THE LENARD RECURSION RELATION AND A FAMILY OF SINGULARLY PERTURBED MATRIX MODELS[*]

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We review some aspects of recent work concerning double scaling limits of singularly perturbed Hermitian random matrix models and their connection to Painlevé equations. We present new results showing how a Painlevé III hierarchy recently proposed by the author can be connected to the Lenard recursion formula used to construct the Painlevé I and II hierarchies.

1. Introduction

The semi-classical one-Hermitian matrix model is defined as the measure

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM \tag{1}$$

on the set $\mathcal{H}^{n \times n}(J)$ of $n \times n$ Hermitian matrices whose spectrum is a subset of J. Here the potential V is such that V' is a rational function. The angular degrees of freedom of M may be integrated out of this model to give a jpdf for the eigenvalues x_i of M,

$$\frac{1}{\hat{Z}_n} \Delta(x)^2 \prod_{i=1}^{n} w(x_i) dx_i \,, \tag{2}$$

where $w(x) := e^{-nV(x)}$ and Δ is the Vandermonde determinant. Such models are a very general class of models for which the method of orthogonal

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polynomials can be used to give a solution. The method of orthogonal polynomials expresses the eigenvalue k-point correlation functions in terms of the correlation kernel,

$$K_n(x,y) = h_{n-1}^{-1} \frac{\sqrt{w(x)w(y)}}{x-y} \left(p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x) \right) , \qquad (3)$$

where p_j, $j = 0, 1, \ldots$ are a family of monic polynomials of degree j characterised by the relations

$$\int_0^\infty p_j(x)p_m(x)w(x)dx = h_j \delta_{jm} . \qquad (4)$$

The limiting mean eigenvalue density is given by

$$\rho(x) := \lim_{n\to\infty} \frac{1}{n} K_n(x,x) , \qquad (5)$$

and describes the macroscopic behaviour of the eigenvalues for large n.

Such models have been studied at finite n in [1] and their relation to integrable systems fully described. What is less known are the types of critical behaviour in such a model. In the case that V is polynomial and $J = \mathbb{R}$, a number of distinct critical points have been identified and studied over the last two decades. These have been classified as:

— Type I — The spectral density acquires extra zeros at the edge a of its support. The usual behaviour of ρ near a is $\rho(x) \propto (x-a)^{\frac{1}{2}}$, however, when extra zeros are present, the possible behaviours of ρ are $\rho(x) \propto (x-a)^{\frac{k}{2}}$ with $k \in \mathbb{N}$. The limiting kernel in the neighbourhood of such points is constructed in terms of solutions to the kth Painlevé I equation.

— Type II — The spectral density acquires new zeros at some point a in the bulk of its support. The behaviour of ρ near a is $\rho(x) \propto (x-a)^{2k}$ with $k \in \mathbb{N}$. The limiting kernel in the neighbourhood of such points is constructed from solutions to the kth Painlevé II equation.

— Type III — The spectral density acquires a new cut in its support. This transition is known as a "birth of a cut" [8]. Here, the limiting kernel is constructed from Hermite polynomials and the local behaviour in the new cut mimics a GUE matrix model.

In the more general case of the semi-classical model, no such classification exists, however a number of special cases have been investigated in the recent literature.

— The effect of logarithmic singularities in the potential have been investigated in a number of works. In [7], the effect of a singularity in the bulk results in a kernel constructed with Bessel functions. In [9], the situation of a logarithmic singularity coinciding with a Type II critical point was found to lead to kernels containing solutions to the general Painlevé II equation. Finally, logarithmic singularities at the edge of the spectrum results in general Painlevé XXXIV equations.

— The addition of a hard edge also results in a new behaviour. It has long been known that the kernel near the hard edge can be constructed in terms of Bessel functions. More recent work [10] has considered the case of a hard edge meeting a soft edge, with the resulting kernel constructed in terms of Painlevé XXXIV transcendents. This was further extended in [11] to a hard edge meeting a Type I critical point in which it was shown that the associated Painlevé transcendents satisfy the kth member of the Painlevé XXXIV hierarchy.

— Finally, very recently, the behaviour of eigenvalues near poles in the potential have been studied. The case of a simple pole both in the bulk and at the hard edge have been investigated in [2–5] and it was shown that the kernel can be constructed using solutions of Painlevé III. The case of higher order poles at a hard edge was studied in [6] by the author and collaborators and the kernel in the neighbourhood of the pole was constructed using solutions of a member of a Painlevé III hierarchy.

In this short note, we report on some new aspects of the work in [6]. In particular, we give a relation between the Painlevé III hierarchy defined in [6] and the Lenard recursion relations that are ubiquitous in the Painlevé I and II hierarchies.

2. A Painlevé III hierarchy

In [6], the kth member of the Painlevé III hierarchy was defined as the system of k ODEs ($p = 1, \ldots, k$),

$$\sum_{q=0}^{p} \left(\ell_{k-p+q+1} \ell_{k-q} - (\ell_{k-p+q} \ell_{k-q})'' + 3\ell'_{k-p+q} \ell'_{k-q} - 4u\ell_{k-p+q} \ell_{k-q} \right) = \tau_p \,, \tag{6}$$

for k unknown functions $\ell_1 = \ell_1(s), \ldots, \ell_k = \ell_k(s)$, with $\ell_{k+1}(s) = 0$ and $\ell_0(s) = \frac{s}{2}$. The τ_ps are constants that act as times. The quantity $u = u(s)$ is defined by

$$u(s) = -\frac{1}{4\ell_k^2} \left(\left(\ell_k^2 \right)'' - 3 \left(\ell'_k \right)^2 + \tau_0 \right) \,. \tag{7}$$

Example 2.1. For $k = 1$, we have the equation

$$\ell_1''(s) = \frac{\ell_1'(s)^2}{\ell_1(s)} - \frac{\ell_1'(s)}{s} - \frac{\ell_1(s)^2}{s} - \frac{\tau_0}{\ell_1(s)} + \frac{\tau_1}{s}, \tag{8}$$

which we identify as a special case of the Painlevé III equation.

Example 2.2. If $k = 2$, we have a system of two ODEs:

$$\frac{\tau_1}{2\ell_1(s)\ell_2(s)} - \frac{\tau_0}{\ell_2(s)^2} + \frac{\ell_2'(s)^2}{\ell_2(s)^2} - \frac{\ell_1'(s)\ell_2'(s)}{\ell_1(s)\ell_2(s)} + \frac{\ell_1''(s)}{\ell_1(s)} - \frac{\ell_2''(s)}{\ell_2(s)} - \frac{\ell_2(s)}{2\ell_1(s)} = 0, \tag{9}$$

and

$$\frac{\ell_1(s)^2\ell_2'(s)^2}{\ell_2(s)^2} - \ell_1'(s)^2 + \frac{s\ell_2'(s)^2}{\ell_2(s)} - \ell_2'(s) - \frac{\tau_0\ell_1(s)^2}{\ell_2(s)^2} - \frac{s\tau_0}{\ell_2(s)} - \tau_2$$
$$= \frac{2\ell_1(s)^2\ell_2''(s)}{\ell_2(s)} - 2\ell_1(s)\ell_1''(s) + s\ell_2''(s) + 2\ell_2(s)\ell_1(s). \tag{10}$$

3. A Riemann–Hilbert problem for the Painlevé III hierarchy

In [6], it was shown that a solution to the kth Painlevé III equation may be extracted from the following RH problem:

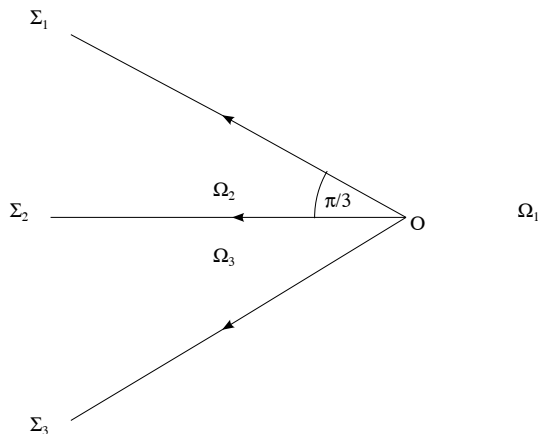(a) $\Phi : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ analytic. See figure 1.

Fig. 1. The jump contour Γ for the model RH problem for Ψ. Contours are labelled by Σ and sectors by Ω.

(b) Φ has the jump relations $\Phi_+(z) = \Phi_-(z)j_i$ for $z \in \Sigma_i$

$$j_1 = \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix}, \qquad j_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad j_3 = \begin{pmatrix} 1 & 0 \\ -e^{-\pi i \alpha} & 1 \end{pmatrix}.$$
(11)

(c) As $z \to \infty$, Φ has the asymptotic behaviour

$$\Phi(z) = \begin{pmatrix} 1 & 0 \\ v(s) & 1 \end{pmatrix} \left(I + \frac{1}{z} \begin{pmatrix} w(s) & v(s) \\ h(s) & -w(s) \end{pmatrix} + \mathcal{O}\left(z^{-2}\right) \right)$$
$$\times e^{\frac{1}{4} i \pi \sigma_3} z^{-\frac{1}{4} \sigma_3} N e^{s z^{1/2} \sigma_3},$$
(12)

where $N = \frac{1}{\sqrt{2}}(I + i\sigma_1)$ and v, h and w are functions of s.

(d) As $z \to 0$, there exists a matrix $\Phi_0(s)$, independent of z, such that Φ has the asymptotic behaviour

$$\Phi(z) = \Phi_0(s)(I + \mathcal{O}(z))e^{\frac{(-1)^{k+1}}{z^k}\sigma_3} z^{\frac{\alpha}{2}\sigma_3} H_j$$
(13)

for $z \in \Omega_j$, where H_1, H_2, H_3 are given by

$$H_1 = I, \qquad H_2 = \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix}, \qquad H_3 = \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}.$$
(14)

We then have as a corollary of [6] Theorem 1:

Theorem 3.1. *Let $\alpha > -1$, and let $\Phi(z; s)$ be the unique solution of the above model RH problem for $s > 0$. Then, the limit*

$$y(s) = -2\frac{d}{ds}\left[\lim_{z \to \infty} s\Phi(z, s)e^{-sz^{1/2}\sigma_3} N^{-1} z^{\frac{1}{4}\sigma_3} e^{-\frac{1}{4}i\pi\sigma_3}\right]_{21}$$
(15)

is a solution of the kth member of the Painlevé III hierarchy.

Remark 3.2. The proof of this theorem identifies y with ℓ_1 in the Painlevé III hierarchy. Furthermore, it also demonstrates that the Lax pair for Φ takes the form:

$$A(z, s) = a(z, s)\sigma_3 + b(z, s)\sigma_+ + c(z, s)\sigma_-,$$
(16)
$$B(z; s) = (z - u(s))\sigma_- + \sigma_+,$$
(17)

where a, b and c are related by

$$a(z, s) = -\tfrac{1}{2}\partial_s b(z, s),$$
(18)
$$c(z, s) = (z - u)b(z, s) - \tfrac{1}{2}\partial_s^2 b(z, s),$$
(19)
$$\partial_s c(z, s) = 1 + 2(z - u(s))a(z, s).$$
(20)

Substituting (18) and (19) into (20) yields

$$z\partial_s b(z,s) = \tfrac{1}{4}\left(\partial_s^3 b(z,s) + 4u(s)\partial_s b(z,s) + 2u'(s)b(z,s)\right) + \tfrac{1}{2}\,. \qquad (21)$$

We may compute $b(z,s)$ by substituting $b(z,s) = \frac{4}{(4z)^{k+1}}\sum_{j=0}^{k}\ell_{k-j}(s)(4z)^j$ into the above equation to get,

$$\ell'_{j+1}(s) = \ell'''_j(s) + 4u(s)\ell'_j(s) + 2u'(s)\ell_j(s)\,. \qquad (22)$$

4. Integration of Lenard-type recursion relations

The recursion relation (22) is the Lenard recursion relation appearing in the Painlevé I and II hierarchies. In those cases, the initial condition is $\ell_0 = \frac{1}{2}$ whereas here we have $\ell_0 = s/2$. Let us consider the general case where ℓ_0 and ℓ_{k+1} are known functions. The fact that ℓ_{k+1} is known implies $u(s)$ satisfies an integro-differential equation of the order of $3k+1$. The following lemma gives $k+1$ constants of motion, *i.e.* functions of $u(s)$ and its derivatives which are constant in s. This allows the equation for $u(s)$ to be reduced to an ODE of the order of $2k$ and we will see that these constants of motion are precisely the ODEs appearing in (6).

Theorem 4.1. *Let ℓ_j be the integro-differential polynomials in u generated by the Lenard recursion relation (22) together with an initial condition for ℓ_0. Furthermore, let ℓ_{k+1} also have a given form. The integro-differential equation corresponding to ℓ_{k+1} has the following constants of motion:*

$$\tau_p = -\ell_{k+1}\ell_{k-p} + \sum_{q=0}^{p}\left(\ell_{k-q}\ell_{k-p+q+1} - \Omega_{k-p+q,k-q}\right)\,, \qquad 0 \le p \le k \quad (23)$$

if $\ell'_{k+1} = 0$, and

$$\sigma_p = -\ell_0\ell_p + \sum_{q=0}^{p-1}\left(\Omega_{p-1-q,q} - \ell_{p-1-q}\ell_{q+1}\right)\,, \qquad 0 \le p \le k \qquad (24)$$

if $\ell'_0 = 0$. In the above expressions, we have introduced

$$\Omega_{n,m}(s) := (\ell_n\ell_m)'' - 3\ell'_n\ell'_m + 4u\ell_n\ell_m\,. \qquad (25)$$

Proof. We begin with the following identity

$$\ell_m\ell'_{n+1} + \ell_n\ell'_{m+1} = \Omega'_{n,m}\,. \qquad (26)$$

This identity can be established by the following argument

$$
\begin{align}
\ell_m(\ell_{n+1})' &= \ell_m\ell_n''' + 4u\ell_m\ell_n' + 2u'\ell_m\ell_n \tag{27}\\
&= \ell_m\ell_n''' + 4u\ell_m\ell_n' + 4u'\ell_m\ell_n - \ell_n\left(\ell_{m+1}' - 4u\ell_m' - \ell_m'''\right) \tag{28}\\
&= \ell_m\ell_n''' + \ell_n\ell_m''' + 4(u\ell_m\ell_n)' - \ell_n\ell_{m+1}' \tag{29}\\
\Rightarrow\; &\ell_m\ell_{n+1}' + \ell_n\ell_{m+1}' = \Omega_{n,m}', \tag{30}
\end{align}
$$

where (27) is the Lenard-type recursion relation multiplied by ℓ_m, (28) is obtained by grouping terms and applying the recursion relation for ℓ_m. (30) is obtained from the elementary identity

$$
\ell_m\ell_n''' + \ell_n\ell_m''' = (\ell_n\ell_m)''' - 3\left(\ell_n'\ell_m'\right)'. \tag{31}
$$

Now note that integrating (26) by parts and letting $n \mapsto n-1$ gives

$$
\ell_m\ell_n' = \ell_{m+1}\ell_{n-1}' + \left[\Omega_{n-1,m} - \ell_{n-1}\ell_{m+1}\right]'. \tag{32}
$$

Geometrically, this equation says that the quantity $\ell_m\ell_n'$ only picks up total derivatives if we move along the anti-diagonals of the lattice points labelled (m, n) for $0 \le m, n \le k + 1$. We can now see how the constants of motion arise. If, due to the boundary conditions on ℓ_0 and ℓ_{k+1}, the quantity $\ell_m\ell_n'$ is a total derivative on the border of the lattice (*i.e.* when m or n are 0 or $k + 1$), then we can produce a total derivative equal to zero by using (32) to move across the lattice from one border to another. Explicitly, moving a distance r along an anti-diagonal gives

$$
\ell_m\ell_n' = \ell_{m+r}\ell_{n-r}' + \left[\sum_{q=0}^{r-1}\Omega_{n-q-1,m+q} - \ell_{n-q-1}\ell_{m+q+1}\right]'. \tag{33}
$$

If $\ell_{k+1}' = 0$, then let $m = k - p$, $n = k + 1$ and $r = p + 1$. This gives

$$
\frac{d}{ds}\left[\ell_{k+1}\ell_{k-p} + \sum_{q=0}^{p}\left(\Omega_{k-p+q,k-q} - \ell_{k-q}\ell_{k-p+q+1}\right)\right] = 0 \tag{34}
$$

which integrating gives the first part of the theorem. If $\ell_0' = 0$, then let $m = 0$ and $r = n$. This gives

$$
\frac{d}{ds}\left[\ell_0\ell_p - \sum_{q=0}^{p-1}\left(\Omega_{p-1-q,q} - \ell_{p-1-q}\ell_{q+1}\right)\right] = 0, \tag{35}
$$

which integrating gives the second part of the theorem. $\qquad\square$

Remark 4.2. Setting $\ell_{k+1} = 0$ and $\ell = s/2$ in the Lenard recursion relation implies the τ_p are constant and we recover (6).

Remark 4.3. Consider the standard Lenard differential polynomials obtained with the boundary condition $\ell_0 = \frac{1}{2}$ and setting all integration constants to zero. From the above theorem, we find that σ_p are constants, which by the definition of the standard Lenard differential polynomials must be zero. We therefore have, after some rearranging,

$$\ell_p = \sum_{q=0}^{p-2} \left(\Omega_{p-1-q,q} - \ell_{p-1-q}\ell_{q+1} \right) + \Omega_{0,p-1} \,. \tag{36}$$

The right-hand side of the above equations only contains ℓ_n with $n < p$, while the left-hand side only contains ℓ_p. Hence, we can use these equations to recursively determine ℓ_p, which are exactly the Lenard differential polynomials. Note that this shows each Lenard differential polynomial is, indeed, a differential polynomial, a fact that is not obvious from their usual definition (22).

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