SUPERSYMMETRY FOR PRODUCTS OF RANDOM MATRICES*

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We consider the singular value statistics of products of independent random matrices. In particular, we compute the corresponding averages of products of characteristic polynomials. To this aim, we apply the projection formula recently introduced for chiral random matrix ensembles which serves as a shortcut of the supersymmetry method. The projection formula enables us to study the local statistics where free probability currently fails. To illustrate the projection formula and underlining the power of our approach, we calculate the hard edge scaling limit of the Meijer G-ensembles comprising the Wishart–Laguerre (chiral Gaussian), the Jacobi (truncated orthogonal, unitary or unitary symplectic) and the Cauchy–Lorentz (heavy tail) random matrix ensembles. All calculations are done for real, complex, and quaternion matrices in a unifying way. In the case of real and quaternion matrices, the results are completely new and hint to the universality of the hard edge scaling limit for a product of these matrices, too. Moreover, we identify the non-linear σ-models to the local statistics of product matrices at the hard edge.

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1. Introduction

Sums and products of random matrices are the simplest generalization of Random Matrix Theory (RMT) to introduce some kind of dimension. Sums of random matrices can be understood as a convolution and regularly appear in the field of Dyson's Brownian motion [1]. Product matrices are versatile as well. Applications of them can be found in mesoscopic physics [2], QCD [3], and wireless telecommunication [4, 5]. In the past years, a lot of progress was made on products of random matrices, see the new review [6] reporting

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on this progress. For example, free probability has been proven as an efficient tool for calculating the macroscopic level density [7]. With the help of orthogonal polynomials, one could calculate algebraic structures like determinants and Pfaffians, their kernels, and certain universal statistics on the local scale of the spectrum [5, 8–11]. In particular, products of random matrices drawn from Meijer G-ensembles exhibit a new kind of universal kernel in the hard scaling limit (microscopic limit around the origin). This limit is called Meijer G-kernel. Its name is reminiscent to the fact that the kernel essentially depends on Meijer G-functions, see [12] for a definition of these functions. The "standard candles" of RMT, the Wishart–Laguerre ensemble [13] (χGβE), the Cauchy–Lorentz ensemble [14] (LβE), and the Jacobi ensemble [15] (JβE) are particular cases of Meijer G-ensembles. Also products of matrices drawn from these three ensembles are Meijer G-ensembles since this class of ensembles is expected to be closed under matrix products.

Most results on the singular value statistics about product matrices are known for complex matrices ($\beta = 2$), only. The only exception, the macroscopic level density, can be computed for real ($\beta = 1$) and quaternion ($\beta = 4$) matrices with free probability [7] because they share the level density with $\beta = 2$. However, the local statistics of the singular values is still highly involved for $\beta = 1, 4$ due to unknown group integrals like the Itzykson–Zuber integral [16] and its polynomial counterpart [11, 17]. The projection formula recently proposed [18] circumvents such problems. This formula is a short-cut of the supersymmetry method [19, 20] and directly relates the original probability density with the weight in the dual superspace.

After introducing the required notation in Sec. 2, we briefly review the projection formula in Sec. 3. Thereby, we only consider the average of a product of characteristic polynomials to keep the calculation simple. We emphasize that the projection formula holds for all three Dyson indices $\beta = 1, 2, 4$ which is the strength of this approach.

In Sec. 4, we demonstrate via the three ensembles, χGβE, LβE, and JβE, how the projection formula works. Thereby, we explicitly compute the well-known orthogonal polynomials for $\beta = 2$ and show that the average of one characteristic polynomial for $\beta = 1$ and the square root of a characteristic polynomial for $\beta = 4$ is, apart from some shifts in the parameters, the same as in the case of $\beta = 2$. Another example is presented in Sec. 5 where we generalize the approach to a product of independently distributed matrices. Also for product matrices, we explicitly calculate the orthogonal polynomials in the case of $\beta = 2$. However, the completely new results are the ones for $\beta = 1, 4$ which are expressed in terms of integrals over Dyson's circular ensembles (CβE) [21]. In this way, we show in Sec. 6 that the universality in the hard edge scaling limit holds for real and quaternion product matrices, too. We are also able to identify the non-linear σ-models which are necessary when comparing the universal results with physical field theories.

2. Preliminaries

We consider rectangular random matrices which are either real ($\beta = 1$), complex ($\beta = 2$), or quaternion ($\beta = 4$). We are particularly interested in the singular value statistics of a random matrix

$$W \in \mathrm{gl}^{(\beta)}(n; n+\nu) = \left\{ \begin{array}{ll} \mathbb{R}^{n\times(n+\nu)}, & \beta = 1\,, \\ \mathbb{C}^{n\times(n+\nu)}, & \beta = 2\,, \\ \mathbb{H}^{n\times(n+\nu)}, & \beta = 4 \end{array} \right. \tag{1}$$

distributed by $P(WW^\dagger)$. We assume $\nu = 0$ in the following to keep the computations simple such that we choose the abbreviation $\mathrm{gl}^{(\beta)}(n) = \mathrm{gl}^{(\beta)}(n; n)$. Nonetheless, this restriction is not that strong since a product of rectangular matrices can be always rephrased to a product of square matrices [10]. Examples of such induced measures resulting from rectangular matrices are given in Sec. 5.

Since we choose the complex representation of the quaternion numbers $\mathbb{H}$ in terms of Pauli matrices, we introduce the convenient parameters

$$\widetilde{\beta} = \frac{4}{\beta}\,, \qquad \gamma = \left\{ \begin{array}{ll} 1\,, & \beta = 1, 2\,, \\ 2\,, & \beta = 4\,, \end{array} \right. \qquad \widetilde{\gamma} = \left\{ \begin{array}{ll} 2\,, & \beta = 1\,, \\ 1\,, & \beta = 2, 4\,. \end{array} \right. \tag{2}$$

For the sake of readability, we restrict ourselves to partition functions of the form

$$Z(M) = \int d[W] P\left(WW^\dagger\right) \det{}^{1/(\gamma\widetilde{\gamma})}\left(WW^\dagger \otimes \mathbf{1}_{\widetilde{\gamma}k} - M\right)\,. \tag{3}$$

The fixed matrix $M = \{M_{ab,ij}\}$ has the dimension $(\gamma n \times \gamma n) \otimes (\widetilde{\gamma}k \times \widetilde{\gamma}k) = \gamma\widetilde{\gamma}nk \times \gamma\widetilde{\gamma}nk$. It has to satisfy the symmetry

$$M^T = \left\{ \begin{array}{ll} \mathbf{1}_n \otimes [\tau_2 \otimes \mathbf{1}_k]\, M\, \mathbf{1}_n \otimes [\tau_2 \otimes \mathbf{1}_k]\,, & \beta = 1\,, \\ [\tau_2 \otimes \mathbf{1}_n] \otimes \mathbf{1}_k\, M\, [\tau_2 \otimes \mathbf{1}_n] \otimes \mathbf{1}_k\,, & \beta = 4\,, \end{array} \right. \tag{4}$$

where τ_2 is the second Pauli matrix. Other properties of M are not required.

The partition function (3) needs an explanation. The determinant acts on the tensor space of $(\gamma n \times \gamma n)$ matrices containing the matrix $WW^\dagger$ and a space of dimension $(\widetilde{\gamma}k \times \widetilde{\gamma}k)$. In the case that $M = \mathbf{1}_{\gamma n} \otimes \mathrm{diag}\,(m_1, \ldots, m_{\widetilde{\gamma}k})$, the determinant is a short-hand notation for a product of characteristic polynomials of $WW^\dagger$ which is a well-known partition function in Random Matrix Theory [22, 23]. The reason why we wrote this product in such an uncommon, compact form is the application we aim at, namely the singular value statistics of matrix products. Then the matrix M does not take such a simple form.

Another particularity of Eq. (3) which needs an explanation is the exponent of the determinant, $1/(\gamma\widetilde{\gamma})$ and the matrix dimensions. In the case of complex matrices ($\beta=2$), the exponent and the dimensions become self-explanatory since they become trivial, *e.g.* $1/(\gamma\widetilde{\gamma})|_{\beta=2}=1$. When W is real ($\beta=1$) then $WW^\dagger$ is real symmetric and $n\times n$ dimensional. The space dual to the polynomials consists of self-dual matrices. The resulting Kramers degeneracy cancels the exponent $1/2$ and doubles the dimension, $k\to 2k$. Exactly the opposite happens in the case of a quaternion matrix W ($\beta=4$). Due to its quaternion structure, the dimension is doubled, $n\to 2n$. However, the dual space consists of symmetric matrices. Since symmetric matrices may have also odd dimensions, we do not need a doubling of the dimension k. The corresponding square roots of the characteristic polynomials are exact and, thus, a polynomial because the spectrum of $WW^\dagger$ is Kramers degenerate. Such a square root is known as quaternion determinant and is equivalent to a Pfaffian determinant [23].

An important ingredient needed for the supersymmetry method is the invariance of the probability density P under the transformation $P(WW^\dagger)=P(UWW^\dagger U^\dagger)$ for all $U\in \mathrm{U}^{(\beta)}(n)$, where

$$\mathrm{U}^{(\beta)}(n)=\left\{\begin{array}{ll}\mathrm{O}(n)\,, & \beta=1\,,\\ \mathrm{U}(n)\,, & \beta=2\,,\\ \mathrm{USp}(2n)\,, & \beta=4\,.\end{array}\right. \tag{5}$$

Only due to this invariance, it is possible to find an integral over a supermatrix whose dimension is independent of the ordinary dimension n and which yields exactly the same partition function as Eq. (3). This can be achieved in four steps which we briefly sketch in Section 3.

For this purpose, we have to introduce two supermatrix spaces and one ordinary matrix space. Let $p,q,N\in\mathbb{N}$, and $\mathrm{U}(p|q)$ and $\mathrm{UOSp}(p|2q)$ be the unitary and the unitary ortho-symplectic supergroup, respectively, see [24–27]. The space of rectangular supermatrices is defined by

$$\mathrm{gl}^{(\beta)}\left(p|q;p'|q'\right)=\mathrm{u}^{(\beta)}\left(p+p'|q+q'\right)/\left[\mathrm{u}^{(\beta)}(p|q)\times\mathrm{u}^{(\beta)}\left(p'|q'\right)\right]\,, \tag{6}$$

where $\mathrm{u}^{(\beta)}(p|q)$ is the Lie superalgebra of the supergroup

$$\mathrm{U}^{(\beta)}(p|q)=\left\{\begin{array}{ll}\mathrm{UOSp}^{(+)}(p|2q)\,, & \beta=1\,,\\ \mathrm{U}(p|q)\,, & \beta=2\,,\\ \mathrm{UOSp}^{(-)}(2p|q)\,, & \beta=4\,.\end{array}\right. \tag{7}$$

The coset is taken via the addition as a group action on the Lie superalgebra. Therefore, a matrix $\rho\in\mathrm{gl}^{(\beta)}(p|q;N)$ is $(\gamma p|\widetilde{\gamma}q)\times(\gamma p|\widetilde{\gamma}q)$ dimensional and has the following form

$$\rho = \begin{bmatrix} \rho_{\mathrm{BB}} & \rho_{\mathrm{BF}} \\ \rho_{\mathrm{FB}} & \rho_{\mathrm{FF}} \end{bmatrix} . \tag{8}$$

The $\gamma p \times \gamma p$ dimensional boson–boson block ρ_{BB} and the $\widetilde{\gamma} q \times \widetilde{\gamma} q$ dimensional fermion–fermion block ρ_{FF} comprise commuting variables, while the other two blocks contain anti-commuting ones.

We employ the same notation for the two inequivalent fundamental representations of the supergroup $\mathrm{UOSp}(p|2q)$ as in [25–27] where the superscripts indicate the transformation property under the complex conjugation, *i.e.*

$$\rho^* = \begin{cases} \mathrm{diag}\,(\mathbf{1}_p, -\imath\tau_2 \otimes \mathbf{1}_q)\, \rho \,\mathrm{diag}\, \left(\mathbf{1}_{p'}, \imath\tau_2 \otimes \mathbf{1}_{q'}\right) , & \beta = 1 , \\ \mathrm{diag}\,(-\imath\tau_2 \otimes \mathbf{1}_p, \mathbf{1}_q)\, \rho \,\mathrm{diag}\, \left(\imath\tau_2 \otimes \mathbf{1}_{p'}, \mathbf{1}_{q'}\right) , & \beta = 4 \end{cases} \tag{9}$$

for $\rho \in \mathrm{gl}^{(\beta)}(p|q; p'|q')$ and

$$U^* = \begin{cases} \mathrm{diag}\,(\mathbf{1}_p, -\imath\tau_2 \otimes \mathbf{1}_q)\, U \,\mathrm{diag}\, (\mathbf{1}_p, \imath\tau_2 \otimes \mathbf{1}_q) , & \beta = 1 , \\ \mathrm{diag}\,(-\imath\tau_2 \otimes \mathbf{1}_p, \mathbf{1}_q)\, U \,\mathrm{diag}\, (\imath\tau_2 \otimes \mathbf{1}_p, \mathbf{1}_q) , & \beta = 4 \end{cases} \tag{10}$$

for $U \in \mathrm{U}^{(\beta)}(p|q) \subset \mathrm{U}(\gamma p|\widetilde{\gamma} q)$. The two relations (9) and (10) are generalization of the definitions of real and quaternion matrices to superspace.

The ordinary matrix space announced is the coset

$$\mathrm{C}\beta\mathrm{E}(\gamma k) = \begin{cases} \mathrm{U}(k)/\mathrm{O}(k) , & \beta = 1 , \\ [\mathrm{U}(k) \times \mathrm{U}(k)]/\mathrm{U}(k) \widehat{=} \mathrm{U}(k) , & \beta = 2 , \\ \mathrm{U}(2k)/\mathrm{USp}(2k) , & \beta = 4 \end{cases} \tag{11}$$

equipped with a normalized Haar measure $d\mu(U)$ induced by the Haar measures on the defining groups. These three sets are the circular ensembles first studied by Dyson [21]. These cosets are also the fermionic part of the supermatrices involved in the superbosonization formula [20]. Since we only discuss the average of products of determinants and not ratios superbosonization reduces to bosonization only involving the circular ensembles (11). Let us recall the properties of a matrix $U \in \mathrm{C}\beta\mathrm{E}(\gamma k)$. The matrix U is unitary and satisfies the symmetries $U^T = U$ for $\beta = 1$ and $U^T = (\tau_2 \otimes \mathbf{1}_k) U (\tau_2 \otimes \mathbf{1}_k)$ for $\beta = 4$.

Also the superdeterminant and the supertrace play an important role in the ensuing calculations. They are defined via the ordinary determinant and trace, and explicitly read

$$\mathrm{Sdet}\,\rho = \frac{\det\left(\rho_{\mathrm{BB}} - \rho_{\mathrm{BF}} \rho_{\mathrm{FF}}^{-1} \rho_{\mathrm{FB}}\right)}{\det \rho_{\mathrm{FF}}} , \qquad \mathrm{Str}\rho = \mathrm{tr}\,\rho_{\mathrm{BB}} - \mathrm{tr}\,\rho_{\mathrm{FF}} \tag{12}$$

for an arbitrary square supermatrix $\rho \in \mathrm{gl}^{(\beta)}(p|q;p|q)$ whose fermion–fermion block ρ_{FF} is invertible. The definitions are chosen in such a way that many properties of the trace and the determinant carry over to superspace. For example, the circularity $\mathrm{Str}AB = \mathrm{Str}BA$, the factorization $\mathrm{Sdet}\, AB = \mathrm{Sdet}\, A\, \mathrm{Sdet}\, B$, and the relation $\ln \mathrm{Sdet}\, A = \mathrm{Str} \ln A$ still hold for two arbitrary invertible square supermatrices A and B. The circularity property of the supertrace works for rectangular supermatrices, as well. A more profound introduction in supersymmetric analysis and algebra can be found in [28].

3. What is the projection formula?

The projection formula in its general form projects functions living on a very large superspace to functions on a much smaller superspace [18]. In this way, it directly relates the original weight P to a weight Q in the smaller superspace. Hence, the projection formula is a shortcut of the supersymmetry method [18]. For our particular purposes, the large superspace is $\mathrm{gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l;n|0)$ with l being an integer larger than or equal to k/γ. The enlargement of the dimensions $k \to 2l$ in the case of k odd and $\beta = 4$ is crucial. The reason is a Cauchy-like integration theorem [26, 29] first derived in a general framework by Wegner [30] which only applies to an even dimensional reduction of a matrix space in the case of $\beta = 1, 4$.

In the first step of deriving the projection formula, we need the following version of this Cauchy-like theorem [18]

$$P\left(WW^{\dagger}\right) = \frac{\int d\left[\widehat{\varOmega}\right] P\left(\varOmega\varOmega^{\dagger}\right)}{\int d\left[\widehat{\varOmega}\right] \exp\left[-\mathrm{Str}\widehat{\varOmega}\widehat{\varOmega}^{\dagger}\right]} \tag{13}$$

with $W \in \mathrm{gl}^{(\beta)}(n) = \mathrm{gl}^{(\beta)}(n|0;n|0)$ and $\widehat{\varOmega} \in \mathrm{gl}^{(\beta)}(\widetilde{\gamma}l|\gamma l;n|0)$. The matrices are embedded as follows

$$\varOmega = \left[\begin{array}{c} W \\ \widehat{\varOmega} \end{array}\right] = \left[\begin{array}{c} W' \\ \varOmega' \end{array}\right] \in \mathrm{gl}^{(\beta)}\left(n+\widetilde{\gamma}l|\gamma l;n\right) . \tag{14}$$

The second splitting in $W' \in \mathrm{gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l-k;n|0)$ and $\varOmega' \in \mathrm{gl}^{(\beta)}(0|k;n|0)$ becomes relevant in the third step of the derivation of the projection formula. The measure $d[\widehat{\varOmega}]$ is the product of all differentials of independent matrix entries of $\widehat{\varOmega}$. The normalization with a Gaussian is true because the proportionality constant is independent of P and thus can be fixed by any weight.

In Eq. (13), we have chosen a supersymmetric extension of P to the superspace $\mathrm{gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l;n|0)$ which is by far not unique. However, the final result is independent of this choice as already discussed in [27]. Such an extension indeed exists for a smooth distribution P. Since P is invariant under the group $\mathrm{U}^{(\beta)}(n)$, we can apply the Cayley–Hamilton theorem implying that P can be expressed in matrix invariants like traces and determinants of $WW^\dagger$. Those invariants have invariant extensions, namely the supertrace and the superdeterminant, *cf.* Eq. (12).

In the next step, we rewrite the determinant in Eq. (3) as a Gaussian integral over a matrix $V=\{V_{aj}\}\in\mathrm{gl}^{(\beta)}(0|k;n|0)$ which only consists of Grassmann (anti-commuting) variables [28]

$$\begin{aligned}&\det{}^{1/(\gamma\widetilde{\gamma})}\left(WW^\dagger\otimes\mathbf{1}_{\widetilde{\gamma}k}-M\right)\\&=\frac{\int d[V]\exp\left[\mathrm{tr}\,VWW^\dagger V^\dagger-\sum_{a,b=1}^{\widetilde{\gamma}k}\sum_{i,j=1}^{\gamma n}M_{ab,ij}V_{ai}V_{bj}^*\right]}{\int d[V]\exp\left[\mathrm{tr}\,VV^\dagger\right]}\,.\end{aligned}\tag{15}$$

Then, the partition function is up to a constant

$$Z(M)\propto\int d[\Omega]d[V]P\left(\Omega\Omega^\dagger\right)\exp\left[-\mathrm{Str}\Omega\Omega^\dagger\widehat{V}^\dagger\widehat{V}-\sum_{a,b=1}^{\widetilde{\gamma}k}\sum_{i,j=1}^{\gamma n}M_{ab,ij}V_{ai}V_{bj}^*\right]\tag{16}$$

with

$$\widehat{V}=\begin{bmatrix}0&0\\V&0\end{bmatrix},\qquad\widehat{V}^\dagger=\begin{bmatrix}0&V^\dagger\\0&0\end{bmatrix}\in\mathrm{gl}^{(\beta)}\left(n+\widetilde{\gamma}l|\gamma l;n+\widetilde{\gamma}l|\gamma l\right)\,.\tag{17}$$

The first $(\gamma n+2\gamma\widetilde{\gamma}l-\widetilde{\gamma}k)$ rows and the last $2\gamma\widetilde{\gamma}l$ columns of $\widehat{V}$ are equal to 0. The change of the sign in front of the first term in the exponential function relates to the fact that Grassmann variables are anti-commuting.

The integrals over V and Ω can be interchanged such that we find the function

$$\widehat{P}\left(\widehat{V}^\dagger\widehat{V}\right)=\int d[\Omega]P\left(\Omega\Omega^\dagger\right)\exp\left[-\mathrm{Str}\Omega\Omega^\dagger\widehat{V}^\dagger\widehat{V}\right]\,.\tag{18}$$

The invariance of $P(\Omega\Omega^\dagger)=P(U\Omega\Omega^\dagger U^\dagger)$ for all $U\in\mathrm{U}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l)$ carries over to a symmetry for $\widehat{P}(\widehat{V}^\dagger\widehat{V})=\widehat{P}(U\widehat{V}^\dagger\widehat{V}U^\dagger)$ for all $U\in\mathrm{U}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l)$. Therefore, the following duality holds

$$\widehat{P}\left(\widehat{V}^\dagger\widehat{V}\right)=\widehat{P}\left(\widehat{V}\widehat{V}^\dagger\right)\tag{19}$$

which is the third important step of the derivation. Employing the definition (18) backwards, the partition function is

$$\begin{aligned} Z(M) &\propto \int d[\Omega]d[V]P\left(\Omega\Omega^\dagger\right)\exp\left[\operatorname{tr}V^\dagger\Omega'\Omega'^\dagger V-\sum_{a,b=1}^{\widetilde{\gamma}k}\sum_{i,j=1}^{\gamma n}M_{ab,ij}V_{ai}V_{bj}^*\right] \\ &\propto \int d\left[\Omega'\right]Q\left(\Omega'\Omega'^\dagger\right)\det{}^{1/(\gamma\widetilde{\gamma})}\left(\mathbf{1}_{\gamma n}\otimes\Omega'\Omega'^\dagger-M\right). \end{aligned} \tag{20}$$

In the last step, we integrated over the remaining degrees of freedom W', *cf.* the splitting (14), which do not show up in the determinant. This integration yields the function

$$Q\left(\Omega'\Omega'^\dagger\right)\propto\int d\left[W'\right]P\left(\left[\begin{array}{cc} W'W'^\dagger & W'\Omega'^\dagger \\ \Omega'W'^\dagger & \Omega'\Omega'^\dagger \end{array}\right]\right). \tag{21}$$

This equation is the essence of the projection formula. The remaining things to do are cosmetics.

We want to express the dyadic matrix $\Omega'\Omega'^\dagger$ as a single square matrix U which is an element in $\mathrm{C}\widetilde{\beta}\mathrm{E}(\gamma k)$. Note that the circular ensemble really relates to the Dyson index $\widetilde{\beta}=4/\beta$ and not β which originates from the symmetries fulfilled by V.

Exactly this is done in the last step. We apply the superbosonization formula [20] which reduces to pure bosonization in our case. This yields the partition function

$$Z(M)=\int d\mu(U)Q(U)\det{}^{1/(\gamma\widetilde{\gamma})}(\mathbf{1}_{\gamma n}\otimes U-M)\det{}^{-n/\widetilde{\gamma}}U \tag{22}$$

with the normalized distribution

$$Q(U)=\frac{\int d[W_1]d[W_2]P\left(\left[\begin{array}{cc} W_1W_1^\dagger+W_2W_2^\dagger & W_2U^{1/2} \\ U^{1/2}W_2^\dagger & U \end{array}\right]\right)}{\int d\mu(U)d[W_1]d[W_2]\det{}^{-n/\widetilde{\gamma}}U\exp\left[-\operatorname{Str}\left(W_1W_1^\dagger+W_2W_2^\dagger\right)+\operatorname{tr}U\right]}. \tag{23}$$

The reduction of the integral (21) to the final expression (23) as an integral over the two matrices $W_1\in\mathrm{gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l-k;n+\widetilde{\gamma}l|\gamma l-k)$ and $W_2\in\mathrm{gl}^{(\beta)}(n+\widetilde{\gamma}l|\gamma l-k;0|k)$ was done in [18] and is skipped here due to the lack of space.

We remark that apart from the case of k odd and $\beta=4$, the auxiliary parameter l can be chosen $l=k/\gamma$. Then, the matrix W_1 is an ordinary square matrix and W_2 is a rectangular matrix only consisting of Grassmann variables.

4. Application to standard random matrix ensembles

Three particular cases of Meijer G-ensembles are the Gaussian $\chi G\beta E$, the heavy-tailed $L\beta E$, and the compactly supported $J\beta E$. We discuss them in Subsections 4.1, 4.2, and 4.3, respectively. These ensembles play important roles in a vast of applications and cover a broad range of systems [8, 9, 13–15, 22].

4.1. Wishart–Laguerre (Gaussian) ensemble

The first ensemble we consider is the $\chi G\beta E$

$$P_{\rm WL}\left(WW^\dagger\right) \propto \det{}^{\nu/\widetilde{\gamma}} WW^\dagger \exp\left[-{\rm tr}\, WW^\dagger/\mathit{\Gamma}^2\right] \tag{24}$$

with $\nu \in \mathbb{N}_0$ and $\mathit{\Gamma} > 0$. It is the oldest random matrix ensemble first studied by Wishart [13]. The determinant in front of the Gaussian originates from a transformation of a rectangular matrix $W' \in {\rm gl}^{(\beta)}(n, n+\nu)$ to the square matrix $W \in {\rm gl}^{(\beta)}(n)$. Therefore, one can understand Eq. (24) as an induced measure [10]. The corresponding weight $Q_{\rm WL}$ is given by Eq. (23)

$$\begin{aligned} Q_{\rm WL}(U) \;\propto\; & \int d[W_1]d[W_2]{\rm Sdet}^{\,\nu/\widetilde{\gamma}} \left[\begin{array}{cc} W_1W_1^\dagger + W_2W_2^\dagger & W_2U^{1/2} \\ U^{1/2}W_2^\dagger & U \end{array}\right] \\ & \times \exp\left[-{\rm Str}\left(W_1W_1^\dagger + W_2W_2^\dagger\right) + {\rm tr}\, U/\mathit{\Gamma}^2\right] \\ \propto\; & \det{}^{-\nu/\widetilde{\gamma}} U e^{{\rm tr}\, U/\mathit{\Gamma}^2} . \end{aligned} \tag{25}$$

Therefore, the partition function (3) for $P_{\rm WL}(WW^\dagger)$ reads

$$Z_{\rm WL}(M) = \frac{\int d\mu(U) \det^{-(n+\nu)/\widetilde{\gamma}} U \det^{1/(\gamma\widetilde{\gamma})}(\mathbf{1}_{\gamma n} \otimes U - M) e^{{\rm tr}\, U/\mathit{\Gamma}^2}}{\int d\mu(U) \det^{-(n+\nu)/\widetilde{\gamma}} U e^{{\rm tr}\, U/\mathit{\Gamma}^2}} . \tag{26}$$

This result agrees with the one derived in [26]. The normalization can be fixed by considering the expansion of the partition function for large M.

The result (26) exhibits nice implications. For example, the case $k = \gamma$ and $M = m\mathbf{1}_{\gamma^2\widetilde{\gamma}n}$ is equal to the orthogonal polynomials for $\beta = 2$ and to the skew-orthogonal polynomials of even order for $\beta = 1, 4$, see [23]. Hence the contour for $\beta = 2$ is a representation of the modified Laguerre polynomials $L_n^{(\nu)}$, see [31], *i.e.*

$$\begin{aligned} Z_{\rm WL}^{(\beta=2,k=1)}(m\mathbf{1}_n) \;\propto\; & \oint dz z^{-(n+\nu+1)}(z-m)^n e^{z/\mathit{\Gamma}^2} \\ \propto\; & \sum_{j=0}^{n} \frac{1}{j!(n-j)!(\nu+j)!}\left(-\frac{m}{\mathit{\Gamma}^2}\right)^j \propto L_n^{(\nu)}\left(\frac{m}{\mathit{\Gamma}^2}\right) . \end{aligned} \tag{27}$$

These polynomials also appear for $\beta = 1, 4$ if we set $k = 1$. Only the argument m is modified to $\widetilde{\gamma} m$. Interestingly, the case $\beta = 4$ is an average over a square root of a determinant which is equivalent to a Pfaffian.

For the case of $k = 2\gamma$ and $M = \mathbf{1}_{\gamma^2\widetilde{\gamma}n} \otimes \mathrm{diag}\,(m_1, m_2)$, we find one of the kernels corresponding to the χGβE [23]. When computing the contour integral (26), we immediately find the corresponding Christoffel–Darboux formulas.

4.2. Cauchy–Lorentz ensemble

The LβE is the next case we want to study. It is defined by the probability density [14, 18]

$$P_{\mathrm{CL}}\left(WW^\dagger\right) \propto \det{}^{\nu/\widetilde{\gamma}} WW^\dagger \det{}^{-\mu}\left(\Gamma^2 \mathbf{1}_{\gamma n} + WW^\dagger\right) \tag{28}$$

with $\Gamma > 1$, $\nu \in \mathbb{N}_0$ and $\mu > k/\gamma + (2n+\nu)/\widetilde{\gamma} - (\gamma\widetilde{\gamma} - 1)/2$ for guaranteeing the convergence of the integral (3)[1]. It is a heavy-tailed distribution and was employed for modelling financial correlations [14].

The choice $\Gamma > 1$ is convenient for the projection formula but is not a restriction at all because it only rescales the ensemble. The term $\det{}^{\nu/\widetilde{\gamma}} WW^\dagger$ can be again understood as a remnant of a rectangular matrix $W' \in \mathrm{gl}^{(\beta)}(n, n+\nu)$. However, we underline that such a transformation from W' to W also changes the exponent μ.

The weight for the dual space is calculated with the help of Eq. (23)

$$\begin{aligned} Q_{\mathrm{CL}}(U) \propto & \int d[W_1] d[W_2] \mathrm{Sdet}\,{}^{\nu/\widetilde{\gamma}} \left[\begin{array}{cc} W_1 W_1^\dagger + W_2 W_2^\dagger & W_2 U^{1/2} \\ U^{1/2} W_2^\dagger & U \end{array}\right] \\ & \times \mathrm{Sdet}\,{}^{-\mu} \left[\begin{array}{cc} \Gamma^2 \mathbf{1}_{\gamma n + \gamma\widetilde{\gamma} l | \gamma\widetilde{\gamma} l - \widetilde{\gamma} k} + W_1 W_1^\dagger + W_2 W_2^\dagger & W_2 U^{1/2} \\ U^{1/2} W_2^\dagger & \Gamma^2 \mathbf{1}_{\widetilde{\gamma} k} + U \end{array}\right] \\ \propto & \det{}^{-\nu/\widetilde{\gamma}} U \det{}^{\mu}\left(\Gamma^2 \mathbf{1}_{\widetilde{\gamma} k} + U\right) \int d[W_1] d[W_2] \mathrm{Sdet}\,{}^{\nu/\widetilde{\gamma}} W_1 W_1^\dagger \\ & \times \mathrm{Sdet}\,{}^{-\mu}\left[\Gamma^2 \mathbf{1}_{\gamma n + \gamma\widetilde{\gamma} l | \gamma\widetilde{\gamma} l - \widetilde{\gamma} k} + W_1 W_1^\dagger + \Gamma^2 W_2 \left(\Gamma^2 \mathbf{1}_{\widetilde{\gamma} k} + U\right)^{-1} W_2^\dagger\right] \\ \propto & \det{}^{-\nu/\widetilde{\gamma}} U \det{}^{\mu - k/\gamma - n/\widetilde{\gamma}}\left(\Gamma^2 \mathbf{1}_{\widetilde{\gamma} k} + U\right) . \end{aligned} \tag{29}$$

In the last step, we have rescaled $W_2 \to W_2(\Gamma^2 \mathbf{1}_{\widetilde{\gamma} k} + U)^{1/2}$ such that the remaining integrals are independent of U. Thereby, we recall that the Berezinian (Jacobian in superspace) is $\det^{-k/\gamma - n/\widetilde{\gamma}}(\Gamma^2 \mathbf{1}_{\widetilde{\gamma} k} + U)$ because W_2

[1] Note that the inequality satisfied by μ in [18] contains a mistake which we have corrected here. The inequality can be found by performing a singular value decomposition of W and then reading off the algebraic behaviour at infinity.

comprises Grassmann variables, only. Hence, we end up with the partition function

$$Z_{\rm CL}(M)=\frac{\int d\mu(U){\rm det}^{-\frac{n+\nu}{\widetilde{\gamma}}}U{\rm det}^{\mu-\frac{k}{\gamma}-\frac{n}{\widetilde{\gamma}}}\left(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k}+U\right){\rm det}^{\frac{1}{\gamma\widetilde{\gamma}}}(\mathbf{1}_{\gamma n}\otimes U-M)}{\int d\mu(U){\rm det}^{-(n+\nu)/\widetilde{\gamma}}U{\rm det}^{\mu-k/\gamma-n/\widetilde{\gamma}}\left(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k}+U\right)}\,. \tag{30}$$

Starting from this formula, one can again easily deduce the orthogonal or skew-orthogonal polynomials, the kernel involving two characteristic polynomials, and the Christoffel–Darboux formula associated to this kernel. For example, the orthogonal polynomials corresponding to the complex LβE = LUE is

$$\begin{aligned} Z_{\rm CL}^{(\beta=2,k=1)}(m\mathbf{1}_n) &\propto \oint dz z^{-(n+\nu+1)}\left(\Gamma^2+z\right)^{\mu-n-1}(z-m)^n \\ &\propto \sum_{j=0}^{n}\frac{1}{j!(n-j)!(\nu+j)!\Gamma[\mu-n-\nu-j]}\left(-\frac{m}{\Gamma^2}\right)^j . \end{aligned} \tag{31}$$

This polynomial can be understood as a Jacobi polynomial when analytically continuing the parameters to negative values, *cf.* Eq. (37). The same polynomials pop up for $\beta=1,4$ when setting $k=1$. This time, we have only to change the exponent $\mu\to\widetilde{\gamma}\mu-\widetilde{\gamma}/\gamma+1$.

4.3. Jacobi (truncated unitary) ensemble

The JβE is defined by [15]

$$P_{\rm J}\left(WW^\dagger\right)\propto{\rm det}^{\nu/\widetilde{\gamma}}WW^\dagger{\rm det}^{\kappa}\left(\Gamma^2\mathbf{1}_{\gamma n}-WW^\dagger\right)\Theta\left(\Gamma^2\mathbf{1}_{\gamma n}-WW^\dagger\right), \tag{32}$$

where $\nu\in\mathbb{N}_0$, $\kappa>-1/(2\gamma)$. The Heaviside step function Θ for matrices is unity if the matrix is positive definite and otherwise vanishes. Again, the scaling $\Gamma>1$ is only introduced to avoid problems with the contour integrals in the dual space. In the case of $\gamma\widetilde{\gamma}\mu\in\mathbb{N}_0$, the random matrix W distributed by Eq. (32) can be understood as a truncation of an orthogonal ($\beta=1$), a unitary ($\beta=2$), or a unitary symplectic ($\beta=4$) matrix, respectively, see [10, 15].

To apply the projection formula, we have first to find the supersymmetric generalization of the Heaviside step function. For this reason, we write this function as $\Theta(\Gamma^2\mathbf{1}_{\gamma n}-W^\dagger W)$. Then, it is clear that this function reads in terms of the supermatrix Ω as $\Theta(\Gamma^2\mathbf{1}_{\gamma n}-\Omega^\dagger\Omega)$ because the dyadic matrix $\Omega^\dagger\Omega$ has still an ordinary dimension and can be embedded in the space of

$\gamma n \times \gamma n$ matrices by a Taylor expansion in the Grassmann valued matrix entries. Such a Taylor expansion is always finite since Grassmann variables are nilpotent. Hence, we do not have to fear any problems of convergence.

Let $n,p,q \in \mathbb{N}$ and $V \in \mathrm{gl}^{(\beta)}(p|q;n|0)$. Then, the extension of the Heaviside step function is done by a limit

$$\begin{aligned} \Theta\left(\mathbf{1}_{\gamma n} - V^\dagger V\right) &= \lim_{\epsilon\to\infty} \det{}^{-1}\left(\mathbf{1}_{\gamma n} + e^{-\epsilon} e^{\epsilon V^\dagger V}\right) \\ &= \lim_{\epsilon\to\infty} \exp\left[\sum_{j=1}^{\infty} \frac{(-1)^j e^{-j\epsilon}}{j} \mathrm{tr}\, e^{j\epsilon V^\dagger V}\right] . \end{aligned} \tag{33}$$

This limit vanishes if one or more eigenvalues of the numerical part of the dyadic matrix $V^\dagger V$ is larger than 1. We emphasize that indeed only the numerical part matters and not the nilpotent terms because of the Taylor expansion in the latter. In the next step, we employ the duality $\mathrm{tr}\, e^{j\epsilon V^\dagger V} = \gamma(n-p) + \widetilde{\gamma} q + \mathrm{Str} e^{j\epsilon V V^\dagger}$. We have

$$\begin{aligned} \Theta\left(\mathbf{1}_{\gamma n} - V^\dagger V\right) &= \lim_{\epsilon\to\infty} (1+e^{-\epsilon})^{\gamma(p-n)-\widetilde{\gamma} q} \mathrm{Sdet}^{\,-1}\left(\mathbf{1}_{\gamma p|\widetilde{\gamma} q} + e^{-\epsilon} e^{\epsilon V V^\dagger}\right) \\ &= \lim_{\epsilon\to\infty} \det{}^{-1}\left(\mathbf{1}_{\gamma p|\widetilde{\gamma} q} + e^{-\epsilon}\left\{e^{\epsilon V V^\dagger}\right\}_{\mathrm{BB}}\right) \\ &= \Theta\left(\mathbf{1}_{\gamma p} - \left\{V V^\dagger\right\}_{\mathrm{BB}}^{\mathrm{num}}\right) . \end{aligned} \tag{34}$$

The Heaviside step function is only taken for the numerical part $\{VV^\dagger\}_{\mathrm{BB}}^{\mathrm{num}}$ of the boson–boson block of the dyadic matrix $VV^\dagger$. Any expansion in the nilpotent terms yields a polynomial in ϵ which are suppressed by the exponential $e^{-\epsilon}$. This implies that the other three blocks of the supermatrix $e^{\epsilon VV^\dagger}$ cannot contribute because they are polynomials in ϵ. The boson–boson block is $\{e^{\epsilon VV^\dagger}\}_{\mathrm{BB}} = e^{\epsilon \{VV^\dagger\}_{\mathrm{BB}}^{\mathrm{num}}}(1+f(\epsilon))$ with f a polynomial and $f(0)=0$. Therefore, Eq. (34) is the correct generalization of the Heaviside step function to the superspace. Interestingly, the Taylor expansion in the nilpotent terms have no influence on the Heaviside step function. But this behaviour has to be expected because the Taylor expansion can only have an effect on the boundary. Only there, one or more eigenvalues of the numerical part $\{VV^\dagger\}_{\mathrm{BB}}^{\mathrm{num}}$ are equal to 1 where the value of the function may change. However, the supersymmetric Heaviside step function vanishes at the boundary, too, due to the expansion in the nilpotent terms yielding an inverted polynomial in ϵ, *e.g.* $\det^{-1}(\mathbf{1}_{\gamma p|\widetilde{\gamma} q} + e^{-\epsilon}\{e^{\epsilon VV^\dagger}\}_{\mathrm{BB}}) \overset{\{VV^\dagger\}_{\mathrm{BB}}\to\mathbf{1}_{\gamma p}}{\longrightarrow} 1/f(\epsilon) \overset{\epsilon\to\infty}{\to} 0$ with f a polynomial.

We employ Eq. (34) in our setting and recognize that the matrix U is not a part of the boson–boson block of the matrix argument of $P_{\rm J}$ in Eq. (23). Hence, the function in the dual space is

$$\begin{aligned} Q_{\rm J}(U) \ &\propto \int d[W_1]d[W_2]{\rm Sdet}^{\,\nu/\widetilde{\gamma}} \left[\begin{array}{cc} W_1W_1^\dagger + W_2W_2^\dagger & W_2U^{1/2} \\ U^{1/2}W_2^\dagger & U \end{array} \right] \\ &\quad \times{\rm Sdet}^{\,\kappa} \left[\begin{array}{cc} \Gamma^2\mathbf{1}_{\gamma n+\gamma\widetilde{\gamma}l|\gamma\widetilde{\gamma}l-\widetilde{\gamma}k} - W_1W_1^\dagger - W_2W_2^\dagger & W_2U^{1/2} \\ U^{1/2}W_2^\dagger & \Gamma^2\mathbf{1}_{\widetilde{\gamma}k} - U \end{array} \right] \\ &\quad \times\Theta\left(\Gamma^2\mathbf{1}_{\gamma n+\gamma\widetilde{\gamma}l} - \left\{W_1W_1^\dagger + W_2W_2^\dagger\right\}_{\rm BB}^{\rm num}\right) \\ &\propto \ \det{}^{-\nu/\widetilde{\gamma}}U\det{}^{-\kappa-k/\gamma-n/\widetilde{\gamma}}\left(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k} - U\right) . \end{aligned} \tag{35}$$

We underline that the boson–boson block of $W_2W_2^\dagger$ only consists of nilpotent parts such that it does not contribute to the Heaviside step function. The corresponding partition function is

$$Z_{\rm J}(M) = \frac{\int d\mu(U){\rm det}^{-\frac{n+\nu}{\widetilde{\gamma}}}U{\rm det}^{-\kappa-\frac{k}{\gamma}-\frac{n}{\widetilde{\gamma}}}\left(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k} - U\right){\rm det}^{\frac{1}{\gamma\widetilde{\gamma}}}(\mathbf{1}_{\gamma n}\otimes U - M)}{\int d\mu(U){\rm det}^{-(n+\nu)/\widetilde{\gamma}}U{\rm det}^{-\kappa-k/\gamma-n/\widetilde{\gamma}}\left(\Gamma^2\mathbf{1}_{\widetilde{\gamma}k} - U\right)} . \tag{36}$$

One can readily check the correctness of this result by calculating the orthogonal or skew-orthogonal polynomials and the kernel involving two characteristic polynomials. For example, with the help of the residue theorem we generate the polynomials,

$$\begin{aligned} Z_{\rm J}^{(\beta=2,k=1)}(m\mathbf{1}_n) \ &\propto \oint dz z^{-(n+\nu+1)}\left(\Gamma^2 - z\right)^{-(n+\kappa+1)}(z-m)^n \\ &\propto \sum_{j=0}^{n}\frac{\Gamma[n+\kappa+\nu+j+1]}{j!(n-j)!(\nu+j)!}\left(-\frac{m}{\Gamma^2}\right)^j \propto P_n^{(\kappa,\nu)}\left(\frac{2m}{\Gamma^2}-1\right) , \end{aligned} \tag{37}$$

where $P_n^{(\kappa,\nu)}$ are the Jacobi polynomials with respect to the weight $(1-x)^\kappa$ $(1+x)^\nu\Theta(1-x^2)$, see [31]. As in the case of the LβE, we find the same polynomials for $\beta = 1, 4$ and $k = 1$ when replacing the exponent $\kappa \to \widetilde{\gamma}\kappa + \widetilde{\gamma}/\gamma - 1$.

We also obtain the well-known Christoffel–Darboux formula of the Jacobi polynomials by setting $k = 2$, $\beta = 2$, and $M = \mathbf{1}_n \otimes {\rm diag}\,(m_1, m_2)$. Then, the integral reduces to a double contour integral after diagonalizing U.

5. Application to product matrices

The computation of the partition function for a product of L matrices $W \to W^{(L)} = \prod_{j=1}^{L} W_j = W_1 \dots W_L$ independently distributed by $P(WW^\dagger) \to \prod_{j=1}^{L} P_j(W_j W_j^\dagger)$ works in a similar way as for a single matrix. Starting from the partition function

$$\begin{aligned} Z_\Pi(M) &= \int \left(\prod_{j=1}^{L} d[W_j] P_j\left(W_j W_j^\dagger\right) \right) \det{}^{1/(\gamma\widetilde{\gamma})} \left[W^{(L)} \left(W^{(L)}\right)^\dagger \otimes \mathbf{1}_{\widetilde{\gamma}k} - M \right] \\ &= \int \left(\prod_{j=1}^{L} d[W_j] P_j\left(W_j W_j^\dagger\right) \right) \det{}^{k/\gamma} W^{(L-1)} \left(W^{(L-1)}\right)^\dagger \\ &\quad \times \det{}^{1/(\gamma\widetilde{\gamma})} \left[W_L W_L^\dagger \otimes \mathbf{1}_{\widetilde{\gamma}k} - X_{L-1}^{-1} M Y_{L-1}^{-1} \right] \end{aligned} \tag{38}$$

with $X_{L-1} = W^{(L-1)} \otimes \mathbf{1}_{\widetilde{\gamma}k}$ and $Y_{L-1} = (W^{(L-1)})^\dagger \otimes \mathbf{1}_{\widetilde{\gamma}k}$, we apply the projection formula for W_L after replacing the matrix $M \to X_{L-1}^{-1} M Y_{L-1}^{-1}$. Then, we obtain

$$\begin{aligned} Z_\Pi(M) =& \int \left(\prod_{j=1}^{L-1} d[W_j] P_j\left(W_j W_j^\dagger\right) \right) d\mu(U_L) Q_L(U_L) \det{}^{-n/\widetilde{\gamma}} U_L \\ & \times \det{}^{1/(\gamma\widetilde{\gamma})} \left[W^{(L-1)} \left(W^{(L-1)}\right)^\dagger \otimes U_L - M \right] \\ =& \int \left(\prod_{j=1}^{L-1} d[W_j] P_j\left(W_j W_j^\dagger\right) \right) d\mu(U_L) Q_L(U_L) \det{}^{k/\gamma} W^{(L-2)} \left(W^{(L-2)}\right)^\dagger \\ & \times \det{}^{\frac{1}{\gamma\widetilde{\gamma}}} \left[W_{L-1} W_{L-1}^\dagger \otimes \mathbf{1}_{\widetilde{\gamma}k} - X_{L-2}^{-1} M Y_{L-2}^{-1} \right] , \end{aligned} \tag{39}$$

where $X_{L-2} = W^{(L-2)} \otimes \sqrt{U_L}$, $Y_{L-2} = (W^{(L-2)})^\dagger \otimes \sqrt{U_L}$, and Q_L is computed as in the projection formula (23). This procedure yields a recursion resulting in the following expression for the partition function,

$$\begin{aligned} Z_\Pi(M) &= \int \left(\prod_{j=1}^{L} d\mu(U_j) Q_j(U_j) \right) \det{}^{-n/\widetilde{\gamma}} U_L \dots U_1 \\ &\quad \times \det{}^{1/(\gamma\widetilde{\gamma})} \left[\mathbf{1}_{\gamma n} \otimes \sqrt{U_L} \dots \sqrt{U_2} U_1 \sqrt{U_2} \dots \sqrt{U_L} - M \right] , \end{aligned} \tag{40}$$

where each matrix U_j is an element in the circular ensemble $\mathrm{C}\widetilde{\beta}\mathrm{E}(\widetilde{\gamma}k)$.

In the final step, we replace $U'_j = \sqrt{U_L}\ldots\sqrt{U_{j+1}}U_j\sqrt{U_{j+1}}\ldots\sqrt{U_L}$ which preserves the symmetries such that $U'_j \in \mathrm{C}\widetilde{\beta}\mathrm{E}(\widetilde{\gamma}k)$. For this purpose, we use two facts. First, the Haar measure is invariant under $d\mu(U) = d\mu(VUV^T)$ for all $V \in \mathrm{U}(\widetilde{\gamma}k)$ resulting from the fact that the explicit form of the Haar measure of $\mathrm{C}\widetilde{\beta}\mathrm{E}(\widetilde{\gamma}k)$ is $d\mu(U) \propto \det^{-k/\gamma-(\gamma-\widetilde{\gamma})/2}U d[U]$ with $d[U]$ the product of the differentials of all independent matrix entries [20, 21]. Second, the weights Q_j are also invariant under $Q_j(U) = Q_j(VUV^\dagger)$ for all $V \in \mathrm{U}^{(\beta)}(k)$. Hence, these weights have an expression in terms of functions of matrix invariants. With the help of a slight abuse of notation, one can say that the weights Q_j satisfy a cyclic permutation symmetry, $Q_j(AB) = Q_j(BA)$ for any two matrices $A, B \in \mathrm{U}(\widetilde{\gamma}k)$.

Finally, we find the result

$$Z_{\Pi}(M)=\int\left(\prod_{j=1}^{L}d\mu\left(U'_j\right)Q_j\left(U'_jU'^{-1}_{j+1}\right)\right)\det{}^{-n/\widetilde{\gamma}}U'_1\det{}^{1/(\gamma\widetilde{\gamma})}\left[\mathbf{1}_{\gamma n}\otimes U'_1-M\right] \tag{41}$$

with $U'_{L+1} = \mathbf{1}_{\widetilde{\gamma}k}$. This result is surprisingly compact. It also reflects the nature of the original product of matrices which is equivalent to a Mellin-like convolution in a matrix space. Also the dual space exhibits this structure of a Mellin-like convolution.

As an example, we calculate the orthogonal polynomials ($k = 1$) of a product of L_{WL} complex $\chi\mathrm{G}\beta\mathrm{E} = \chi\mathrm{GUE}$, Eq. (24), L_{CL} complex $\mathrm{L}\beta\mathrm{E} = \mathrm{LUE}$, Eq. (28), and L_{J} complex $\mathrm{J}\beta\mathrm{E} = \mathrm{JUE}$, (32). We assume this product to be ordered, *i.e.* first the Wishart–Laguerre, then the Cauchy–Lorentz, and finally the Jacobi matrices. The result does not depend on this ordering, see the discussion in [10]. Then the orthogonal polynomials are

$$\begin{aligned}
Z^{(\beta=2,k=1)}_{L_{\mathrm{WL}}L_{\mathrm{CL}}L_{\mathrm{J}}}(m\mathbf{1}_n) \;\propto\; & \oint\left(1-mz_1^{-1}\right)^n\left(\prod_{j=1}^{L_{\mathrm{WL}}}\frac{dz_j}{z_j}\left[\frac{z_{j+1}}{z_j}\right]^{\nu_j}e^{\frac{1}{\Gamma_j^2}\frac{z_j}{z_{j+1}}}\right)\\
&\times\left(\prod_{j=L_{\mathrm{WL}}+1}^{L_{\mathrm{WL}}+L_{\mathrm{CL}}}\frac{dz_j}{z_j}\left[\frac{z_{j+1}}{z_j}\right]^{\nu_j}\left[\Gamma_j^2+\frac{z_j}{z_{j+1}}\right]^{\mu'_j-1}\right)\\
&\times\left(\prod_{j=L_{\mathrm{WL}}+L_{\mathrm{CL}}+1}^{L_{\mathrm{WL}}+L_{\mathrm{CL}}+L_{\mathrm{J}}}\frac{dz_j}{z_j}\left[\frac{z_{j+1}}{z_j}\right]^{\nu_j}\left[\Gamma_j^2-\frac{z_j}{z_{j+1}}\right]^{-\kappa'_j-1}\right)\\
\propto\; & \sum_{j=0}^{n}\frac{\prod_a\Gamma[n+\kappa_a+\nu_a+j+1]}{j!(n-j)!(\prod_a(\nu_a+j)!)(\prod_a\Gamma[\mu_a-n-\nu_a-j])}\\
&\times\left(-\frac{m}{\Gamma^2}\right)^j
\end{aligned} \tag{42}$$

with $z_{L_{\rm WL}+L_{\rm CL}+L_{\rm J}+1}=1$, $\mu_j'=\mu_j-n$, $\kappa_j'=\kappa_j+n$, and $\mathit{\Gamma}^2=\prod_j\mathit{\Gamma}_j^2$. The product of the Gamma functions runs over the possible values for ν_a, κ_a, and μ_a. The polynomial (42) is a hypergeometric function and, thus, a Meijer G-function [12]. It agrees for certain values of the parameters $L_{\rm WL}$, $L_{\rm CL}$, and $L_{\rm J}$ with known results [9, 11]. What is completely new are the results for $\beta=1,4$ and $k=1$ which are essentially the same polynomials. Here, the other approaches failed because of unknown group integrals.

6. Hard edge scaling limit of product matrices

Up to now, every calculation was done for finite n such that we made no approximation and the projection formula was exact. However, to make contact to physical systems and universality, we have to zoom onto the local scale somewhere of the spectrum. A very prominent scaling is the one to a vicinity around the origin also known as the hard edge scaling limit.

As a simple but non-trivial example, we choose the matrix product of the previous section with the source $M=\widetilde{\gamma}(\prod_j\mathit{\Gamma}_j^2)\mathbf{1}_{\gamma n}\otimes\widehat{m}/[n(\prod_a(\mu_a-n/\widetilde{\gamma}))(\prod_a(\kappa_a+n/\widetilde{\gamma}))]$. In particular, we consider the scaling limit $n\to\infty$ and ν_j, $\widehat{\mu}_j=(\mu_j/n-1/\widetilde{\gamma})$, $\widehat{\kappa}_j=(\kappa_j/n+1/\widetilde{\gamma})$, and $\widehat{m}$ fixed. Then, one can easily show that the asymptotics of each weight, regardless what kind of random matrix we consider, is

$$Q_j(\alpha U)\overset{n\gg1}{\propto}\det{}^{\nu_j/\widetilde{\gamma}}U e^{\operatorname{tr}U}\tag{43}$$

with $\alpha=\mathit{\Gamma}_j^2$ for χGβE, $\alpha=\mathit{\Gamma}_j^2/(n\widehat{\mu}_j)$ for LβE, and $\alpha=\mathit{\Gamma}_j^2/(n\widehat{\kappa}_j)$ for LβE. After a proper rescaling of the matrices U_j, the partition function (41) takes the asymptotic form

$$Z_{\mathit{\Pi}}(M)\overset{n\gg1}{\propto}\int\left(\prod_{j=1}^{L}d\mu\left(U_j'\right)\det{}^{\nu_j/\widetilde{\gamma}}U_j\right)e^{\operatorname{tr}U_L+\sum_{j=1}^{L-1}\operatorname{tr}U_jU_{j+1}^{-1}-\operatorname{tr}\widehat{m}U_1^{-1}}\tag{44}$$

with $L=L_{\rm WL}+L_{\rm CL}+L_{\rm J}$. We underline that no saddle point approximation is needed for this limit. Hence, the matrices U_j are still elements of the circular ensemble C$\widetilde{\beta}$E$(\widetilde{\gamma}k)$.

For $\beta=2$, the partition function (44) yields the Meijer G-kernel of a product of matrices drawn from χGβEs, *cf.* [9]. This can be seen by diagonalizing the unitary matrices, applying the Itzykson–Zuber integral [16] and, finally, integrating over a determinantal point process. The entries of the resulting determinant are Meijer G-functions. Our result emphasizes the conjecture that also this kernel is universal. Indeed, we could also have chosen another scaling which still leads to a hard edge scaling limit. Then, we would get finite rank deformations of the result (44) which was recently discovered for a product of truncated unitary matrices in [11]. Nevertheless, the limiting kernel is still a Meijer G-kernel but with other parameters.

From a physical point of view, one can ask for the non-linear σ-model corresponding to the partition function (44). In this framework, the function in the exponential function is identified as the potential. The integration domain $\mathrm{C}\widetilde{\beta}\mathrm{E}^L(\widetilde{\gamma}k)$ is the coset of the "flavour" group which keeps the "massless Lagrangian" ($\widehat{m} = 0$) in the full theory at finite "volume" n invariant divided by the group which keeps the ground state invariant. As in the case of $L = 1$, the theory is spontaneously broken. For a product matrix, the "flavour" symmetry at finite "volume" n is $\mathrm{U}^L(\widetilde{\gamma}k)$ for $\beta = 1, 4$ and $[\mathrm{U}(k) \times \mathrm{U}(k)]^L$ for $\beta = 2$ which can be readily checked by linearising the product $W^{(L)}$ in the matrices W_j. This group is spontaneously broken to $[\mathrm{U}^{(\widetilde{\beta})}(k)]^L$ and the source term for its condensate is the "mass" $\widehat{m}$. This non-linear σ model generalizes the one for the Wishart–Laguerre ensemble which were found in QCD [32] and mesoscopic systems [24].

7. Conclusions

We briefly presented the projection formula [18] for averages over products of characteristic polynomials which is a shortcut of the supersymmetry method [19, 20, 27]. The general results found by this approach were demonstrated in the case of Wishart–Laguerre (χGβE), Cauchy–Lorentz (LβE), and Jacobi (JβE) ensembles, in particular, we rederived the corresponding orthogonal polynomials for $\beta = 2$. These polynomials are essentially the same when averaging over one characteristic polynomial for $\beta = 1$ and over a square root of a characteristic polynomial for $\beta = 4$.

Moreover, we generalized the projection formula to products of matrices. Since the projection formula works in a unifying way for all three Dyson indices $\beta = 1, 2, 4$, this approach is an ideal alternative compared to other methods like orthogonal polynomials and free probability when studying real or quaternion matrices. Note that up to now free probability only applies to global spectral properties and to use orthogonal polynomials we need to know group integrals like the Itzykson–Zuber integral [16] or its polynomial counterpart [11, 17]. The projection formula circumvents this problem. In particular, we were able to show that the spectral statistics at the hard edge are the same for products of completely different random matrices only depending on the number of matrices defined and their indices $\nu_1, \ldots, \nu_l$ encoding the rectangularity of the matrices. This was done for all three cases $\beta = 1, 2, 4$ and underlines the strength of the projection formula where other methods fail. In the complex case ($\beta = 2$), we easily deduce from our results those for the Meijer G-ensembles studied in [9, 11].

The projection formula also enabled us to identify the non-linear σ-models and the symmetry breaking pattern for product matrices and derived the potential of the Goldstone manifold. This result is completely new and shows what the effective theory associated to such a product matrix would look like. In particular, one can understand a product matrix by itself as a discrete one-dimensional system. Therefore, our results shows one way to generalize the zero-dimensional RMT to a one-dimensional theory.

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