example [5, 6], where the main anti-diagonal has the variable h_{n+1} which changes as n changes, in our case the main anti-diagonal has elements of $g_0(\cdot)$ which is a fixed function. So the labelling is different.

We show that $\{H_{k,n}\}$ converge jointly and the limit variables are $\{g_k(U)a_k\}$, where U is uniformly distributed on $[-1/2, 1/2]$ and is tensor independent of $\{a_k\}$. As a consequence, for any $K\geq 1$, $\sum_{|k|\leq K} H_{k,n}$ converges in the (algebraic sense) and the LSD of this real symmetric matrix exists and equals $\sum_{|k|\leq K} g_k(U)a_k$ with distribution $\tilde{F}_K$ say.

Finally, consider the full Hankel-type matrix $H_n=\sum_{|k|\leq n} H_{k,n}$. By imposing suitable restrictions on the functions $\{g_k\}$, H_n is approximated by the finite-diagonal matrix $\sum_{|k|\leq K} H_{k,n}$ in an appropriate metric and this helps us to conclude that the LSD of H_n exists under these conditions on $\{g_k\}$. The limit distribution function is the weak limit of $\tilde{F}_K$ as $K\to\infty$ and may be formally expressed as $\sum_{k=-\infty}^{\infty} g_k(U)a_k$. There does not seem to be any analytic description of the limit distribution function.

The case when the $\{g_k\}$ are not symmetric leads to a non-symmetric H_n. Studying the LSD of this matrix is an extremely difficult problem. We have made some elementary remarks on some special cases at the end of the article.

2. Preliminaries

A *non-commutative probability space* is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a unital algebra (with unity $\mathbf{1}$) and $\phi : \mathcal{A}\to\mathbb{C}$ is a linear functional satisfying $\phi(\mathbf{1}) = 1$. Elements of a non-commutative probability space will also be called (*non-commutative*) *random variables*. If an appropriate $*$ operation is defined on $\mathcal{A}$, then $(\mathcal{A}, \phi)$ is called a $*$-probability space[1]. A random variable $a \in \mathcal{A}$ is said to be self-adjoint if $a = a^*$ and unitary if $aa^* = a^*a = \mathbf{1}$. It is called Haar unitary if $\phi(a^k) = \mathbb{I}_{\{k=0\}}$.

For our purposes, we need the following $*$-probability space. Let $\mathcal{A}_n$ be the space of $n \times n$ symmetric random matrices with elements which are real numbers or are random variables with all moments finite. Then ϕ_n equal to $\frac{1}{n} \mathrm{E}_\mu[\mathrm{Tr}(\cdot)]$ or $\frac{1}{n}[\mathrm{Tr}(\cdot)]$ both yield a $*$-probability space.

For $\{b_i\}_{i \in J} \subset \mathcal{A}$, their *joint moments* is the collection $\{\phi(b_{i_1} b_{i_2} \dots b_{i_k}),\ k \geq 1\}$, where each $b_{i_j} \in \{b_i\}_{i \in J}$.

Random variables $\{b_{i,n}\}_{i \in J} \subset (\mathcal{A}_n, \phi_n)$ are said to *converge in law* to $\{b_i\}_{i \in J} \subset (\mathcal{A}, \phi)$ (as $n \to \infty$) if each joint moment of $\{b_{i,n}\}_{i \in J}$ converges to the corresponding joint moment of $\{b_i\}_{i \in J}$. That is, if for $k \geq 1$,

$$\phi_n \left[P\left(b_{i_1,n}, b_{i_2,n}, \dots, b_{i_k,n}\right)\right] \to \phi\left[P\left(b_{i_1}, b_{i_2}, \dots, b_{i_k}\right)\right]$$

for all polynomials P. If this happens, we write

$$\{b_{i,n}\}_{i \in J} \xrightarrow{\phi_n} \{b_i\}_{i \in J}\,.$$

If the random variables $\{b_{i,n}\}_{i \in J}$ are $n \times n$ (non-random) matrices, then the above convergence is assumed to be with respect to $\phi_n = \frac{1}{n}\mathrm{Tr}$. If, instead, they are random matrices, then the above convergence is in one of the following two senses:

(i) We say that $\{b_{i,n}\}_{i \in J}$ converges to $\{b_i\}_{i \in J}$, if the convergence holds with respect to $\phi_n = \frac{1}{n}\mathrm{E}\,\mathrm{Tr}$.

(ii) We say $\{b_{i,n}\}_{i \in J}$ converges almost surely to $\{b_i\}_{i \in J}$, if the convergence holds with respect to $\phi_n = \frac{1}{n}\mathrm{Tr}$, almost surely.

3. Hankel-type finite-diagonal matrices

Let $\{g_k\}_{-\infty < k < \infty}$ be a two-sided sequence of functions, such that for each k, $g_k : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$, g_k is continuous and symmetric about 0. Let $H_{k,n}$ be the $n \times n$ Hankel-type matrix defined in (1.2). When $g_k \equiv 1$, $H_{k,n}$ is the Hankel matrix with all entries 0, except the entries on the k^{th} anti-diagonal which are all assumed to be 1. Note that counted from the main anti-diagonal, k positive (negative) refers to the lower (respectively upper) anti-diagonal. We call this matrix $D_{k,n}$.

To describe the joint limit of $H_{k,n}$, let $(\mathcal{A}, \phi)$ be a $*$-probability space, and let $\{a_i\}_{i \in \mathbb{Z}} \subset \mathcal{A}$ be a sequence of self-adjoint and unitary elements such that $\phi(a_{i_1} \dots a_{i_k})$ is as defined in (1.1).

[1] Since we shall, mostly, be dealing with only real symmetric matrices, all our algebras, unless otherwise stated, are real.

It is then not hard to see that, a_i's are distributed as symmetric Bernoulli and are pair-wise freely independent but not totally free.

Theorem 3.1. *For any K, $(H_{k,n}, |k| \leq K)$ jointly converge to $(g_k(U)a_k, |k| \leq K)$ where $\{a_k\}$ are elements of a $*$-probability space $(\mathcal{A}, \phi)$, ϕ as defined in (1.1) and U is a random variable, uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$ and independent (in the classical sense) of $\mathcal{A}$. In particular, the LSD of $\sum_{|j|\leq K} H_{j,n}$ equals $\sum_{|j|\leq K} g_j(U)a_j$.*

Before we prove the above theorem, we state and prove a corollary.

Corollary 1. *$(D_{k,n}, |k| \leq K)$ converge jointly to $(a_k, |k| \leq K))$ where a_k are as in (1.1). In particular, for real numbers $\{h_k, |k| \leq K)$, the LSD of $\sum_{|k|\leq K} h_k D_{k,n}$ equals $\sum_{|k|\leq K} h_k a_k$. For any $s \neq t$, the LSD of $D_{s,n} + D_{t,n}$ is the arc-sine law and $D_{s,n}D_{t,n}$ is asymptotically Haar unitary.*

Proof. The joint convergence follows from Theorem 3.1. By that theorem, all moments of the ESD converge. Note that these moments determine a distribution uniquely which is as given in the statement of the corollary. Finally, it just suffices to observe that for any $s \neq t$, the $*$-distribution of $a_s + a_t$ is the arcsine law as it is a free convolution of symmetric Bernoulli (see, [7, pp. 200–202]) and that $a_s a_t$ is Haar unitary.

Proof of Theorem 3.1. First note that if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $D_{k,n}$, then $\forall i = 1, \ldots, n$, $\lambda_i \in \{-1, 0, 1\}$ and 0 has algebraic multiplicity $|k|$ and multiplicity of 1 and -1 are equal as $n \to \infty$. So ESD of $D_{k,n}$ converges to the random variable $a_k = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, *i.e.*, a_k is symmetric Bernoulli.

Let, for any fixed s, $T_{s,n}$ denote the $n \times n$ Toeplitz matrix whose entries are all zero except those on the s^{th} diagonal which equal 1, index s being counted from the main diagonal ($s = 0$) and $s = \pm 1, \ldots$ above and below the main diagonal respectively.

If r and s are any two integers, then (for large enough n), the product $D_{r,n}D_{s,n}$ equals $T_{s-r,n}$ except for s many rows and r many columns which are zero. Consequently, $D^2_{k,n}$ is an identity matrix except whose k rows and k columns are zero. Thus for asymptotic purposes, we may treat $D^2_{k,n}$ as an identity matrix.

Now, consider $T_{s,n}$ and $D_{r,n}$. Then, the $(i,j)^{\text{th}}$ entry of the product $T_{s,n}D_{r,n}$ equals $= \sum_{j_1} t_{i,j_1} h_{j_1,j}$ which is $\neq 0$ iff $j = n + r + 1 - i - s$. Since there are only finitely many such possibilities, $\lim_{n\to\infty} \frac{1}{n} \operatorname{Tr}(T_s D_r) = 0$.

Finally, note that $T_{r,n}T_{s,n} = T_{r+s,n}$ except for a finitely many entries.

Using the above facts repeatedly, it is easy to see that

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{Tr}\left(H_{k_1,n}H_{k_2,n}\dots H_{k_s,n}\right)=0\,,\qquad \text{if } s=2m-1 \text{ for some } m\geq 1\,.$$

So assume, $s=2m$. For convenience, we will write n_k for $n+k+1$ for any integer k

$$\begin{aligned}
&\frac{1}{n}\operatorname{Tr}\left(H_{k_1,n}H_{k_2,n}\dots H_{k_{2m},n}\right)\\
&=\frac{1}{n}\sum_{i,j_1,\dots,j_{2m-1}}\left(H_{k_1,n}(i,j_1)H_{k_2,n}(j_1,j_2)\dots H_{k_{2m},n}(j_{2m-1},i)\right)\\
&=\frac{1}{n}\sum_i H_{k_1,n}(i,n_{k_1}-i)\dots H_{k_{2m},n}\left(\sum_{j=1}^{2m-1}(-1)^{j+1}n_{k_j}-i,\sum_{j=1}^{2m}(-1)^j n_{k_j}+i\right)
\end{aligned}$$

(to satisfy trace condition the last index must be i, *i.e.*, $k_1+k_3+\dots$
$=k_2+k_4+\dots$)

$$=\frac{1}{n}\sum_i g_{k_1}\left(\frac{i-\left[\frac{n_{k_1}}{2}\right]}{n}\right)\dots g_{k_{2m}}\left(\frac{i-\left[\frac{n_{k_{2m}}}{2}\right]}{n}\right)\mathbb{I}_{k_1+k_3+\dots+k_{2m-1}=k_2+k_4+\dots+k_{2m}}$$

(this is a Riemann sum and using uniform continuity of g_k's)

$$\to\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(g_{k_1}\dots g_{k_{2m}}\right)(x)dx\right)\mathbb{I}_{k_1+k_3+\dots+k_{2m-1}=k_2+k_4+\dots+k_{2m}}\,.$$

Thus, $\lim\frac{1}{n}\operatorname{Tr}(H_{k_1,n}H_{k_2,n}\dots H_{k_s,n}) = \mathbb{E}_U\otimes\phi(g_{k_1}(U)a_1,g_{k_2}(U)a_2,\dots,g_{k_s}(U)a_s)$, where $\mathbb{E}_U$ is the usual expectation with respect to Lebesgue measure on $[-\frac{1}{2},\frac{1}{2}]$ and ϕ is a linear functional on $\mathcal{A}$ as defined in (1.1) and they act independently (classical sense) on

$$\begin{aligned}
&\mathcal{C}\left(\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\right)\otimes\mathcal{A}\\
&:=\left\{f(U)a : f \text{ continuous real-valued function on } \left[-\tfrac{1}{2},\tfrac{1}{2}\right]\,,\quad a\in\mathcal{A}\right\}\,.
\end{aligned}$$

This completes the proof of the theorem.

Remark 3.1. Let us define an $n\times n$ k-diagonal random Hankel matrix $\tilde{H}_{k,n}$ whose $(i,j)^{\text{th}}$ entry is $g_k(U)\mathbb{I}_{i+j=n+k+1}$ where U is a random variable uniformly distributed on $I:=[-\frac{1}{2},\frac{1}{2}]$. Suppose $g_k(\cdot)$ are continuous even functions on I. Following arguments similar to that given in the proof of Theorem 3.1, one can show that *(i)* For fixed $K > 0$,

$$\left(\tilde{H}_{-K,n}, \tilde{H}_{-K+1,n}, \ldots, \tilde{H}_{K-1,n}\tilde{H}_{K,n}\right) \xrightarrow{\frac{1}{n}\,\mathrm{E}\,\mathrm{Tr}} (g_{-K}(U)a_{-K}, \ldots, g_K(U)a_K)\ ,$$

where $\{a_i\}_{|i|\leq K} \subset (\mathcal{A}, \phi)$ are as defined in (1.1) and U is independent of $(\mathcal{A}, \phi)$. As a consequence, the expected ESD of $\sum_{i=-K}^{K} \tilde{H}_{i,n}$ converges weakly to $\sum_{j=-K}^{K} g_j(U)a_j$.

(ii) For almost every value of U,

$$\left(\tilde{H}_{-K,n}, \tilde{H}_{-K+1,n}, \ldots, \tilde{H}_{K,n}\right) \xrightarrow{\frac{1}{n}\,\mathrm{Tr}} (g_{-K}(U)a_{-K}, \ldots, g_K(U)a_K)$$

and hence for fixed $K > 0$, for almost every given ω, the ESD of $\sum_{i=-K}^{K} \tilde{H}_{i,n}$ converges weakly to $\sum_{j=-K}^{K} g_j(U(\omega))a_j$. Note that this is a random limit depending on ω (a typical point in the probability space where U is defined).

4. When all diagonals are present

Now, for U as previously defined, let $(\mathcal{C}(U), \mathbb{E}_U)$ be a classical probability space where $\mathcal{C}(U) := \{f(U) : f$ is continuous on $I\}$ and $\mathbb{E}_U$ is the usual expectation on I with respect to Lebesgue measure. Then consider the non-commutative probability space $(\tilde{\mathcal{A}}, \tilde{\phi})$ where $\tilde{A}$ is the algebra generated by $\{f(U)a : f(U) \in \mathcal{C}(U), a \in \mathcal{A}\}$ and $\tilde{\phi}$ acts on $f(U)a \in \tilde{\mathcal{A}}$ as $\tilde{\phi}(f(U)a) = (\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx)\phi(a)$ which is extended linearly on $\tilde{\mathcal{A}}$. $(\tilde{\mathcal{A}}, \tilde{\phi})$ is a $*$-probability space where $(f(U)a)^* = f(U)a^*$.

Let

$$\sum_{j=-k}^{k} g_j(U)a_j =: b_k \in \tilde{\mathcal{A}}\,.$$

We have seen that $\sum_{j=-k}^{k} H_{j,n}$ converges to b_k which is self-adjoint. It is also easy to see that $\{\tilde{\phi}(b_k^m)\}_{m\geq 1}$ defines a unique distribution function $\tilde{F}_k$ (say) which is the LSD of $\sum_{j=-k}^{k} H_{j,n}$.

To deal with matrices which may have all diagonals non-zero, we need some additional conditions on $\{g_j\}$ and an appropriate metric which will allow such matrices to be approximated by Hankel-type matrices with finitely many non-zero anti-diagonals.

The Mallow's metric is defined on the space of all probability distributions with finite second moment. Let F and G be two distribution functions with finite second moment. Then, the Mallow's distance between F and G is defined as

$$d_M^2(F,G) := \inf_{X\sim F, Y\sim G} \mathrm{E}\,|X-Y|^2\,. \tag{4.1}$$

It is known that $d_M(F_n,F)\to 0$ if and only if $\int x^2 dF_n(x)\to\int x^2 dF(x)$ and F_n converges to F weakly.

We need the following upper bound of this metric between the ESD of two matrices: let A,B be two $n\times n$ real symmetric matrices with eigenvalues $\lambda_1\le\lambda_2\le\cdots\le\lambda_n$ and $\beta_1\le\beta_2\le\cdots\le\beta_n$, respectively. Then,

$$d_M^2(F_A,F_B) \le \frac{1}{n}\sum_{j=1}^{n}(\lambda_j-\beta_j)^2 \le \frac{1}{n}\,\mathrm{Tr}(A-B)^2\,. \tag{4.2}$$

The first inequality is obvious and the last inequality above is a standard result in matrix algebra; one can see a proof of this in Lemma 2.3 of [8].

Theorem 4.1. *Suppose $\{g_j\}$ are continuous even functions on $I := [-\frac{1}{2},\frac{1}{2}]$. Suppose*

(i) $\sum_{k=-\infty}^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} g_j^2(x)dx<\infty,$ *and*

(ii) $\sum_{j=-n+1}^{n-1}\sum_{i=1}^{n}\int_{\frac{i-1-[\frac{n_j}{2}]}{n}}^{\frac{i-[\frac{n_j}{2}]}{n}}\left(g_j^2\left(\frac{i-[\frac{n_j}{2}]}{n}\right)-g_j^2(x)\right)dx\to 0$ *as* $n\to\infty$, *where* $n_j := n+j+1$.

Then, LSD of $H_n := \sum_{j:|j|\le n} H_{j,n}$ exists and equals $\lim_{k\to\infty}\tilde F_k =: \tilde F_\infty$. The limiting random variable may be written as $\sum_{-\infty}^{\infty} g_j(U)a_j$ where U and a_j are as defined in Theorem 3.1 and has distribution function $\tilde F_\infty$.

Proof. Let F_n and $F_{k,n}$ denote respectively the ESD of H_n and $\sum_{j=-k}^{k} H_{j,n}$. First, we will show that $\{\tilde F_k\}$ is weakly convergent. For that, it is enough to show that $\{\tilde F_k\}$ is Cauchy in d_M. Let $n>k_2>k_1$. Then,

$$\begin{aligned} d_M\left(\tilde F_{k_1},\tilde F_{k_2}\right) &\le d_M\left(\tilde F_{k_1},F_{k_1,n}\right)+d_M\left(\tilde F_{k_2},F_{k_2,n}\right)+d_M\left(F_{k_1,n},F_{k_2,n}\right)\\ &= d_1+d_2+d_3 \text{ (say)}\,. \end{aligned}$$

But by Theorem 3.1, $d_1 + d_2 \to 0$ as $n \to \infty$. For d_3, observe that

$$\begin{aligned}
d_3^2 := d_M^2\left(F_{k_1,n}, F_{k_2,n}\right) &\leq \frac{1}{n}\operatorname{Tr}\left(\sum_{k_1<|j|\leq k_2} H_{j,n}\right)^2 \quad \text{(by (4.2))} \\
&\leq \frac{1}{n}\sum_{j:k_1<|j|\leq k_2}\sum_i g_j^2\left(\frac{i-\left[\frac{n_j}{2}\right]}{n}\right) \\
&\leq \sum_{j:k_1<|j|\leq k_2}\left[\frac{1}{n}\sum_{i=1}^{n} g_j^2\left(\frac{i-\left[\frac{n_j}{2}\right]}{n}\right) - \int_{-\frac{1}{2}}^{\frac{1}{2}} g_j^2(x)dx\right] \\
&\quad + \sum_{j:k_1<|j|\leq k_2}\int_{-\frac{1}{2}}^{\frac{1}{2}} g_j^2(x)dx\,, \\
&\to 0 \quad \text{as} \quad k_1, k_2 \to \infty \text{ (by Conditions } (i) \text{ and } (ii)).
\end{aligned}$$

This implies that $\tilde{F}_k$ converges weakly to a distribution function $\tilde{F}_\infty$ (say). Now, to prove the theorem, consider

$$d_M\left(F_n, \tilde{F}_\infty\right) \leq d_M\left(F_n, F_{k,n}\right) + d_M\left(F_{k,n}, \tilde{F}_k\right) + d_M\left(\tilde{F}_k, \tilde{F}_\infty\right).$$

Since $\tilde{F}_k$ converges weakly to $\tilde{F}_\infty$, for a fixed $\varepsilon > 0$, there exists a $K \in \mathbb{N}$ such that

$$d_M\left(\tilde{F}_k, \tilde{F}_\infty\right) \leq \varepsilon \quad \text{for all} \quad k \geq K\,.$$

Now, for any fixed $k \geq K$, by Theorem 3.1 $d_M(F_{k,n}, \tilde{F}_k) \to 0$ as $n \to \infty$. Finally, again using (4.2), we have

$$\begin{aligned}
d_M^2(F_n, F_{k,n}) &\leq \frac{1}{n}\operatorname{Tr}\left(H_n - \sum_{j=-k}^{k} H_{j,n}\right)^2 \leq \frac{1}{n}\sum_{j:|j|>k}\sum_i g_j^2\left(\frac{i-\left[\frac{n_j}{2}\right]}{n}\right) \\
&\leq \sum_{j:|j|>k}\left[\frac{1}{n}\sum_{i=1}^{n} g_j^2\left(\frac{i-\left[\frac{n_j}{2}\right]}{n}\right) - \int_{-\frac{1}{2}}^{\frac{1}{2}} g_j^2(x)dx\right] \\
&\quad + \sum_{j:|j|>k}\int_{-\frac{1}{2}}^{\frac{1}{2}} g_j^2(x)dx\,,
\end{aligned}$$

and due to Conditions *(i)*, *(ii)*, right-hand side goes to zero as $n \to \infty$. Hence, $d_M(F_n, \tilde{F}_\infty) \to 0$ as $n \to \infty$. This completes the proof of the theorem.

Remark 4.1. The study of the LSD of $H_{k,n}$ and H_n, when the symmetry assumption is removed, does not seem to be easy. This requires further investigation. The following simple observations though may be made. Let $N_{k,n} = (a_{ij}))_{1 \le i,j \le n}$ be the $n \times n$ non-symmetric Hankel-type matrix,

$$a_{i,j} = \begin{cases} 1 & \text{if} \quad i + j = n + k + 1 \quad \text{and} \quad i \le [(n+k)/2]\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Then clearly, the LSD of both $N_{k,n}$ and $N^*_{k,n}$ are the point mass at zero, δ_0. The LSD of the symmetric matrix $N_{k,n}N^*_{k,n}$ converges in distribution to the Bernoulli random variable $(1/2)\delta_0 + (1/2)\delta_1$. Since $N_{k,n} + N^*_{k,n} = D_{k,n}$, its LSD is the symmetric Bernoulli $(1/2)\delta_{-1} + (1/2)\delta_1$. The limiting moment of any monomial in $(N_{k,n}, N^*_{k,n})$ is zero unless $N_{k,n}$ and $N^*_{k,n}$ appear alternately in the monomial. However, the limiting joint free cumulants of $(N_{k,n}, N^*_{k,n})$ may be non-zero even if they do not appear alternately. For example, it can be checked that

$$\lim \kappa_4 \left(N_{k,n}, N_{k,n}, N^*_{k,n}, N^*_{k,n}\right) = 1/4\,.$$

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