

RECENT EXACT AND ASYMPTOTIC RESULTS FOR PRODUCTS OF INDEPENDENT RANDOM MATRICES*

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In this review, we summarise recent results for the complex eigenvalues and singular values of finite products of finite size random matrices, their correlation functions and asymptotic limits. The matrices in the product are taken from ensembles of independent real, complex, or quaternionic Ginibre matrices, or truncated unitary matrices. Additional mixing within one ensemble between matrices and their inverses is also covered. Exact determinantal and Pfaffian expressions are given in terms of the respective kernels of orthogonal polynomials or functions. Here, we list all known cases and some straightforward generalisations. The asymptotic results for large matrix size include new microscopic universality classes at the origin and a generalisation of weak non-unitarity close to the unit circle. So far, in all other parts of the spectrum, the known standard universality classes have been identified. In the limit of infinite products, the Lyapunov and stability exponents share the same normal distribution. To leading order, they both follow a permanent point processes. Our focus is on presenting recent developments in this rapidly evolving area of research.

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1. Introduction

The study of products of random matrices goes back to the early days of Random Matrix Theory, when in 1960 Furstenberg and Kesten [1] studied them in the context of dynamical systems and their Lyapunov exponents. When multiplying matrices generically, the product becomes non-Hermitian. Hence, it is natural to choose each factor independently to be non-Hermitian, taken from a Gaussian distribution in the simplest case, the Ginibre ensembles [2] with real, complex or quaternionic matrix elements. These are the analogues of the three classical Wigner–Dyson ensembles, labelled $\beta = 1, 2, 4$, respectively. When modelling unitary time evolution, the

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choice of matrices to be multiplied is slightly less trivial. Taking factors from ensembles of Haar distributed orthogonal, unitary or symplectic matrices does not lead out of the one matrix case, due to the invariance of the Haar measure. For the product to become non-trivial, instead, one may choose to multiply truncated unitary matrices which are distributed according to the Jacobi measure, leading to a sub-unitary evolution. Other choices are of course possible, *e.g.* by multiplying unitary matrices that are not Haar distributed [3–5] giving a matrix-diffusion on the unitary group. Another approach to matrix-diffusion was considered in [6] where also a discretisation resulting in a product of Hermitian matrices was considered.

In any case, one has two choices in studying spectral properties of the product matrix: either the eigenvalues which are generally complex, or the singular values which are real positive. The latter were studied first in order to define and study the Lyapunov exponents. Their application to dynamical systems has led to the activities summarised in [7]. Examples for more recent applications include wireless communication [8] and combinatorics [9]. Although, in general, complex eigenvalues and singular values are not related individually, in the limit of infinite products the radii and singular values become identical, in other words, the stability and Lyapunov exponents have the same normal distribution [10, 11], as first conjectured in [12].

It is quite surprising that the spectral properties of products of random matrices were first determined in the infinite product case, using mainly probabilistic tools, and in the limit of infinite matrix dimension (keeping the number of factors fixed), using planar Feynman diagrams [13] and probability theory [8, 14, 15], see [16] for a very recent review on the free probability approach. The common point to the latter two approaches is that they contain information about global spectral properties. These were found to be universal when multiplying matrices from various different ensembles [17], including analytical results for factors from the elliptic ensemble [18].

Apart from special cases for 2×2 matrices, see *e.g.* [19], explicit results for the product of M matrices of size $N \times N$ with M and N finite are very recent, starting from [20–22] for the complex eigenvalues, and from [23, 24] for the singular values. They reveal a determinantal and Pfaffian structure and will be the main topic of this review. Inherited from the fact that the product of normal or gamma random variables is distributed according to the Meijer G -function [25], these special functions appear in the weight and determinantal expressions for the products of random matrices too. Obviously, the detailed knowledge of all eigenvalue and singular value correlation functions has opened up the possibility to study local spectral properties, and we will also review the recent progress on finding known and new universality classes in the large- N limit.

In [20], the joint density of complex eigenvalues for products of M matrices of size $N \times N$ was derived and the kernel of orthogonal polynomials (OP) in the complex plane that determines all correlation functions was given. While in the bulk and edge large- N scaling limit at fixed M the respective Ginibre universality classes for a single matrix were recovered, at the origin a new universality class labelled M with a hypergeometric kernel appeared [20]. These results were generalised to products of rectangular matrices (these are also called induced Ginibre matrices [26]) for $\beta = 4$ in [22], to the rectangular $\beta = 2$ case [27], and to the joint density for $\beta = 1$ in [28]. Here also products of rectangular truncated unitary matrices and products of mixed type were considered, for which a weak commutation relation was proved. In all these results, the change of variables from matrices to complex eigenvalues uses a generalised Schur decomposition.

This decomposition was well-known for $M = 2$ and $\beta = 1, 2$, see *e.g.* [29], whereas its generalisation to an arbitrary number of matrices M was given in [27, 30] for $\beta = 2$, and was extended to $\beta = 4$ in [31]. A similar extension exists for $\beta = 1$ and was implicitly used in [28, 32]. In [27], also mixed products of Ginibre and inverse Ginibre matrices, and of truncated unitary matrices and their inverses were considered, and their joint densities and corresponding kernels were given. The large- N limit for such products of unitary matrices truncated from $U(N + \kappa)$ to $U(N)$, also called random contractions, was studied in [33]. At strong non-unitarity in the bulk and edge scaling limit, the universal result for a single Ginibre matrix was recovered. In the origin limit, the same new class as for products of M Ginibre matrices [20] mentioned above was found. Very recently, a more rigorous derivation of this bulk and edge universality was presented in [34], including the case of products of rectangular Ginibre matrices when both the inner and outer edge become soft. At weak non-unitarity, the result for a single matrix [35] was extended to a new kernel labelled $M\kappa$ in [33]. A discussion for $M > 1$ products of truncated orthogonal (and unitary symplectic) matrices was given in [28], albeit an understanding of the universal kernels is still lacking — for a single truncated orthogonal matrix see, however, [36].

The distribution of radii of complex eigenvalues of products of Ginibre matrices is given by permanents [21, 31], *cf.* [37] for $M = 1$. The corresponding gap and overcrowding probabilities were derived in [21] for $\beta = 2$ and in [31] for $\beta = 4$, including the corresponding asymptotic expansions for finite and infinite point processes, for fixed M . The limiting distribution of the largest radius of all eigenvalues was shown to interpolate between the log-normal and Gumbel distribution (valid for a single Ginibre matrix [38]) in a double scaling limit in [39], where both N and M become infinite.

Turning to singular values, their joint density and corresponding kernel of orthogonal functions were derived in [23] for square and [24] for rectangular matrices with $\beta = 2$. These enjoy a relation to multiple OP [40] and a universal correlation kernel labelled M was found in the local origin scaling limit [41]. For $M = 2$, it agrees with the limiting kernel in a Cauchy two-matrix model [42], a correspondence that was extended to the Cauchy multi-matrix model and general M very recently [43], *cf.* [44]. This result was generalised in [45] to a mixed product of Ginibre matrices times a single truncated unitary matrix, introducing the notion of polynomial ensembles. There, their relation to a certain type of biorthogonal ensembles previously studied in [46, 47] was pointed out. Polynomial ensembles enjoy special invariance properties, related to products of random matrices [48]. The same limiting kernel as in [41] was also found in [49] for products of Ginibre and inverse Ginibre matrices, where the average characteristic polynomial was computed as well. The zeros and asymptotic analysis of the average characteristic polynomial was performed in [50]. The Fredholm determinant for the gap probability at the origin was shown to satisfy a system of non-linear ordinary differential equations [51]. In the bulk and at the soft edge, it was conjectured in [23] to find the universal sine and Airy kernel respectively, see [52] for a very recent proof including mixed products of Ginibre matrices and their inverse. The correlation functions of singular values of truncated unitary matrices is also currently under way [53]. So far, information on local properties of the singular values for the $\beta = 1, 4$ classes has been very difficult to access, due to the absence of the corresponding Harish–Chandra integral. Very recent progress has been possible using supersymmetric techniques, see [54].

The limiting positions for the Lyapunov exponents of products of $\beta = 2$ [55] and $\beta = 4$ [56] Ginibre matrices were derived using probabilistic methods, including correlated Gaussian distributions for each factor. The detailed knowledge for the joint densities described above allowed to show [33] for uncorrelated complex Ginibre matrices that each Lyapunov exponent becomes normally distributed, following a permanental point process. The same leading order behaviour holds taking the same limit of the radii of complex eigenvalues, called stability exponents. This leads to a one-to-one correspondence between limiting radii and singular values. The expressions for the stability exponents were extended to $\beta = 4$ and $\beta = 1$ most recently in [11], where the latter case was obtained under the assumption that for large M , all eigenvalues of the real product matrix become real. Such a behaviour was previously observed in [57] and proved for general N in [32].

All aforementioned finite- N and $-M$ results have assumed that each factor in the product is independent¹. To date, only complex eigenvalues of products of $M = 2$ matrices that are coupled through an Itzykson–Zuber term and thus not independent have been studied for finite- N at $\beta = 2$ [58], $\beta = 4$ [59], and $\beta = 1$ [60]. They are given in terms of parameter-dependent families of Laguerre polynomials in the complex plane, and we refer to the review [61] where all these results are summarised. Results for singular values of products of such dependent matrices are currently under way [62].

The remaining content of this paper is organised as follows. In Section 2, we will present exact results for finite N and M including the joint densities, kernels and correlation functions; the section is divided into two Subsections 2.1 and 2.2 discussing results for complex eigenvalues and singular values, respectively. In both cases, we consider Ginibre and inverse Ginibre matrices as well as truncated unitary matrices. We turn to the local large- N limits at fixed M depending on the location in the spectrum in Section 3. Here, we will be very brief and only give more details for the limiting kernels that were not previously known. Again, we will first discuss complex eigenvalues in Subsection 3.1 and then discuss singular values in Subsection 3.2. The following Section 4 is devoted to the opposite infinite product limit at fixed N . Our discussion of open problems follows in Section 5.

2. Exact results for finite products of finite size matrices

2.1. Complex eigenvalues

2.1.1. Ginibre matrices

We begin with the simplest case: A *product of rectangular Ginibre matrices*. We emphasise that this special case is very illustrative, since the main ideas from the treatment of products of Ginibre matrices extend to all the other examples described in this review.

We are seeking the statistical properties of complex eigenvalues of the product matrix

$$\Pi_M \equiv X_M X_{M-1} \dots X_2 X_1, \quad (2.1)$$

where each X_j is an $N_j \times N_{j-1}$ matrix distributed according to a Gaussian density

$$P(X_j) = \left(\frac{\beta}{2\pi} \right)^{\beta N_j N_{j-1}/2} \exp \left[-\frac{\beta}{2\gamma} \text{Tr } X_j X_j^\dagger \right], \quad j = 1, \dots, M. \quad (2.2)$$

Here, the matrices X_j ($j = 1, \dots, M$) are independent and the index $\beta = 1, 2, 4$ denotes whether real, complex or quaternionic matrices are considered

¹ Of course, powers of a single matrix form an exception, leading to the same global density as the product of independent matrices, see *e.g.* [14].

(we only multiply matrices with the same β). The parameter

$$\gamma = \begin{cases} 1 & \text{for } \beta = 1, 2 \\ 2 & \text{for } \beta = 4 \end{cases} \quad (2.3)$$

is related to the fact that quaternions are given by their 2×2 matrix representation, which implies that the eigenvalues come in complex conjugate pairs (see *e.g.* [63]). If we disregard eigenvalues which are trivially zero, then we can choose $N \equiv N_0 \leq N_1 \leq \dots \leq N_M$ without loss of generality; this is due to a weak commutation relation [28]. Furthermore, we can parametrise the matrix product as

$$X_M X_{M-1} \dots X_2 X_1 = U \begin{pmatrix} \tilde{X}_M \tilde{X}_{M-1} \dots \tilde{X}_2 \tilde{X}_1 \\ 0 \end{pmatrix}, \quad (2.4)$$

where U is an $N_M \times N_M$ orthogonal ($\beta = 1$), unitary ($\beta = 2$), or unitary symplectic ($\beta = 4$) matrix and each \tilde{X}_j is an $N \times N$ matrix. This parameterisation results in the measure [28]

$$\prod_{j=1}^M dX_j P(X_j) = d\mu(U) \prod_{j=1}^M d\tilde{X}_j \tilde{P}_{\nu_j}(\tilde{X}_j), \quad (2.5)$$

where dX_j and $d\tilde{X}_j$ are the flat measures over all independent matrix elements, $d\mu(U)$ is the normalised Haar measure, and each \tilde{X}_j is distributed according to the induced density

$$\tilde{P}_{\nu_j}(\tilde{X}_j) \propto \det[\tilde{X}_j \tilde{X}_j^\dagger]^{\beta \nu_j / (2\gamma)} \exp \left[-\frac{\beta}{2\gamma} \text{Tr } \tilde{X}_j \tilde{X}_j^\dagger \right]. \quad (2.6)$$

Here, we have introduced the convenient notation $\nu_j \equiv N_j - N$ for the differences between matrix dimensions. We emphasise that (2.5) enables us to reduce the problem involving rectangular matrices to the problem involving square matrices, which is a considerable simplification (we refer to [28] for a more thorough discussion). It should be noted that this simplification is possible for all examples mentioned in this review. Finally, we note that the induced densities (2.6) are isotropic, *i.e.* they are invariant under $X_j \rightarrow UX_jV$, where U and V are orthogonal ($\beta = 1$), unitary ($\beta = 2$), or unitary symplectic ($\beta = 4$) matrices. This is an extremely important observation needed for the weak communication relation [28], which states that any averaged property of such a product matrix (depending only on the product matrix itself, not on the individual matrices) is independent of the ordering of the factors. For products of rectangular Ginibre matrices,

this implies that all averaged properties (*e.g.* correlations between eigenvalues or singular values) are invariant under permutations of the indices ν_m ($m = 1, \dots, M$). The reader might note that all products we consider in this review are independent of the ordering of the factors. This originates from the fact that *all ensembles we consider are isotropic*.

We are now ready to state the normalised *joint probability density function* (jpdf) of complex eigenvalues for the product matrix Π_M . For $\beta = 2$, the jpdf reads [20, 27, 28]

$$\mathcal{P}_{\text{jpdf}, \nu}^{\beta=2}(z_1, \dots, z_N) = \frac{1}{\mathcal{Z}_{N, \nu}^{\beta=2}} \prod_{n=1}^N w_{\nu}^{\beta=2}(z_n) \prod_{1 \leq j < l \leq N} |z_l - z_j|^2. \quad (2.7)$$

Apart from the corresponding *weight function* given by a Meijer G -function

$$w_{\nu}^{\beta}(z) = G_{0, M}^{M, 0} \left(\begin{matrix} - \\ \frac{\beta}{2} \nu_1, \dots, \frac{\beta}{2} \nu_M \end{matrix} \middle| \left(\frac{\beta}{2} \right)^M |z|^2 \right), \quad (2.8)$$

it agrees with the jpdf for a single Ginibre matrix $M = 1$, where the repulsion of eigenvalues is given by the squared absolute value of the Vandermonde determinant. The *normalisation constant* is given for all three $\beta = 1, 2, 4$ by

$$\mathcal{Z}_{N, \nu}^{\beta} = \frac{N! \pi^{N(\beta-\gamma)/\gamma}}{2^{(2-\beta)\gamma MN(N+1)/4}} \prod_{n=1}^N \prod_{j=1}^M \Gamma \left(\frac{\beta}{2} (\nu_j + n) \right). \quad (2.9)$$

The following steps are needed to derive the jpdf in (2.7) for $\beta = 2$ for general M (and subsequently, for the other β s): (*i*) a generalised Schur decomposition [20, 27, 30] of the individual factors distributed according to (2.6), and (*ii*) the fact that the upper triangular matrices from this decomposition decouple. The remaining integrals can be performed [20, 27, 28] and lead to the Meijer G -function as the weight in (2.8). For the most general set of indices, the Meijer G -function is defined as [64]

$$G_{p, q}^{m, n} \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) \equiv \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{ds z^s \prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)}. \quad (2.10)$$

The integration contour \mathcal{C} depends on the poles of the Gamma functions, cf. [64]. In the simplest case with $p + q < 2(m + n)$ and z in the upper half plane, it runs from $-i\infty$ to $+i\infty$ leaving all poles of $\Gamma(b_j - s)$ to the right and of $\Gamma(1 - a_j + s)$ to the left. Empty products are defined as unity. For the special case (2.8), an alternative multiple integral representation of the

Meijer G -function exists [20, 27]

$$G_{0,M}^{M,0}\left(\begin{matrix} - \\ \nu_1, \dots, \nu_M \end{matrix} \middle| |z|^2\right) = \pi^{1-M} \prod_{j=1}^M \int_{\mathbb{C}} d^2 z_j |z_j|^{2\nu_j} e^{-|z_j|^2} \delta^2(z - z_M \dots z_1). \quad (2.11)$$

Note that the right-hand side corresponds to the density of a product of (unnormalised) scalar-valued random variables. Here, we get the explanation why the Meijer G -function appears: it was previously known for Gaussian or gamma random variables that their product is distributed according to the Meijer G -function [25]. This property is passed on to products of random matrices as well. It follows immediately from the representation (2.11) that the (bi-)moments of the weight functions are given by

$$\int_{\mathbb{C}} d^2 z w_{\nu}^{\beta=2}(z) z^k z^{*\ell} = \delta_{k,\ell} \pi \prod_{j=1}^M \Gamma(\nu_j + k + 1). \quad (2.12)$$

Here, the moments are zero for $k \neq \ell$ since the weight function (2.8) is invariant under rotation in the complex plane.

The k -point density correlation functions are defined following [63]

$$R_k^{\beta}(z_1, \dots, z_k) \equiv \frac{N!}{(N-k)!} \int_{\mathbb{C}} d^2 z_{k+1} \dots d^2 z_N \mathcal{P}_{\text{jpdf}}^{\beta}(z_1, \dots, z_N). \quad (2.13)$$

We will consider jpdfs with the following form:

$$\mathcal{P}_{\text{jpdf}}^{\beta=2}(z_1, \dots, z_N) = \frac{1}{\mathcal{Z}_N^{\beta=2}} \prod_{n=1}^N w^{\beta=2}(|z_n|) \prod_{1 \leq \ell < k \leq N} |z_k - z_{\ell}|^2, \quad (2.14)$$

where $w^{\beta=2}(|z|)$ is an arbitrary weight function that only depends on the modulus. Then, it is well-known that the corresponding OPs are monomials

$$\int_{\mathbb{C}} d^2 z w^{\beta=2}(|z|) z^k z^{*\ell} = \delta_{k,\ell} h_k. \quad (2.15)$$

Furthermore, it follows from general considerations that the correlation functions, including the jpdf for $k = N$, form a *determinantal point process* [63]

$$R_k^{\beta=2}(z_1, \dots, z_k) = \prod_{j=1}^k w^{\beta=2}(|z_j|) \det_{1 \leq i, j \leq k} \left[K_N^{\beta=2}(z_i, z_j) \right], \quad (2.16)$$

where the *kernel* is given by

$$K_N^{\beta=2}(z, u) = \sum_{n=0}^{N-1} \frac{(zu^*)^n}{h_n}. \quad (2.17)$$

Here, h_n denotes the squared norm determined through the orthogonality relation (2.15). The normalisation constant in (2.14) is uniquely determined by these squared norms as well. Explicitly, we have

$$\mathcal{Z}_N^\beta = N! \prod_{n=0}^{N-1} h_n. \quad (2.18)$$

For the product of complex ($\beta = 2$) rectangular Ginibre matrices, the jpdf is given by (2.7) with weight (2.8) and the squared norms are immediately obtained from (2.12). Thus the *kernel for the product of rectangular Ginibre matrices* is given by [20, 27]

$$K_{N,\nu}^{\beta=2}(z, u) = \frac{1}{\pi} \sum_{n=0}^{N-1} \frac{(zu^*)^n}{\prod_{m=1}^M \Gamma(n+1+\nu_m)}, \quad (2.19)$$

while the corresponding *normalisation constant* follows, cf. (2.9)

$$\mathcal{Z}_{N,\nu}^\beta = N! \pi^N \prod_{n=0}^{N-1} \prod_{j=1}^M \Gamma(n+1+\nu_j). \quad (2.20)$$

This determines all k -point correlation functions for $\beta = 2$ via (2.16) with weight (2.8).

The second set of results that follows from Eq. (2.14) for $\beta = 2$ is the *joint density of the radii* r_j of the complex eigenvalues, $z_j = r_j e^{i\theta_j}$. It is obtained by integrating over all angles θ_j leading to a permanent [21]

$$\prod_{n=1}^N \int_0^{2\pi} d\theta_n r_n \mathcal{P}_{\text{jpdf}}^{\beta=2}(z_1, \dots, z_N) = \frac{\prod_{n=1}^N 2\pi w^{\beta=2}(r_n)}{\mathcal{Z}_N^{\beta=2}} \text{per}_{1 \leq j, l \leq N} \left[r_{j-1}^{2l-1} \right], \quad (2.21)$$

where we have included the factors r_n from the radial measure $dr_n r_n$. The radii thus become independent random variables, generalising the result in [65]. Moreover, the *hole probability* that a disc of radius r centred at

the origin is empty of eigenvalues becomes

$$\text{Prob}[\forall j : r_j > r] \equiv \prod_{n=1}^N \int_r^\infty dr_n r_n \int_0^{2\pi} d\theta_n \mathcal{P}_{\text{jpdf}}^{\beta=2}(z_1, \dots, z_N) \quad (2.22)$$

$$= \prod_{n=1}^N \frac{G_{1,M+1}^{M+1,0} \left(\begin{matrix} 1 \\ 0, n + \nu_1, \dots, n + \nu_M \end{matrix} \middle| r^2 \right)}{\prod_{j=1}^M \Gamma(n + \nu_j)}. \quad (2.23)$$

The second line was derived in [31]. It agrees with [63] for $M = 1$ with $\nu_1 = 0$. An alternative result for $M = 2$ was previously obtained in [66].

The quaternionic k -point correlation functions for $\beta = 4$ are defined identically to the complex case (2.13). Here, we will consider jpdfs of the form:

$$\mathcal{P}_{\text{jpdf}}^{\beta=4}(z_1, \dots, z_N) = \frac{1}{\mathcal{Z}_N^{\beta=4}} \prod_{n=1}^N w^{\beta=4}(|z_n|) |z_n - z_n^*|^2 \prod_{1 \leq k < \ell \leq N} |z_\ell - z_k|^2 |z_\ell - z_k^*|^2, \quad (2.24)$$

where the weight function is invariant under rotation in the complex plane. We restrict ourselves to the upper half plane \mathbb{C}_+ , due to complex conjugated pairing of eigenvalues for $\beta = 4$. It should be noted that except for the weight function, the jpdf is identical to that of a single Ginibre matrix $M = 1$. Additionally, we define the *skew-symmetric product*

$$\langle f, g \rangle_s \equiv \int_{\mathbb{C}_+} d^2 z w^{\beta=4}(|z|) (z^* - z) (f(z)g(z^*) - f(z^*)g(z)). \quad (2.25)$$

It follows from general considerations that the corresponding k -point correlation functions including the jpdf form a *Pfaffian point process* [63, 67] reading

$$\begin{aligned} R_k^{\beta=4}(z_1, \dots, z_k) \\ = \prod_{j=1}^k w^{\beta=4}(z_j) (z_j^* - z_j) \text{Pf}_{1 \leq i, j \leq k} \begin{bmatrix} K_N^{\beta=4}(z_i, z_j^*) & -K_N^{\beta=4}(z_i^*, z_j^*) \\ K_N^{\beta=4}(z_i, z_j) & -K_N^{\beta=4}(z_i^*, z_j) \end{bmatrix}, \end{aligned} \quad (2.26)$$

with the *kernel* given in terms of monic skew-orthogonal polynomials

$$K_{N,\nu}^{\beta=4}(z, u) = \sum_{n=0}^{N-1} \frac{p_{2n+1}(z)p_{2n}(u) - p_{2n+1}(u)p_{2n}(z)}{h_n}. \quad (2.27)$$

Here, the *skew-orthogonal polynomials* $p_n(z)$ are defined such that they satisfy the *skew-orthogonality relations* [63, 67]

$$\langle p_{2k+1}, p_{2\ell+1} \rangle_s = \langle p_{2k}, p_{2\ell} \rangle_s = 0, \quad \langle p_{2k+1}, p_{2\ell} \rangle_s = h_k \delta_{k,\ell}. \quad (2.28)$$

We know that the weight function in the jpdf (2.24) is invariant under rotation in the complex plane. This implies that

$$\int_{\mathbb{C}_+} d^2 z \, w^{\beta=4}(|z|) z^k z^{*\ell} = s_k \delta_{k,\ell}, \quad (2.29)$$

where s_k are constants depending on the weight. It was pointed out in [22] that in order to find the skew-orthogonal polynomials, it is sufficient to calculate the constants, s_k . Explicitly, we have

$$p_{2n}(z) = \sum_{k=0}^n \left[\prod_{\ell=k+1}^n \frac{s_{2\ell}}{s_{2\ell-1}} \right] z^{2k}, \quad p_{2n+1}(z) = z^{2n+1}, \quad h_n = 2 s_{2n+1}. \quad (2.30)$$

The main idea behind this result is that the odd polynomials are always monomials, due to the rotational invariance of the weight.

We can now turn to *products of rectangular Ginibre matrices with quaternionic matrix elements* ($\beta = 4$), where we refer to [22, 28, 31] for details. The derivation of the jpdf for the complex eigenvalues is based on the generalised Schur decomposition for quaternionic matrices, see [31]. With the notation given above, we have

$$\mathcal{P}_{\text{jpdf}, \nu}^{\beta=4}(z_1, \dots, z_N) = \frac{1}{\mathcal{Z}_{N, \nu}^{\beta=4}} \prod_{n=1}^N w_{\nu}^{\beta=4}(z_n) |z_n - z_n^*|^2 \prod_{1 \leq k < l \leq N} |z_l - z_k|^2 |z_k - z_l^*|^2 \quad (2.31)$$

with the weight and normalisation constant defined in (2.8) and (2.9), respectively. Inserting (2.8) with $\beta = 4$ into (2.29) yields

$$s_n = \frac{\pi}{2^{M(n+1)+1}} \prod_{m=1}^M \Gamma(2\nu_m + n + 1). \quad (2.32)$$

This determines the skew-orthogonal polynomials (2.30) and, therefore, all correlation functions via (2.26), with weight and kernel given by (2.8) and (2.27), respectively.

Turning to the *distribution of radii*, we obtain up to factors of 2 the same result as in Eq. (2.21) after integration

$$\prod_{n=1}^N \int_0^{2\pi} d\theta_n r_n \mathcal{P}_{\text{jpdf},\nu}^{\beta=4}(z_1, \dots, z_N) = \frac{\prod_{n=1}^N 4\pi w_\nu^{\beta=4}(r_n)}{\mathcal{Z}_{N,\nu}^{\beta=4}} \text{per}_{1 \leq j, l \leq N} \left[r_{j-1}^{4l-3} \right]. \quad (2.33)$$

This is true despite the initial repulsion of eigenvalues from the real line in Eq. (2.31). The radii are once again independent random variables, generalising the results of [38]. Likewise, the $\beta = 4$ *hole probability* defined as in Eq. (2.22) was obtained in [31]

$$\text{Prob}[\forall j : r_j > r] = \prod_{n=1}^N \frac{G_{1,M+1}^{M+1,0} \left(0, 2n + 2\nu_1, \dots, 2n + 2\nu_M \middle| 2^M r^2 \right)}{\prod_{j=1}^M \Gamma(2n + 2\nu_j)}. \quad (2.34)$$

For $M = 1$, it agrees with [63]; for $M = 2$, see [66] for a different expression.

Finally, we state the jpdf for *products of rectangular real Ginibre matrices* ($\beta = 1$). It is well-known that the eigenvalues of a real matrix are either real or come in complex conjugate pairs. The main difficulty for real matrices is that a complete triangularisation is possible if and only if all the eigenvalues are real [29]. Typically, a real Ginibre matrix will have both real and complex eigenvalues which prevents this. However, it is always possible to make an incomplete triangularisation involving 2×2 matrices. If N is even, then it is possible to write down the *jpdf in terms of 2×2 matrices* [28]

$$\mathcal{P}_{\text{jpdf},\nu}^{\beta=1}(Z_1, \dots, Z_{N/2}) = \frac{1}{\mathcal{Z}_{N,\nu}^{\beta=1}} \prod_{n=1}^{N/2} W_\nu^{\beta=1}(Z_n) \prod_{1 \leq k < \ell \leq \frac{N}{2}} |\det [Z_\ell^T \otimes \mathbb{I}_2 - \mathbb{I}_2 \otimes Z_k]|, \quad (2.35)$$

where each Z_n is a real 2×2 matrix, hence its eigenvalues are either real or a complex conjugate pair. These eigenvalues can be identified with the eigenvalues of the original $N \times N$ matrix Π_M . A similar expression for N odd was given in [28], but will not be repeated here. The *weight function with a matrix argument* is given by

$$W_\nu^{\beta=1}(Z) = \prod_{j=1}^M \int_{\mathbb{R}^{2 \times 2}} dZ_j |\det Z_j|^{\nu_j} e^{-\frac{1}{2} \text{Tr} Z_j^T Z_j} \delta^{2 \times 2}(Z - Z_M \dots Z_1), \quad (2.36)$$

which should be compared to expression (2.11). The fact that the jpdf (2.35) is only known up to 2×2 matrix integrals plus the fact that one has to

distinguish real and complex conjugate eigenvalue pairs has so far prevented progress in computing density correlation functions or the distribution of radii. In [28], a general approach was presented using the fact that the eigenvalues of a 2×2 matrix can be linked to its singular values. This approach led to an M -fold integral representation of the two-point weight, but the expression was too complicated for any further calculations to be tractable.

As mentioned above, a complete triangularisation is possible when all eigenvalues are real, $z_j = x_j \in \mathbb{R}$ ($j = 1, \dots, N$). In this special case, the approach used for $\beta = 2, 4$ can be extended in a straightforward manner leading to a *jpdf with all eigenvalues real* [32],

$$\mathcal{P}_{\text{jpdf}, \nu}^{\beta=1, \mathbb{R}}(x_1, \dots, x_N) = \frac{1}{\mathcal{Z}_{N, \nu}^{\beta=1}} \prod_{n=1}^N w_{\nu}^{\beta=1}(x_n) \prod_{1 \leq j < l \leq N} |x_l - x_j|, \quad (2.37)$$

with the weight and normalisation given in Eqs. (2.8) and (2.9), respectively. If N is even, then the probability that all eigenvalues are real is given by [32]

$$\begin{aligned} & \prod_{n=1}^N \int_{\mathbb{R}} dx_n \mathcal{P}_{\text{jpdf}, \nu}^{\beta=1, \mathbb{R}}(x_1, \dots, x_N) \\ &= \frac{\det_{1 \leq k < \ell \leq \frac{N}{2}} \left[G_{M+1, M+1}^{M+1, M} \left(\frac{3}{2} - \frac{\nu_1}{2} - \ell, \dots, \frac{3}{2} - \frac{\nu_M}{2} - \ell, 1 \middle| 1 \right) \right]}{\prod_{n=1}^N \prod_{m=1}^M \Gamma\left(\frac{\nu_m}{2} + \frac{n}{2}\right)}. \end{aligned} \quad (2.38)$$

A similar expression holds for N odd.

2.1.2. Ginibre and inverse Ginibre matrices

Next, we consider *the generalised eigenvalue problem* given by the characteristic equation

$$\det[Y_1 \dots Y_L \lambda - X_M \dots X_1] = 0, \quad (2.39)$$

where each X_j is an $N_j \times N_{j-1}$ matrix and each Y_j is an $N_{M+j-1} \times N_{M+j}$ matrix. If each X_j and Y_j is distributed independently according to (2.2), then the generalised eigenvalue problem may formally be thought of finding the eigenvalues of a *mixed product of Ginibre and inverse Ginibre matrices*

$$(Y_1 \dots Y_L)^{-1} X_M \dots X_1. \quad (2.40)$$

This was studied for square matrices in [27]; the *jpdf of complex eigenvalues* reads

$$\mathcal{P}_{\text{jpdf}, \nu, \mu}^{\beta=2}(z_1, \dots, z_N) = \frac{1}{\mathcal{Z}_{N, \nu, \mu}^{\beta=2}} \prod_{n=1}^N w_{\nu, \mu}^{\beta=2}(z_n) \prod_{1 \leq j < l \leq N} |z_l - z_j|^2 \quad (2.41)$$

with *weight function*

$$w_{\nu,\mu}^{\beta=2}(z) = G_{L,M}^{M,L} \left(\begin{matrix} -N - \mu_1, \dots, -N - \mu_L \\ \nu_1, \dots, \nu_M \end{matrix} \middle| |z|^2 \right). \quad (2.42)$$

Here, we use the notation $\nu_i = N_i - N$ for $i = 1, \dots, M$ and $\mu_j = N_{M+j} - N$ for $j = 1, \dots, L$. Equivalently, the weight function can be written in the following integral representation

$$\begin{aligned} w_{\nu,\mu}^{\beta=2}(z) &= \pi^{1-M-L} \prod_{\ell=1}^L \int_{\mathbb{C}} d^2 u_{\ell} e^{-|u_{\ell}|^2} |u_{\ell}|^{2(N+\mu_{\ell}-2)} \\ &\times \prod_{m=1}^M \int_{\mathbb{C}} d^2 z_m e^{-|z_m|^2} |z_m|^{2\nu_m} \delta^2 \left(z - \frac{z_M \dots z_1}{u_L \dots u_1} \right). \end{aligned} \quad (2.43)$$

It follows that the squared norms defined via (2.15) are given by

$$h_n^{\nu,\mu} = \pi \prod_{m=1}^M \Gamma(\nu_m + n + 1) \prod_{\ell=1}^L \Gamma(N + \mu_{\ell} - n). \quad (2.44)$$

Thus the *normalisation constant* directly follows from (2.18), while all k -point correlation functions are given via (2.16) with weight (2.42) and kernel (2.17). Note that in both (2.18) and (2.17), we have $0 \leq n \leq N-1$, which ensures that the gamma functions in (2.44) are well-defined. For $L = 0$, the jpdf (2.41) reduces to the previous results (2.7), while for $M = L = 1$, it reduces to the spherical ensemble introduced and solved in [68].

The joint density of the radii is given by (2.21) and a simple calculation starting from the definition (2.22) yields the *hole probability*

$$\text{Prob}[\forall j : r_j > r] = \prod_{n=1}^N \frac{G_{L+1,M+1}^{M+1,L} \left(\begin{matrix} n - N - \mu_1, \dots, n - N - \mu_L, 1 \\ 0, n + \nu_1, \dots, n + \nu_M \end{matrix} \middle| r^2 \right)}{\prod_{k=1}^M \Gamma(n + \nu_k + 1) \prod_{\ell=1}^L \Gamma(N + \mu_{\ell} - n)}. \quad (2.45)$$

The result for $\beta = 2$ can be extended to $\beta = 4$ in a simple manner. The *jpdf of complex eigenvalues* is of the same form as (2.24) except that the weight function and the normalisation constant changes. The *weight function* for $\beta = 4$ can be expressed in terms of (2.42)

$$w_{\nu,\mu}^{\beta=4}(z) = w_{2\nu,2\mu+N}^{\beta=2} \left(2^{(M-L)/2} z \right). \quad (2.46)$$

It follows from (2.29) that

$$s_n = \frac{\pi}{2^{(M-L)(n+1)+1}} \prod_{m=1}^M \Gamma(2\nu_m + n + 1) \prod_{\ell=1}^L \Gamma(2N + 2\mu_{\ell} - n + 1). \quad (2.47)$$

This determines the skew-orthogonal polynomials (2.30) and, therefore, all correlation functions via (2.26) with the kernel given by (2.27). The corresponding spherical ensemble ($M = L = 1$) for $\beta = 4$ was studied in [69]. For $\beta = 1$, the spherical ensemble was studied in [70] but we will not discuss the generalisation to arbitrary M and L for $\beta = 1$ here.

2.1.3. Truncated unitary matrices

Now, we turn to *products of truncated unitary matrices*. Consider M independent orthogonal ($\beta = 1$), unitary ($\beta = 2$), or unitary symplectic ($\beta = 4$) matrices U_j of different sizes K_j for $j = 1, \dots, M$. Let the unitary matrices be uniformly distributed with respect to their corresponding Haar measure; we seek the truncation of each matrix U_j to its upper-left sub-block X_j of size $N_j \times N_{j-1}$. We are interested in the complex eigenvalues of the following product of M truncated unitary matrices

$$\Pi_M \equiv X_M \dots X_1. \quad (2.48)$$

For $K_j - N_j - N_{j-1} > 0$, projection of the Haar measure on the original group to a measure for the sub-block gives

$$P_j(X_j) \propto \Theta \left[1 - X_j^\dagger X_j \right] \det \left[1 - X_j^\dagger X_j \right]^{\beta(K_j - N_j - N_{j-1} + 1 - 2/\beta)/(2\gamma)} \quad (2.49)$$

for each X_j . Here, Θ denotes the Heaviside theta function of matrix argument. In the more general case, see [33] for an integral representation of the measure.

For $\beta = 2$, the *jpdf of complex eigenvalues* derived in [27, 28, 33] is of the same form as Eq. (2.7), with the *weight function* given by

$$w_{\nu, \kappa}^{\beta=2}(z) = G_{M,M}^{M,0} \left(\begin{matrix} \kappa_1, \dots, \kappa_M \\ \nu_1, \dots, \nu_M \end{matrix} \middle| |z|^2 \right). \quad (2.50)$$

Here, we have $\nu_j \equiv N_j - N$ and $\kappa_j \equiv K_j - N_{j-1}$, with the restriction $\kappa_j - \nu_j > 0$ for $j = 1, \dots, M$. The theta function in (2.49) is included in the Meijer G -function, because the weight function (2.50) is strictly zero outside the unit disc. For example, we have [64]:

$$G_{1,1}^{1,0} \left(\begin{matrix} 1 \\ 0 \end{matrix} \middle| |z|^2 \right) = \Theta(1 - |z|^2) \quad \text{and} \quad G_{2,2}^{2,0} \left(\begin{matrix} 1, 1 \\ 0, 0 \end{matrix} \middle| |z|^2 \right) = -\log|z|^2 \Theta(1 - |z|^2). \quad (2.51)$$

Again, the weight function can be written via an M -fold integral representation [27, 28, 33],

$$w_{\nu,\mu}^{\beta=2}(z) = \pi^{1-M} \prod_{m=1}^M \frac{1}{\Gamma(\kappa_m - \nu_m)} \int_D d^2 z_m |z_m|^{2\nu_m} (1 - |z_m|^2)^{\kappa_m - \nu_m - 1} \times \delta^2(z - z_M \dots z_1), \quad (2.52)$$

where the integration domain, D , is the unit disk. From this representation, we find the squared norms

$$h_n^{\nu,\mu} = \pi \prod_{m=1}^M \frac{\Gamma(\nu_m + n + 1)}{\Gamma(\kappa_m + n + 1)}. \quad (2.53)$$

The *kernel* and the *normalisation constant* are uniquely determined by (2.17) and (2.18), respectively.

In complete analogy to the Ginibre case, we may go to the generalised eigenvalue problem (2.39) and consider a *mixed product of truncated and inverse truncated matrices*. We introduce L additional independent unitary matrices V_j of different sizes T_j for $j = 1, \dots, L$, which are distributed uniformly with respect to Haar measure. Truncating each matrix V_j to its upper-left sub-block Y_j of size $N_{M+j-1} \times N_{M+j}$, the general eigenvalue problem gives rise to a jpdf with exactly the same structure as (2.7), except that the weight function (2.50) is replaced by [27]

$$w_{\nu,\mu,\kappa,\tau}^{\beta=2}(z) = G_{M+L,M+L}^{M,L} \left(\begin{matrix} -N - \mu_1, \dots, -N - \mu_L, \kappa_1, \dots, \kappa_M \\ \nu_1, \dots, \nu_M, -N - \tau_1, \dots, -N - \tau_L \end{matrix} \middle| |z|^2 \right). \quad (2.54)$$

Here, $\mu_j \equiv N_{M+j} - N$ and $\tau_j \equiv T_j - N_{M+j-1}$ satisfy the restriction $\tau_j - \mu_j > 0$ for $j = 1, \dots, L$. Note that if $M = 0$, then the weight function (2.54) is strictly zero inside the unit disc.

The result for $\beta = 2$ can be extended to $\beta = 4$ in a simple manner. The *jpdf of complex eigenvalues* is of the same form as (2.31) except that the weight function and the normalisation constant changes, see [28]. The *weight function* for the product of truncated unitary symplectic matrices can be expressed in terms of (2.50) as

$$w_{\nu,\kappa}^{\beta=4}(z) = w_{2\nu,2\kappa-1}^{\beta=2}(z). \quad (2.55)$$

It follows from (2.29) that

$$s_n = \pi \prod_{m=1}^M \frac{\Gamma(2\nu_m + n + 1)}{\Gamma(2\kappa_m + n)}. \quad (2.56)$$

This determines the skew-orthogonal polynomials (2.30) and, therefore, all correlation functions via (2.26), with the kernel given by (2.27).

2.1.4. Mixed products

Finally, we mention that no new techniques are required in order to study more complicated products constructed from different combinations of Ginibre, inverse Ginibre, truncated unitary as well as inverse truncated unitary matrices. Such products will have a jpdf with a structure similar to the examples above and the weight function can again be expressed as a Meijer G -function, albeit with more indices. In fact, products of Ginibre and truncated unitary matrices have previously been studied in [28] for all β , although the most general results are restricted to $\beta = 2, 4$ due to the incompleteness of a generalised real Schur decomposition as discussed in Section 2.1.1. We stress again that all matrix ensembles described in this section are isotropic, which implies that the ordering of the matrices is irrelevant for all statistical properties of the eigenvalues, see [28].

2.2. Singular values

In this section, we seek the statistical properties of the singular values rather than of the complex eigenvalues for some random product matrix. Explicitly, we consider products of Ginibre, inverse Ginibre, and truncated unitary matrices. It turns out that all these ensembles are *polynomial ensembles* [45, 48] which are special types of *biorthogonal ensembles* [46, 47]. For this reason, we first recall a few general properties of polynomial ensembles. Let x_n ($n = 1, \dots, N$) be a set of positive variables. We are interested in a *generic jpdf* of the following form

$$\mathcal{P}_{\text{jpdf}}^{\beta=2}(x_1, \dots, x_N) = \frac{1}{\mathcal{Z}_N^{\beta=2}} \prod_{1 \leq i < j \leq N} (x_j - x_i) \det_{1 \leq \ell, k \leq N} \left[w_{k-1}^{\beta=2}(x_\ell) \right], \quad (2.57)$$

where $\{w_k^{\beta=2}(x)\}$ is a collection of weight functions on the positive half-line. The k -point correlation functions are defined by [63]

$$R_k^{\beta}(x_1, \dots, x_k) \equiv \frac{N!}{(N-k)!} \prod_{n=k+1}^N \int_0^\infty dx_n \mathcal{P}_{\text{jpdf}}^{\beta}(x_1, \dots, x_N). \quad (2.58)$$

It follows from the biorthogonal structure of the jpdf (2.57) that the k -point correlation functions including the jpdf form a *determinantal point process* reads

$$R_k^{\beta=2}(x_1, \dots, x_k) = \det_{1 \leq i, j \leq k} \left[K_N^{\beta=2}(x_i, x_j) \right], \quad (2.59)$$

with *kernel*

$$K_N^{\beta=2}(x, y) = \sum_{n=0}^{N-1} \frac{1}{h_n} p_n(x) \psi_n(y). \quad (2.60)$$

The functions $p_n(x)$ and $\psi_n(x)$ must satisfy the *biorthogonality relation*

$$\int_0^\infty dx p_k(x) \psi_\ell(x) = h_k \delta_{k,\ell}. \quad (2.61)$$

Due to the structure of the jpdf (2.57), each $p_n(x)$ is a monic polynomial. For the orthogonal functions $\psi_n(x)$, we require that their linear span given by $\text{span}\{\psi_0(x), \dots, \psi_{N-1}(x)\}$ agrees with that of the weight functions $w_n^{\beta=2}(x)$, $\text{span}\{w_0^{\beta=2}(x), \dots, w_{N-1}^{\beta=2}(x)\}$. The biorthogonal functions as well as the squared norms h_n are uniquely defined. Finally, the normalisation constant in the jpdf (2.57) can be written as a product following [63]

$$Z_N^{\beta=2} = N! \prod_{n=0}^{N-1} h_n. \quad (2.62)$$

Hence, the normalisation is completely determined by the squared norms.

2.2.1. Ginibre matrices

Now, we are ready to discuss *products of rectangular Ginibre matrices*. We keep the notation from Section 2.1.1 and consider a product matrix Π_M given by (2.1), where each X_j is an $(N + \nu_j) \times (N + \nu_{j-1})$ matrix distributed according to (2.2). A derivation of the jpdf for the singular values of the product matrix in the $\beta = 2$ case was presented in [23, 24] and explicitly uses the Itzykson–Zuber integration formula [71]. The absence of similar integration formulae for $\beta = 1$ and $\beta = 4$ have, so far, restricted explicit calculations to $\beta = 2$. For some asymptotic quantities, this difficulty can be circumvented using supersymmetric techniques [54].

Let x_j ($j = 1, \dots, N$) denote the squared singular values of the product matrix Π_M , then the *jpdf for the singular values of the product of complex Ginibre matrices* is given by (2.57) with weight functions [23, 24]

$$w_{\nu;k}^{\beta=2}(x) = G_{0,M}^{M,0} \left(\begin{matrix} - \\ \nu_1, \dots, \nu_{M-1}, \nu_M + k \end{matrix} \middle| x \right). \quad (2.63)$$

The biorthogonal functions corresponding to these weights were first obtained in [23, 24]. It was shown that the *biorthogonal functions* and the

squared norms are given by

$$p_n(x) = \sum_{k=0}^n \frac{(-1)^{n+k} n!}{(n-k)! k!} \left[\prod_{m=1}^M \frac{\Gamma(\nu_m + n + 1)}{\Gamma(\nu_m + k + 1)} \right] x^k, \quad (2.64)$$

$$\begin{aligned} \psi_n(x) &= \sum_{k=0}^n \frac{(-1)^{n+k} n!}{(n-k)! k!} \frac{\Gamma(\nu_M + n + 1)}{\Gamma(\nu_M + k + 1)} \\ &\quad \times G_{0,M}^{M,0} \left(\begin{matrix} - \\ \nu_1, \dots, \nu_{M-1}, \nu_M + k \end{matrix} \middle| x \right), \end{aligned} \quad (2.65)$$

$$h_n = n! \prod_{m=1}^M \Gamma(\nu_m + n + 1). \quad (2.66)$$

It is immediately seen from this representation that the biorthogonal functions are monic, but for further calculation it is often useful to rewrite them in terms of special functions. The polynomial $p_n(x)$ can be written either as a hypergeometric function or as a Meijer G -function, while $\psi_n(x)$ can be written as a single Meijer G -function. Explicitly, we have [23, 24]

$$\begin{aligned} p_n(x) &= (-1)^n \frac{h_n}{n! h_0} {}_1F_M \left(\begin{matrix} -n \\ \nu_1 + 1, \dots, \nu_M + 1 \end{matrix} \middle| x \right) \\ &= -h_n G_{1,M+1}^{0,1} \left(\begin{matrix} n+1 \\ -\nu_1, \dots, -\nu_M, 0 \end{matrix} \middle| x \right), \end{aligned} \quad (2.67)$$

$$\psi_n(x) = G_{1,M+1}^{M+1,0} \left(\begin{matrix} -n \\ 0, \nu_1, \dots, \nu_M \end{matrix} \middle| x \right). \quad (2.68)$$

This determines the normalisation via (2.62) and all correlations via (2.59) with kernel (2.60). Furthermore, the explicit formulation of the biorthogonal functions (2.67) and (2.68) allows a double contour (and a real) integral representation of the kernel [41],

$$\begin{aligned} K_N^{M,\nu}(x, y) &= \frac{1}{(2\pi i)^2} \int_{\mathcal{C}} du \oint_{\Sigma} dv \frac{x^u y^{-v-1}}{u-v} \frac{\Gamma(u-N+1)}{\Gamma(v-N+1)} \frac{\Gamma(u+1)}{\Gamma(v+1)} \prod_{m=1}^M \frac{\Gamma(u+\nu_m+1)}{\Gamma(v+\nu_m+1)} \\ &= \int_0^1 du G_{1,M+1}^{0,1} \left(\begin{matrix} N \\ -\nu_1, \dots, -\nu_M, 0 \end{matrix} \middle| ux \right) G_{1,M+1}^{M+1,0} \left(\begin{matrix} -N \\ 0, \nu_1, \dots, \nu_M \end{matrix} \middle| uy \right), \end{aligned} \quad (2.69)$$

where \mathcal{C} is a straight line from $-\frac{1}{2} - i\infty$ to $-\frac{1}{2} + i\infty$, while Σ encloses $1, \dots, N$ in the positive direction without any intersection with \mathcal{C} . This is

seen by writing the biorthogonal functions (2.67) and (2.68) as their integral representations, see (2.10), and inserting them into the expression for kernel (2.60); the sum can be performed and the first line of (2.69) is obtained. In the second line, we have displayed a second real integral representation from [41]. It is already very reminiscent of the large- N asymptotic result as we will see in Section 3.2.1, see Eq. (3.17).

It was pointed out in [41] that the polynomials (2.64) satisfy a stricter condition than biorthogonality; they are multiple orthogonal polynomials of type II with respect to the weights $w_{\nu;0}^{\beta=2}(x), \dots, w_{\nu;M-1}^{\beta=2}(x)$ (see [72, 73] for a discussion of multiple orthogonal polynomials). This means that

$$\int_0^\infty dz x^\ell p_n(x) w_{\nu;k}^{\beta=2}(x) = 0, \quad (2.70)$$

for $k = 0, \dots, M-1$ and $\ell = 0, \dots, \lceil \frac{n-k}{M} \rceil - 1$, where $\lceil x \rceil$ denote the ceiling function. This multiple orthogonality may be used to establish $M+2$ term recurrence relations for the biorthogonal functions:

$$\begin{aligned} xp_n(x) &= p_{n+1}(x) + \sum_{m=0}^M a_{m,n} p_{n-m}(x), \quad a_{m,n} = \int_0^\infty dx x p_n(x) \frac{\psi_{n-m}(x)}{h_{n-m}}, \\ x \frac{\psi_n(x)}{h_n} &= \frac{\psi_{n-1}(x)}{h_{n-1}} + \sum_{m=0}^M b_{m,n} \frac{\psi_{n+m}(x)}{h_{n+m}}, \quad b_{m,n} = \int_0^\infty dx x p_{n+m}(x) \frac{\psi_n(x)}{h_n}. \end{aligned} \quad (2.71)$$

These recurrence coefficients satisfy $a_{m,n} = b_{m,n-m}$ and were explicitly calculated in [41] but will not be repeated here.

2.2.2. Ginibre and inverse Ginibre matrices

Like for the complex eigenvalues, we will also consider a *mixed product of Ginibre and inverse Ginibre matrices*; here we follow [49]. We seek the *jpdf* for the squared singular values of the matrix

$$Y_L^{-1} \dots Y_1^{-1} X_M \dots X_1, \quad (2.72)$$

where X_i and Y_j are distributed according to the induced density (2.6) with indices ν_i and μ_j , respectively. Recall that any rectangular structure of the product matrix can be incorporated by choosing ν_i ($i = 1, \dots, M$) and μ_j ($j = 1, \dots, L$) to be positive integers. The *jpdf for the mixed product of Ginibre and inverse Ginibre matrices* is given by (2.57) with *weight function* [49]

$$w_{\nu,\mu;k}^{\beta=2}(x) = G_{L,M}^{M,L} \left(\begin{matrix} -N - \mu_1, \dots, -N - \mu_L \\ \nu_1, \dots, \nu_{M-1}, \nu_M + k \end{matrix} \middle| x \right). \quad (2.73)$$

This choice of weight functions requires that $M \geq 1$ but allows $L = 0$. However, there exists an alternative choice which requires that $L \geq 1$ and allows $M = 0$, leading to the same correlation functions.

Similar to the pure Ginibre case, the biorthogonal functions may be expressed neatly in terms of special functions [49]

$$\begin{aligned} p_n(x) &= (-1)^n \frac{h_n}{n! h_0} {}_{L+1}F_M \left(\begin{matrix} -n, 1 - N - \mu_1, \dots, 1 - N - \mu_L \\ \nu_1 + 1, \dots, \nu_M + 1 \end{matrix} \middle| (-1)^L x \right) \\ &= -h_n G_{L+1,M+1}^{0,1} \left(\begin{matrix} n + 1, N + \mu_1, \dots, N + \mu_L \\ -\nu_1, \dots, -\nu_M, 0 \end{matrix} \middle| x \right), \end{aligned} \quad (2.74)$$

$$\psi_n(x) = G_{L+1,M+1}^{M+1,L} \left(\begin{matrix} -N - \mu_1, \dots, -N - \mu_L, -n \\ 0, \nu_1, \dots, \nu_M \end{matrix} \middle| x \right), \quad (2.75)$$

$$h_n = n! \prod_{m=1}^M \Gamma(n + \nu_m + 1) \prod_{\ell=1}^L \Gamma(N + \mu_\ell - n). \quad (2.76)$$

If $L = 0$, then the biorthogonal functions reduce to the Ginibre case and they satisfy the $M + 2$ term recurrence relations (2.71). However, if $L \geq 1$, then the multiple orthogonality relations (2.70) are no longer valid and no recurrence relations are known to date.

As in the Ginibre case, the normalisation constant and the k -point correlations are determined by the biorthogonal functions (2.74) and (2.75) together with the squared norms (2.76) via (2.62) and (2.59), respectively. Also a double contour integral representation is possible

$$\begin{aligned} K_N^{M,L,\nu,\mu}(x,y) &= \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} du \oint_{\Sigma} dv \frac{x^u y^{-v-1}}{u-v} \frac{\Gamma(u-N+1)}{\Gamma(v-N+1)} \frac{\Gamma(u+1)}{\Gamma(v+1)} \\ &\quad \times \prod_{m=1}^M \frac{\Gamma(u+\nu_m+1)}{\Gamma(v+\nu_m+1)} \prod_{\ell=1}^L \frac{\Gamma(N+\mu_m-u)}{\Gamma(N+\mu_m-v)}. \end{aligned} \quad (2.77)$$

The contour Σ is defined as for products of Ginibre matrices and a similar single real integral representation as in (2.69) exists.

2.2.3. Truncated unitary matrices

We now turn to *products of truncated unitary matrices*. The derivation of the jpdf for such products requires an extension of the Itzykson–Zuber

integral which was very recently proved in [53]. Moreover, the authors of [53] showed that the *jpdf for the product of M truncated unitary matrices* was given by (2.57) with *weight function*

$$w_{\nu, \kappa; k}^{\beta=2}(x) = G_{M, M}^{M, 0} \left(\begin{matrix} \kappa_1, \dots, \kappa_{M-1}, \kappa_M - N + k + 1 \\ \nu_1, \dots, \nu_{M-1}, \nu_M + k \end{matrix} \middle| x \right). \quad (2.78)$$

Here, we use the same notation as in Section 2.1.3. The biorthogonal functions are given by

$$\begin{aligned} p_n(x) &= (-1)^n \frac{h_n}{n! h_0} {}_{M+1}F_M \left(\begin{matrix} -n, \kappa_1 + 1, \dots, \kappa_M + 1 \\ \nu_1 + 1, \dots, \nu_M + 1 \end{matrix} \middle| x \right) \\ &= h_n G_{M+1, M+1}^{0, M+1} \left(\begin{matrix} -\kappa_1, \dots, -\kappa_M, n + 1 \\ 0, -\nu_1, \dots, -\nu_M \end{matrix} \middle| x \right), \end{aligned} \quad (2.79)$$

$$\psi_n(x) = G_{M+1, M+1}^{M+1, 0} \left(\begin{matrix} -n, \kappa_1, \dots, \kappa_M \\ \nu_1, \dots, \nu_M, 0 \end{matrix} \middle| x \right), \quad (2.80)$$

$$h_n = n! \prod_{m=1}^M \frac{\Gamma(\nu_m + n + 1)}{\Gamma(\kappa_m + n + 1)}, \quad (2.81)$$

which determines the normalisation (2.62) and the correlations (2.59). Similar to the two previous examples, the kernel can be written as a double contour integral,

$$\begin{aligned} K_N^{M, \nu, \kappa}(x, y) &= \frac{1}{(2\pi i)^2} \int_C du \oint_{\Sigma} dv \frac{x^u y^{-v-1}}{v-u} \frac{\Gamma(u-N+1)}{\Gamma(v-N+1)} \frac{\Gamma(u+1)}{\Gamma(v+1)} \\ &\quad \times \prod_{m=1}^M \frac{\Gamma(u+\nu_m+1)}{\Gamma(v+\nu_m+1)} \frac{\Gamma(v+\kappa_m+1)}{\Gamma(u+\kappa_m+1)}, \end{aligned} \quad (2.82)$$

where the contour Σ is chosen such that it encloses $1, \dots, N$ without intersecting the other contour C encircling the negative real axis in positive sense. For the real integral representation corresponding to (2.69), we refer to [53].

2.2.4. Mixed products

As a final remark, we mention that more complicated products constructed from Ginibre, inverse Ginibre, truncated unitary and inverse truncated unitary matrices can be studied without introducing new techniques. Such a product of Ginibre matrices times a single truncated unitary matrix has previously been studied in [45]. When mixed products are considered, it is important to note that all matrix ensembles described in this section are isotropic, which implies that the ordering of the matrices is irrelevant for the statistical properties of the singular values, see [28].

3. Local universality for large matrix dimensions

In this section, we will discuss the universal limits for products of a finite number M of matrices as the matrix dimension N tends to infinity. Global spectra for both the complex eigenvalues and the singular values of product matrices have been studied intensively in the literature; in particular, using techniques from free probability, see [9, 13–15, 17, 18, 74–77]. Many of these results predate the exactly solvable cases described in Section 2. However, almost nothing was known about the local universality for products of random matrices until recently. One of the main benefits of the matrix products described in Section 2 is that their exact solvability allows a direct study of both global and local universality. Here, we will restrict our attention to results about local universality. In the known cases, it turns out that local correlations in the bulk and at the soft edges correspond to classical universality results from Random Matrix Theory; that is Ginibre-type correlations for the complex eigenvalues and correlations given in terms of the sine and Airy kernel for the singular values. At the origin and at the hard edge, new universality classes arise. In order to make this review both short and concise, our main focus will be on these new universality classes. We stress that this choice does not mean that the results for the bulk and the soft edges are less interesting, neither that these results are easily obtained.

3.1. Complex eigenvalues

3.1.1. Origin

We consider the local correlations at the origin for a product of complex ($\beta = 2$) Ginibre matrices first obtained for square matrices in [20]. The correlations are given by (2.16) with weight (2.8) and kernel (2.19). The local scale at the origin is obtained by keeping the number of matrices M as well as the parameters $\nu_m > -1$ ($m = 1, \dots, M$) fixed, while taking the matrix dimension N to infinity without further rescaling. This is a trivial task since the weight function (2.8) is independent of N , while the kernel (2.19) simply becomes an infinite sum which can be written either as a hypergeometric function or a Meijer G -function [20]

$$\begin{aligned} K_{\text{origin}}^{M,\nu}(u, v) &\equiv \lim_{N \rightarrow \infty} K_{N,\nu}^{\beta=2}(u, v) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(uv^*)^n}{\prod_{m=1}^M \Gamma(\nu_m + n + 1)} \\ &= \frac{1}{\pi} \frac{{}_1F_M \left(\begin{matrix} 1 \\ \nu_1 + 1, \dots, \nu_M + 1 \end{matrix} \middle| uv^* \right)}{\prod_{m=1}^M \Gamma(\nu_m + 1)} \\ &= \frac{1}{\pi} G_{1,M+1}^{1,1} \left(\begin{matrix} 0 \\ 0, -\nu_1, \dots, -\nu_M \end{matrix} \middle| -uv^* \right). \end{aligned} \quad (3.1)$$

The *universal k -point correlation functions at the origin*, therefore, read

$$\rho_{\text{origin};k}^{M,\nu}(z_1, \dots, z_k) = \prod_{\ell=1}^k w_{\nu}^M(z_{\ell}) \det_{1 \leq i, j \leq k} \left[K_{\text{origin}}^{M,\nu}(z_i, z_j) \right], \quad (3.2)$$

with kernel (3.1) and weight

$$w_{\nu}^M(z) = G_{0,M}^{M,0} \left(\nu_1, \dots, \nu_M \left| |z|^2 \right. \right). \quad (3.3)$$

For $M = 1$ and $\nu = 0$, the product ensemble reduces to the classical Ginibre ensemble. Evaluating the special functions (3.1) and (3.3) for $M = 1$ and inserting this into (3.2), we explicitly see that

$$\begin{aligned} \rho_{\text{origin};k}^{M=1,\nu=0}(z_1, \dots, z_k) &= \det_{1 \leq i, j \leq k} \left[\frac{1}{\pi} \exp \left(-\frac{1}{2} |z_i|^2 - \frac{1}{2} |z_j|^2 + z_i z_j^* \right) \right] \\ &= \rho_{\text{bulk};k}^{\beta=2}(z_1, \dots, z_k). \end{aligned} \quad (3.4)$$

These are the known correlation functions for a single complex Ginibre matrix at the origin. Because in the Ginibre ensemble the density is flat, the origin is not special and Eq. (3.4) agrees with the bulk scaling limit, see the next subsection.

For $M \geq 2$, the correlations (3.2) satisfy a reduction relation

$$\lim_{\nu_M \rightarrow \infty} (\nu_M)^k \rho_{\text{origin};k}^{M,\nu}(\sqrt{\nu_M} z_1, \dots, \sqrt{\nu_M} z_k) = \rho_{\text{origin};k}^{M-1,\nu}(z_1, \dots, z_k). \quad (3.5)$$

We recall that the parameters ν_m ($m = 1, \dots, M$) incorporate the rectangular structure of the matrices.

Let us now turn to a mixed product of Ginibre and inverse Ginibre matrices as discussed in Section 2.1.2. Here, the k -point correlation functions are given by (2.16) with weight (2.42) and kernel (2.17), where the squared norms are given by (2.44). It can be seen that with ν_m and μ_m fixed for $m = 1, \dots, M$, we have the following scaling limit for the k -point correlation function

$$\lim_{N \rightarrow \infty} \frac{1}{N^{kL}} R_{\text{origin};k}^{M,L,\nu,\mu} \left(\frac{z_1}{N^{L/2}}, \dots, \frac{z_k}{N^{L/2}} \right) = \rho_{\text{origin};k}^{M,\nu}(z_1, \dots, z_k). \quad (3.6)$$

Here, the right-hand side is given by (3.2), *i.e.* it is the same as for the product without any inverse matrices $L = 0$.

Finally, we will look at products of truncated unitary matrices. We consider the weight function (2.50) and the kernel (2.17) with the squared norms given by (2.53). Let J and L be integers such that $J + L = M$. We take M and J as well as ν_m ($m = 1, \dots, M$) and κ_j ($j = 1, \dots, J$) to be fixed, while $\kappa_\ell = N + O(1)$ as N tends to infinity for $\ell = 1, \dots, L$. The k -point correlation functions (2.16) have a hard edge scaling limit given by

$$\begin{aligned} \rho_{\text{origin};k}^{M,J,\nu,\kappa}(z_1, \dots, z_k) &\equiv \lim_{N \rightarrow \infty} \frac{1}{N^{kL}} R_{\text{origin};k}^{M,J,\nu,\kappa} \left(\frac{z_1}{N^{L/2}}, \dots, \frac{z_k}{N^{L/2}} \right) \\ &= \prod_{\ell=1}^k w_{\nu,\kappa}^{M,J}(z_\ell) \det_{1 \leq i, j \leq k} \left[K_{\text{origin}}^{M,J,\nu,\kappa}(z_i, z_j) \right], \end{aligned} \quad (3.7)$$

where the weight and kernel are given by

$$w_{\nu,\kappa}^{M,J}(z) = G_{J,M}^{M,0} \left(\kappa_1, \dots, \kappa_J \middle| |z|^2 \right), \quad (3.8)$$

$$\begin{aligned} K_{\text{origin}}^{M,J,\nu,\kappa}(u, v) &= \frac{1}{\pi} \frac{J+1 F_M \left(1, \kappa_1 + 1, \kappa_J + 1 \middle| uv^* \right)}{\prod_{m=1}^M \Gamma(\nu_m + 1) / \prod_{j=1}^J \Gamma(\kappa_j + 1)} \\ &= \frac{1}{\pi} G_{J+1,M+1}^{1,J+1} \left(0, -\kappa_1, \dots, -\kappa_J \middle| -uv^* \right). \end{aligned} \quad (3.9)$$

This was first shown in [33] for a product with $J = 0$; in this case, the correlation functions (3.7) reduce to the case with M Ginibre matrices (3.2). It remains to find the local correlations for more general products. It would be natural to look at mixed products constructed from both Ginibre and truncated unitary matrices as well as their inverses.

A more challenging task is to find the local correlations at the origin for products of real ($\beta = 1$) and quaternionic ($\beta = 4$) matrices. For $\beta = 4$, the correlations at the origin were obtained for a single matrix as well as a product of two matrices in [67] and [59], respectively. The main idea presented in [67] and reused in [59] was to write down a set of coupled differential equations for the kernel at the origin. This technique can be extended to products of an arbitrary number of matrices. However, the complicated structure of the differential equations for arbitrary M has prevented an explicit evaluation for $M \geq 3$, so far. The calculations simplify considerably if we integrate out the angular dependence of the eigenvalues; this idea was explicitly used in [22, 28, 31]. The local radial density at the origin for products of Ginibre matrices was explicitly calculated in [22] and a structure closely related to that of complex matrices was found. For $\beta = 1$, the correlations for $M = 2$ were found in [60], but almost nothing is known for $M \geq 3$.

3.1.2. Bulk and soft edge

It was shown in [20, 33, 34] that, under proper rescaling, the local k -point correlation functions of the eigenvalues in the bulk for a product of complex Ginibre or truncated unitary matrices are given by

$$\rho_{\text{bulk};k}^{\beta=2}(z_1, \dots, z_k) = \det_{1 \leq i, j \leq k} \left[\frac{1}{\pi} \exp \left(-\frac{1}{2}|z_i|^2 - \frac{1}{2}|z_j|^2 + z_i z_j^* \right) \right], \quad (3.10)$$

which is the same as for the standard complex Ginibre ensemble, see *e.g.* [63, 78]. In order to study the local correlations in the vicinity of a (soft) edge, we need to choose a point located on the edge. Given such a point, z_0 , it was shown [20, 33, 34] that proper rescaling leads to

$$\begin{aligned} \rho_{\text{soft};k}^{\beta=2}(z_1, \dots, z_k) \\ = \det_{1 \leq i, j \leq k} \left[\frac{1}{2\pi} \exp \left(-\frac{1}{2}|z_i|^2 - \frac{1}{2}|z_j|^2 + z_i z_j^* \right) \operatorname{erfc} \left(\frac{z_0^* z_i + z_j^* z_0}{\sqrt{2}} \right) \right], \end{aligned} \quad (3.11)$$

where $\operatorname{erfc}(x)$ is the complementary error function. Note that this supplements the universality results at strong non-Hermiticity known for the complex eigenvalues of non-Hermitian matrices [79–81], see also [82] for a recent heuristic approach.

3.1.3. Weak non-unitarity limit

In the case of truncated unitary (or orthogonal) matrices, one may consider the particular limit, in which the number of truncations remains finite, while the matrix size(s) go to infinity. Consequently, the resulting truncated matrices become almost unitary, with the macroscopic density of complex eigenvalues condensing on the unit circle. However, locally, the complex eigenvalues may still extend inside the unit disc. For $M = 1$, this limit was studied first in [35] and named weakly non-unitary. For $M > 1$ and a particular choice of parameters, this has been generalised in [33]. Namely, if in Subsection 2.1.3, we truncate all matrices in the product starting from the same size, $K_j = K$, down to square matrices of size $N_j = N$ ($\nu_j = 0$) for all j , we can take the following large- N limit with $K - N = \kappa$ fixed. In order to obtain non-trivial local correlations inside the unit disc, we take k points z_l in the vicinity of a fixed point z_0 on the unit circle:

$$z_j = 1 - \frac{1}{N}(x_j + iy_j), \quad x_j > 0 \quad \text{for} \quad j = 1, 2, \dots, k. \quad (3.12)$$

Here, we have chosen $z_0 = 1$ without loss of generality, due to the rotation invariance. In contrast to the previous sections, here the weight (2.50) and

the kernel (2.17) with norms (2.53) do not converge to a limit individually. Only their combination as it appears in the correlation functions (2.16) has a limit which we give here straight away

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N^{2k}} R_k^{\beta=2} \left(1 - \frac{1}{N}(x_1 + iy_1), \dots, 1 - \frac{1}{N}(x_k + iy_k) \right) \\ &= \det_{1 \leq j, l \leq k} \left[K_{\text{weak}}^{M, \kappa}(x_j + iy_j, x_l + iy_l) \right], \end{aligned} \quad (3.13)$$

where the *weak kernel* is given by [33]

$$\begin{aligned} K_{\text{weak}}^{M, \kappa}(x_j + iy_j, x_l + iy_l) &= \frac{\Theta(x_j)\Theta(x_l)(4x_j x_l)^{(M\kappa-1)/2}}{\pi(M\kappa-1)!} \\ &\times \left(-\frac{\partial}{\partial t} \right)^{M\kappa} \left(\frac{1 - e^{-t}}{t} \right) \Big|_{t=(x_j+x_l+i(y_j-y_l))/M}. \end{aligned} \quad (3.14)$$

This expression generalises the kernel at weak non-unitarity given in [35] for $M = 1$. In order to extend this result to a more general parameter setting with the κ_j being different, the difficulty is to obtain the limiting kernel (and not the weight).

3.1.4. Further limits

In addition to the various limits in different parts of the spectrum we discussed so far, further asymptotic limits can be considered. For example, the large radius limit $r \rightarrow \infty$ of the hole probabilities Eqs. (2.23) and (2.34) at finite and infinite N have been derived in [21] and [31] for $\beta = 2$ and 4, respectively. Furthermore, the infinite- N process can be considered directly and, for example, overcrowding estimates can be made [21, 31]. Similar questions arise in the analysis of zeros of Gaussian analytic functions, cf. [37].

A related question was considered in [39], where the distribution of the largest radius was studied in a double scaling limit with both N and M going to infinity. In the case of products of square Ginibre matrices with $\beta = 2$, a transition between the known Gumbel distribution for $M = 1$ [83] and a lognormal distribution was found. The latter relates to the study of the largest stability exponent which will be introduced in Section 4.

3.2. Singular values

3.2.1. Hard edge

In this section, we return to statistical properties of the squared singular values for products of random matrices. The products described in Section 2.2 were all determinantal point processes (2.59) with a kernel (2.60)

constructed from a set of biorthogonal functions. A study of the local correlations for such products at the hard edge for large matrix dimensions was first undertaken in [41], where it was found that a new family of universal correlation kernels arises. We saw in Section 2.2.1 that the kernel for the product of Ginibre matrices can be written as a double contour integral (2.69). Using that

$$\frac{\Gamma(u - N + 1)}{\Gamma(v - N + 1)} = \frac{\sin \pi u}{\sin \pi v} N^{u-v} (1 + O(N^{-1})) , \quad (3.15)$$

we find the hard edge limit

$$\begin{aligned} K_{\text{Meijer}}^{M,\nu}(x, y) &\equiv \lim_{N \rightarrow \infty} \frac{1}{N} K_N^{M,\nu} \left(\frac{x}{N}, \frac{y}{N} \right) \\ &= \frac{1}{(2\pi i)^2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} du \oint_{\Sigma} dv \frac{x^u y^{-v-1}}{u-v} \frac{\sin \pi u}{\sin \pi v} \frac{\Gamma(u+1)}{\Gamma(v+1)} \\ &\quad \times \prod_{m=1}^M \frac{\Gamma(u + \nu_m + 1)}{\Gamma(v + \nu_m + 1)} . \end{aligned} \quad (3.16)$$

Evaluating the integrals allows a representation in terms of special functions

$$\begin{aligned} K_{\text{Meijer}}^{M,\nu}(x, y) &\equiv \int_0^1 ds \frac{{}_0F_M \left(\nu_1 + 1, \dots, \nu_M + 1 \mid -sx \right)}{\prod_{m=1}^M \Gamma(\nu_m + 1)} G_{0,M+1}^{M,0} \left(\nu_1, \dots, \nu_M, 0 \mid sy \right) \\ &= \int_0^1 ds G_{0,M+1}^{1,0} \left(0, -\nu_1, \dots, -\nu_M \mid sx \right) G_{0,M+1}^{M,0} \left(\nu_1, \dots, \nu_M, 0 \mid sy \right) . \end{aligned} \quad (3.17)$$

We will refer to (3.17) as the *Meijer G-kernel* [53]. The Hamilton equations for the gap probability in this limit were studied in [51].

For $M = 1$, the product consists of a single matrix, hence it reduces to the Wishart–Laguerre ensemble. It is well-known that the hard edge behaviour for the Wishart–Laguerre ensemble is described by the Bessel kernel, see *e.g.* [78]. This is explicitly incorporated in the Meijer G -kernel (3.17) by

$$K_{\text{Meijer}}^{M=1,\nu}(x, y) = \left(\frac{y}{x} \right)^{\frac{\nu}{2}} \int_0^1 ds J_{\nu}(2\sqrt{sx}) J_{\nu}(2\sqrt{sy}) = 4 \left(\frac{y}{x} \right)^{\frac{\nu}{2}} K_{\text{Bessel}}^{\nu}(4x, 4y) , \quad (3.18)$$

where the *Bessel kernel* is defined as

$$K_{\text{Bessel}}^{\nu}(x, y) \equiv \frac{\sqrt{y} J_{\nu}(\sqrt{x}) J_{\nu-1}(\sqrt{y}) - \sqrt{x} J_{\nu-1}(\sqrt{x}) J_{\nu}(\sqrt{y})}{2(x-y)}. \quad (3.19)$$

Note that the x - and y -dependent prefactor to the Bessel kernel in (3.18) cancels out when calculating the correlation functions due to the determinantal structure. For $M \geq 2$, the Meijer G -kernel (3.17) satisfies a reduction relation,

$$\lim_{\nu_M \rightarrow \infty} \nu_M K_N^{M, \nu}(\nu_M x, \nu_M y) = K_{\text{Meijer}}^{M-1, \nu}(x, y), \quad (3.20)$$

where the kernel on the right-hand side depends on the remaining parameters ν_m ($m = 1, \dots, M-1$). Recall that for the product of rectangular Ginibre matrices, the parameters ν_m incorporate the rectangular structure.

Similar results were obtained for mixed products of Ginibre and inverse Ginibre matrices in [49] and for products of truncated unitary matrices in [53]; the product of Ginibre matrices together with a single truncated unitary matrix was studied in [45]. The biorthogonal functions for mixed products of Ginibre and inverse Ginibre matrices are given in Section 2.2.1. For M and L as well as ν_m ($m = 1, \dots, M$) and μ_{ℓ} ($\ell = 1, \dots, L$) fixed, we have [49]

$$\lim_{N \rightarrow \infty} \frac{1}{N^{L+1}} K_N^{M, L, \nu, \mu} \left(\frac{x}{N^{L+1}}, \frac{y}{N^{L+1}} \right) = K_{\text{Meijer}}^{M, \nu}(x, y), \quad (3.21)$$

where the kernel on the left-hand side is given by (2.60).

For products of truncated unitary matrices, the biorthogonal functions as well as a double contour integral representation of the kernel can be found in Section 2.2.3. Let J and L be integers such that $J + L = M$. We take M and J as well as ν_m ($m = 1, \dots, M$) and κ_j ($j = 1, \dots, J$) to be fixed, while $\kappa_{\ell} = N + O(1)$ as N tends to infinity for $\ell = 1, \dots, L$. In this case, we find

$$\lim_{N \rightarrow \infty} \frac{1}{N^{L+1}} K_N^{M, J, \nu, \kappa} \left(\frac{x}{N^{L+1}}, \frac{y}{N^{L+1}} \right) = K_{\text{Meijer}}^{M, J, \nu, \kappa}(x, y), \quad (3.22)$$

where the kernel on the right-hand side is a generalised version of the kernel (3.17) given by [53]

$$\begin{aligned}
& K_{\text{Meijer}}^{M,J,\nu,\kappa}(x,y) \\
&= \int_0^1 ds \frac{{}_JF_M\left(\begin{matrix} \kappa_1+1, \dots, \kappa_J+1 \\ \nu_1+1, \dots, \nu_M+1 \end{matrix} \middle| -sx\right)}{\prod_{m=1}^M \Gamma(\nu_m+1) / \prod_{j=1}^J \Gamma(\kappa_j+1)} G_{J,M+1}^{M,0}\left(\begin{matrix} \kappa_1, \dots, \kappa_J \\ \nu_1, \dots, \nu_M, 0 \end{matrix} \middle| sy\right) \\
&= \int_0^1 ds G_{J,M+1}^{1,J}\left(\begin{matrix} -\kappa_1, \dots, -\kappa_J \\ 0, -\nu_1, \dots, -\nu_M \end{matrix} \middle| sx\right) G_{J,M+1}^{M,0}\left(\begin{matrix} \kappa_1, \dots, \kappa_J \\ \nu_1, \dots, \nu_M, 0 \end{matrix} \middle| sy\right).
\end{aligned} \tag{3.23}$$

Note that this Meijer G -kernel reduces to (3.17) for $J = 0$, exactly like the k -point correlation functions (3.7) reduce to (3.2) for $J = 0$.

In addition to the product ensembles described above, the Meijer G -kernel (3.17) has appeared in the context of Cauchy multi-matrix models [42–44, 84] and Muttalib–Borodin ensembles [46, 47]. The reappearance of the Meijer G -kernel (3.17) in all these models suggests a much stronger underlying universality principle similar to known results for the Bessel kernel [85–89]. In the global regime, such a link is provided by the equilibrium measure [90].

3.2.2. Bulk and soft edge

The evaluation of the local statistics in the bulk and at the soft edge turns out to be much more technically demanding than the evaluation at the hard edge, and has only very recently been obtained for the product of rectangular Ginibre matrices [52]. There, it was shown that classical random statistics is obtained under proper rescaling, *i.e.* the *sine kernel* in the bulk and the *Airy kernel* at the soft edge. Recall that the sine and Airy kernel are defined by (see *e.g.* [63])

$$K_{\text{sine}}(x, y) \equiv \frac{\sin \pi(x - y)}{x - y} \quad \text{and} \quad K_{\text{Airy}}(x, y) \equiv \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}, \tag{3.24}$$

respectively. We emphasise that the derivation of the sine and Airy kernel presented in [52] crucially depends on known results for the global density of the squared singular values [9, 76, 91], and, in particular, an asymptotic expression in terms of elementary functions obtained in [92–94]. However, the strategy for obtaining the local statistics presented in [52] is by no means restricted to the Ginibre case; in fact, Ref. [52] provides similar results for mixed products of Ginibre and inverse Ginibre matrices (see Section 2.2.2) and for Ginibre matrices mixed with a single truncated unitary matrix (see [45]) although the proofs are only sketched.

4. Lyapunov and stability exponents for large products

Until now, we have considered the asymptotic behaviour of matrix products with a finite number of factors M as the matrix dimensions tend to infinity. In this section, we will consider the opposite situation, where the matrix dimensions are kept fixed as the number of factors M tends to infinity. In general, we are interested in the spectral properties of a product matrix

$$\Pi_M \equiv X_M \dots X_1, \quad (4.1)$$

where each X_m ($m = 1, \dots, M$) is an $N \times N$ random matrix, independently chosen from some ensemble. The multiplicative ergodic theorem of Oseledec [95, 96] states that if the second moments of the diagonal entries of $X_m^\dagger X_m$ are finite, then there is a well-defined limiting matrix

$$V \equiv \lim_{M \rightarrow \infty} \left(\Pi_M^\dagger \Pi_M \right)^{1/2M} \quad (4.2)$$

with real eigenvalues e^{λ_n} ($n = 1, \dots, N$). Here, the λ_n are known as Lyapunov exponents. Let x_n ($n = 1, \dots, N$) denote the squared singular values of the product matrix (4.1); Oseledec's theorem tells us that $x_n \sim e^{2M\lambda_n}$ for large M , hence negative (positive) Lyapunov exponents represent exponential decay (growth) and therefore stability (instability). Moreover, we see that all squared singular values diverge exponentially whenever the Lyapunov spectrum is non-degenerate. In such cases, it is expected that the Lyapunov exponents become independent Gaussian random variables with fluctuations of the order of $M^{-1/2}$ for sufficiently large M , see [7, 97, 98]. Together with the symmetry of the Lyapunov exponents under permutations, this statement is equivalent to saying that the jpdf for the squared singular values is given by a permanental point process,

$$\begin{aligned} & \prod_{n=1}^N 2M e^{2M\xi_n} \mathcal{P}_{\text{jpdf}} \left(e^{2M\xi_1}, \dots, e^{2M\xi_N} \right) \\ & \sim 1/N! \text{ per}_{1 \leq i, j \leq N} \left[\sqrt{\frac{M}{2\pi\sigma_i^2}} \exp \left(-\frac{M(\xi_j - \lambda_i)^2}{2\sigma_i^2} \right) \right], \end{aligned} \quad (4.3)$$

with means λ_i and variances σ_i to be determined. The Gaussian approximation is expected to be valid when the distance between Lyapunov exponents is large compared to the size of their fluctuations, *i.e.* $|\lambda_i - \lambda_j| \gg M^{-1/2}$ for all $i \neq j$.

How much of this expectation has been realised? In [99], the mean and variance of the largest Lyapunov exponent has been computed for products of real Ginibre matrices, see (4.4) below. The same ideas were used to calculate all the Lyapunov exponents in [100]. In these papers, probabilistic methods were applied, which we will briefly recall below. Further explicit results for the Lyapunov exponents (again, based on probability theory) were derived much later for factors from correlated Gaussian ensembles in [55] for $\beta = 2$ and [56] for $\beta = 1, 4$, see λ_i in (4.4) for the uncorrelated case.

It is clear from the discussion in Section 2.2 that the explicit results for finite- N and $-M$ should give access to the permanental structure. Indeed, in [10] such a derivation was recently performed for the product of Ginibre matrices with $\beta = 2$. There, the permanental point process (4.3) was derived starting from (2.57), defining $x_j = e^{2M\xi_j}$ and taking M to be large. The following values for the variances (and means) were obtained for $\beta = 2$:

$$\lambda_n^\beta = \frac{1}{2} \log \frac{2}{\beta} + \frac{1}{2} \psi \left(\frac{\beta n}{2} \right) \quad \text{and} \quad (\sigma_n^\beta)^2 = \frac{1}{4} \psi' \left(\frac{\beta n}{2} \right), \quad (4.4)$$

where $\psi(x)$ denotes the digamma function. For further details including the subleading higher order cumulants for $\beta = 2$, we refer to [10].

Previously, a direct access to the jpdf at finite M and N was unknown, and alternative methods for the evaluation of the Lyapunov exponents were developed much earlier [99]. These are valid for the more general class of isotropic matrices (including the Ginibre case) which is why we recall them here. The partial sum of Lyapunov exponents can be written as

$$A_k^\beta \equiv \sum_{n=1}^k \lambda_{N-n+1}^\beta \stackrel{d}{=} \frac{1}{2M} \sum_{m=1}^M \sup_{A \in \mathbb{F}^{N \times k}} \log \frac{\det A^\dagger X_m^\dagger X_m A}{\det A^\dagger A}, \quad (4.5)$$

where λ_{N-n+1}^β denotes the n^{th} largest Lyapunov exponent and the supremum is over all $N \times k$ matrices with entries in $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for $\beta = 1, 2, 4$. For large M , the right-hand side of (4.5) can be evaluated using the law of large numbers. Recently, it was pointed out in [101] that the method of [99] can be also used to extract the variances of the Lyapunov exponents. In this case, formula (4.5) is used to evaluate the average $\langle \rangle$ with respect to the X_j

$$\left\langle \left(A_k^\beta \right)^2 \right\rangle - \left\langle A_k^\beta \right\rangle^2 = \sum_{n=1}^k \text{Var} \left(\lambda_{N-n+1}^\beta \right) + 2 \sum_{1 \leq i < j \leq k} \text{Cov} \left(\lambda_{N-i+1}^\beta, \lambda_{N-j+1}^\beta \right). \quad (4.6)$$

If the Lyapunov spectrum is non-degenerate, then the covariances are expected to decay exponentially at large M , hence they may be neglected at

leading order in M , see [101]. Note that if we are only interested in the largest Lyapunov exponent, *i.e.* $k = 1$, then the second sum on the right-hand side of (4.6) disappears which was used much earlier in [99] to find the variance of the largest Lyapunov exponent.

Mixed products of Ginibre and inverse Ginibre matrices as well as products of truncated unitary matrices can also be evaluated using either the probabilistic methods from [99] or the exactly solvable models from Section 2 as in [10, 11]. The mixed product case can be constructed from (4.4) and will not be repeated here, see [101]. We can consider unitary matrices truncated from $(N + \kappa) \times (N + \kappa)$ Haar distributed matrices to $N \times N$ sub-blocks (see the discussion in Section 2.1.3) leading to [101]

$$\lambda_n^\beta = \frac{1}{2}\psi\left(\frac{\beta n}{2}\right) - \frac{1}{2}\psi\left(\frac{\beta(\kappa + n)}{2}\right), \quad (4.7)$$

$$(\sigma_n^\beta)^2 = \frac{1}{4}\psi'\left(\frac{\beta n}{2}\right) - \frac{1}{4}\psi'\left(\frac{\beta(\kappa + n)}{2}\right). \quad (4.8)$$

These quantities can also be found by direct evaluation of the jpdf from Sections 2.2.3 and 2.1.3. While the discussion given above involves solely square matrices, it is a straightforward task to include the parameters ν_j , see [11].

Although Lyapunov exponents are the standard choice for the characterisation of stability, a different possibility exists. Rather than investigating the singular values of the product matrix (4.1), we could look at the complex eigenvalues z_n ($n = 1, \dots, N$) of the product matrix (4.1). The absolute values of the eigenvalues are expected to grow exponentially for large M , although there is no equivalent of Oseledec's theorem in this case. It was therefore suggested in [10] to parametrise the eigenvalues as $z_n = e^{M\xi_n + i\theta_n}$, with ξ_n and θ_n real. It was previously conjectured that the *stability exponents* driving the exponential growth (decay) are identical to the Lyapunov exponents whenever the spectrum is non-degenerate [12]. For the product of Ginibre matrices, this was proved in [10, 11] starting from Eqs. (2.7), (2.31) and (2.37), leading to (4.3) with identical means and variances (4.4) for the stability exponents for $\beta = 2, 4$ and 1 , respectively. Note, however, that the corrections to this limit which were also computed in [10, 11] differ from the corrections for the Lyapunov exponents known for $\beta = 2$ only.

It is natural to ask about the angular dependence of the complex eigenvalues for M going to infinity as well. For $\beta = 2$, the spectrum is trivially rotational invariant; for products of $\beta = 4$ Ginibre matrices, it was shown in [11] that the eigenvalues have a sine-squared repulsion from the real axis, due to the pairing of complex conjugate eigenvalues. The most interesting case is undoubtedly $\beta = 1$. For the product of real Ginibre matrices all

eigenvalues become real when the number of matrices tends to infinity. This was first observed numerically in [57] and for $N = 2$, and later verified for general N by explicit calculations in [32]. It was found that for large N , the probability that all eigenvalues are real can be estimated by [32]

$$\text{Prob}[\forall j : z_j \in \mathbb{R}] \sim \left(\frac{1}{\sqrt{2}} \left(\frac{\Gamma(1/M + 1)\Gamma(1/2)}{\Gamma(1/M + 1/2)} \right)^{M/4} \right)^{N^2}, \quad (4.9)$$

which tends to unity as M goes to infinity. This was confirmed in [11].

5. Open problems

Several open problems can be easily identified. It would be desirable to allow for a wider choice of ensembles of random matrices that can be multiplied, while maintaining the exact solvability for finite products of M factors at finite matrix size $\sim N$. Apart from allowing for classical Hermitian ensembles, one generalisation relevant for universality is to include invariant non-Gaussian ensembles. As in the truncated unitary ensemble, this immediately drops the independence amongst the matrix elements within each factor. On the other hand, one could also drop the independence among different factors in the product. Furthermore, new limiting kernels may be accessible from the finite N and M results, such as by taking different double scaling limits.

Taking factors from the real Ginibre ensemble or studying truncated orthogonal matrices still presents a challenge. But even for the truncated unitary ensemble, only a very limited parameter range of the exact solution at finite N and M has been explored in the weak non-unitarity limit. This and the orthogonal ensemble may provide a rich study ground, being potentially relevant in systems with absorption or with sources of decoherence. Generally speaking, the plethora of new mathematical results reviewed here has not yet been fully exploited in applications. Especially, the new microscopic classes at the origin labelled M will probably require to go beyond standard areas such as combinatorics or telecommunications. The initial motivation to study toy models of dynamical systems in the beginning of the sixties may be very worthwhile revisiting under this new light of insights.

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