

LIMIT DISTRIBUTIONS OF GAUSSIAN BLOCK ENSEMBLES* **

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It has been shown by Voiculescu that important classes of square independent random matrices are asymptotically free, where freeness is a noncommutative analog of classical independence. Recently, we introduced the concept of matricial freeness, which is similar to freeness in free probability, but it also has some matricial features. Using this new concept of noncommutative independence, we described the asymptotics of blocks and symmetric blocks of certain classes of independent random matrices. In this paper, we present the main results obtained in this framework, concentrating on the ensembles of blocks of Gaussian random matrices.

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1. Introduction

Our main objective is to describe asymptotic joint distributions of rectangular blocks of independent random matrices, called *random blocks*, under the expectation of normalized partial traces. For that purpose, we use a new concept of noncommutative independence called *matricial freeness* and associated arrays of operators which give Hilbert space realizations of these distributions. The well-known connection between free probability and the asymptotics of independent square random matrices under the expectation of normalized trace can also be reproduced in this framework.

The most fundamental results of this nature were obtained by Voiculescu [1], who showed that certain ensembles of independent $n \times n$ random matrices $\{Y(u, n) : u \in \mathcal{U}\}$ were asymptotically free under the expectation of the normalized complete trace, $\tau(n) = \mathbb{E} \otimes \text{Tr}(n)$, where $\text{Tr}(n)$ stands for the

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trace divided by n . In particular, if the entries of $Y(u, n)$ are i.i.d. complex $(0, 1/n)$ -Gaussian random variables, we can symbolically write

$$\lim_{n \rightarrow \infty} Y(u, n) = \eta(u),$$

where $\{\eta(u) : u \in \mathcal{U}\}$ is the standard free circular system of operators and convergence is understood in the sense of mixed moments under $\tau(n)$. The operators $\eta(u)$ live in the free Fock space and have the standard circular distribution (uniform distribution on the unit disc in the complex plane) in the vacuum state. A similar result holds for Hermitian Gaussian random matrices whose limit joint distributions are described by mixed moments of free Gaussian operators with semicircle distributions.

Gaussian random matrices studied by Voiculescu had i.i.d. entries, except that in the case of Hermitian ensembles the matrices were assumed to be Hermitian. If the entries are independent but not identically distributed, standard free probability may not suffice to describe asymptotic distributions of these matrices. One of the approaches is to employ the much more general scheme of freeness with amalgamation as in the papers of Shlyakhtenko [2] and Benaych-Georges [3]. Recently, we have developed another approach [4, 5] in the case when the entries are independent and block-identically distributed (i.b.i.d.). This approach is based on the concept of matricial freeness [6], which can be viewed as a matricial generalization of freeness, lying somewhere between freeness and freeness with amalgamation.

In this context, we decompose matrices $Y(u, n)$ with i.b.i.d. entries into rectangular blocks $\{S_{p,q}(u, n) : 1 \leq p, q \leq r\}$, or symmetric blocks $\{T_{p,q}(u, n) : 1 \leq p \leq q \leq r\}$. The block analog of the above Voiculescu's result for the ensemble of independent Gaussian matrices can then be written in the form

$$\lim_{n \rightarrow \infty} S_{p,q}(u, n) = \zeta_{p,q}(u),$$

where $\zeta_{p,q}(u)$ are certain operators living in the *matricially free Fock space*, a matricial analog of the free Fock space, into which $\eta(u)$ decompose. The convergence is in the sense of mixed $*$ -moments under the expectation of normalized partial traces over the subsets of basis vectors defined by the block decomposition, denoted $\tau_q(n) = \mathbb{E} \circ \text{Tr}_q(n)$, where $1 \leq q \leq r$. In turn, the corresponding $*$ -moments of operators are computed with respect to the family of vacuum states. A similar theorem holds for symmetric blocks of Hermitian and non-Hermitian Gaussian random matrices.

This block model encodes more degrees of freedom than the usual framework of square random matrices with i.i.d. Gaussian entries. Apart from block variances, the most important parameters are the *asymptotic dimensions* of blocks, namely

$$d_q = \lim_{n \rightarrow \infty} \frac{n_q}{n},$$

where $n = n_1 + \dots + n_r$ is the n -dependent partition of n defined by the block decomposition of $Y(u, n)$. We assume that these limits exist, but it is possible that some of them vanish. This leads to three types of blocks or symmetric blocks: *balanced* (if both their asymptotic dimensions are positive), *unbalanced* (if one asymptotic dimension is positive and the other one vanishes) and *evanescent* (if both asymptotic dimensions vanish). For details, see [5].

Using all types of blocks, we can construct random matrix models for other notions of independence, such as monotone independence, Boolean independence, c-freeness (conditional freeness) and s-freeness (freeness with subordination). Moreover, the limit moments can be written in a quite explicit form as polynomials in d_1, \dots, d_r , with block variances as additional parameters. In some cases, this enables us to construct simple random matrix models for certain families of probability measures, like the model for free Meixner laws [7, 8] constructed in [9]. At the same time, studying such polynomials can lead to new results in free probability. For instance, a study of the limit distributions of products of independent rectangular Gaussian random matrices produced polynomials which can be viewed as *multivariate Fuss–Narayana polynomials* and, moreover, turned out to be the moments of the free multiplicative convolution of Marchenko–Pastur distributions with arbitrary shape parameters [10].

In order to make this paper easy to follow, we avoid technical details and only sketch some proofs. For complete proofs, we refer the reader to [4, 5, 9, 10].

2. Matricial operator systems

In the framework of free probability, an important role is played by semi-circular and circular systems of operators. Such systems were introduced by Voiculescu, who used them to construct the generators of free group factors in order to prove certain isomorphisms between them [1].

These generators play also an important role in the framework of matricial freeness. For the definition of that concept, which can be viewed as a matricial generalization of freeness involving a family of states instead of one state, we refer the reader to [6]. In this paper, we prefer to restrict our attention to important examples of arrays of operators [4, 5] which are matricially free with respect to a family of states associated with vacuum vectors in the ‘matricial’ analog of the free Fock space, called the matricially free Fock space.

Definition 2.1. Let $(\mathcal{H}_{p,q}(u))$, $u \in \mathcal{U}$, be a family of $r \times r$ arrays of Hilbert spaces. The *matricially free Fock space* is the Hilbert space direct sum of the form

$$\mathcal{M} = \bigoplus_{q=1}^r \mathcal{M}_q,$$

where

$$\mathcal{M}_q = \mathbb{C}\Omega_q \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{p_1, \dots, p_m \in [r] \\ u_1, \dots, u_m \in \mathcal{U}}} \mathcal{H}_{p_1, p_2}(u_1) \otimes \mathcal{H}_{p_2, p_3}(u_2) \otimes \dots \otimes \mathcal{H}_{p_m, q}(u_m),$$

with $[r] := \{1, 2, \dots, r\}$ and $\{\Omega_1, \dots, \Omega_r\}$ being the set of unit (vacuum) vectors, equipped with the canonical inner product. Let $\{\Psi_1, \dots, \Psi_r\}$ be the corresponding states replacing the single vacuum state in the free Fock space.

In fact, for most purposes, it suffices to take each Hilbert space to be one-dimensional, namely $\mathcal{H}_{p,q}(u) = \mathbb{C}e_{p,q}(u)$, where the vectors $e_{p,q}(u)$ form an orthonormal basis. Arrays of certain partial isometries living in this Fock space which remind free creation operators $\ell(u)$ living in the free Fock space serve as generators of certain Toeplitz–Cuntz–Krieger algebras [11]. These partial isometries, when multiplied by positive scalars, become matricial analogs of free creation operators.

Definition 2.2. Let $B(u) = (b_{p,q}(u))$ be an array of positive real numbers for any $u \in \mathcal{U}$. We associate with each such array the array of *matricially free creation operators* whose non-trivial action onto the basis vectors is

$$\begin{aligned} \wp_{p,q}(u)\Omega_q &= \sqrt{b_{p,q}(u)}e_{p,q}(u), \\ \wp_{p,q}(u)(e_{q,t}(s)) &= \sqrt{b_{p,q}(u)}(e_{p,q}(u) \otimes e_{q,t}(s)), \\ \wp_{p,q}(u)(e_{q,t}(s) \otimes w) &= \sqrt{b_{p,q}(u)}(e_{p,q}(u) \otimes e_{q,t}(s) \otimes w) \end{aligned}$$

for any $p, q, t \in [r]$ and $u, s \in \mathcal{U}$, where $e_{q,t}(s) \otimes w$ is a basis vector. Their actions onto the remaining basis vectors give zero. The corresponding *matricially free annihilation operators* are their adjoints $\wp_{p,q}^*(u)$. In some cases, it will be convenient to use the same notation even if $b_{p,q}(u) = 0$, when we obtain trivial operators.

Certain linear combinations of the matricially free creation and annihilation operators are of special interest. We have studied several matricial systems of operators constructed from the matricially free creation and annihilation operators or their symmetrized counterparts when describing

the asymptotic distributions of blocks or symmetric blocks of Hermitian and non-Hermitian Gaussian random matrices [4, 5]. We define them below, mentioning also the corresponding ensembles of random blocks. We also give their realizations as operator-valued matrices, which identifies some of them with the generators of free group factors used by Voiculescu in [1].

1. Non-trivial matricially free creation and annihilation operators can be realized as operator-valued matrices from the C^* -algebra $\mathcal{A} \otimes M_n(\mathbb{C})$, where \mathcal{A} is the C^* -algebra generated by the family $\{\ell(p, q, u) : 1 \leq p, q \leq r, u \in \mathcal{U}\}$ of $*$ -free creation operators living in the free Fock space. Namely,

$$\begin{aligned}\wp_{p,q}(u) &= \ell(p, q, u) \otimes e(p, q), \\ \wp_{p,q}(u)^* &= \ell(p, q, u)^* \otimes e(q, p),\end{aligned}$$

where $\{e(p, q) : 1 \leq p, q \leq r\}$ is the array of matrix units. If φ is the vacuum state in this free Fock space and ψ_q is the vector state on $M_n(\mathbb{C})$ defined by the basis vector e_q of \mathbb{C}^n , then the state Ψ_q can be identified with $\varphi \otimes \psi_q$, as we showed in [12].

2. *Matricially free Gaussian operators* are self-adjoint operators of the form

$$\omega_{p,q}(u) = \wp_{p,q}(u) + \wp_{p,q}(u)^*,$$

and they are the canonical Gaussian operators in our framework. It turns out that they describe the limit distributions of unbalanced symmetric blocks of Hermitian Gaussian random matrices.

3. It is convenient to introduce the *symmetrized creation operators* as

$$\widehat{\wp}_{p,q}(u) = \begin{cases} \wp_{p,q}(u) + \wp_{q,p}(u) & \text{if } p < q \\ \wp_{q,q}(u) & \text{if } p = q \end{cases}$$

and the *symmetrized annihilation operators* as their adjoints $\widehat{\wp}_{p,q}^*(u)$.

4. In order to describe the limit distributions of balanced symmetric blocks of Hermitian Gaussian random matrices, we need to use *symmetrized Gaussian operators*

$$\widehat{\omega}_{p,q}(u) = \widehat{\wp}_{p,q}(u) + \widehat{\wp}_{p,q}(u)^*,$$

where all operators are non-trivial. It is easy to see that if all creation and annihilation operators involved are non-trivial, then the above ones can be identified with the operator-valued matrices of the form

$$\widehat{\omega}_{p,q}(u) = \begin{cases} g(p, q, u) \otimes e(p, q) + g(p, q, u)^* \otimes e(q, p) & \text{if } p < q \\ s(q, u) \otimes e(q, q) & \text{if } p = q \end{cases},$$

where $\{g(p, q, u) : 1 \leq p \leq q \leq r, u \in \mathcal{U}\}$ is a family of circular operators and $\{s(q, u) : 1 \leq q \leq r, u \in \mathcal{U}\}$ is a family of semicircular operators. These matrices are generators of free group factors introduced by Voiculescu [1].

5. In order to describe the limit distributions of the usual (non-symmetric) blocks of non-Hermitian Gaussian random matrices, we need to use *matricial R -circular operators* [12]

$$\zeta_{p,q}(u) = \wp_{p,q}(u') + \wp_{q,p}(u'')^*,$$

where the notation means that in order to construct one such operator labelled u one needs two operators, which are labelled u' and u'' , thus the index set \mathcal{U} has to be doubled. If the added operators are non-trivial, then we can identify the above ones with operator-valued matrices of the form

$$\zeta_{p,q}(u) = g(p, q, u) \otimes e(p, q)$$

for any $1 \leq p, q \leq r$ and any $u \in \mathcal{U}$, where $\{g(p, q, u) : 1 \leq p, q \leq r, u \in \mathcal{U}\}$ is a family of circular operators (here, the covariances of $\wp_{p,q}(u)$ are symmetric and identical for u' and u''). Sums $\zeta(u) = \sum_{p,q} \zeta_{p,q}(u)$ are R -cyclic matrices introduced in [13].

6. In order to describe the limit joint distributions of the symmetric blocks of non-Hermitian Gaussian random matrices, we need to use *matricial circular operators* [5] which are natural symmetrizations of $\zeta_{p,q}(u)$, namely

$$\eta_{p,q}(u) = \widehat{\wp}_{p,q}(u') + \widehat{\wp}_{p,q}(u'')^*,$$

where $u \in \mathcal{U}$ and $1 \leq p \leq q \leq r$. If the added operators are non-trivial, then the above ones can be identified with the operator-valued matrices of the form

$$\eta_{p,q}(u) = \begin{cases} g(p, q, u) \otimes e(p, q) + g(q, p, u) \otimes e(q, p) & \text{if } p < q \\ s(q, u) \otimes e(q, q) & \text{if } p = q \end{cases},$$

where $\{g(p, q, u) : 1 \leq p, q \leq r, u \in \mathcal{U}\}$ is a family of circular operators and $\{s(q, u) : 1 \leq q \leq r, u \in \mathcal{U}\}$ is a family of semicircular operators.

3. Hermitian Gaussian Block Ensemble

In free probability, we study the mixed moments of independent $n \times n$ random matrices $\{Y(u, n) : u \in \mathcal{U}\}$ under the expectation of the normalized trace

$$\tau(n) = \mathbb{E} \circ \text{Tr}(n),$$

where

$$\mathrm{Tr}(n)(A) = \frac{1}{n} \mathrm{Tr}(A)$$

for any $n \times n$ matrix A as $n \rightarrow \infty$. Let us recall the asymptotic freeness result of Voiculescu for independent Hermitian Gaussian random matrices [14].

Theorem 3.1. *If we are given an ensemble of independent Hermitian $n \times n$ random matrices $\{Y(u, n) : u \in \mathcal{U}\}$, whose entries $Y_{i,j}(u, n)$ satisfy $Y_{i,j}(u, n) = \overline{Y_{j,i}(u, n)}$, are complex $(0, 1/n)$ -Gaussian if $i \neq j$ and real $(0, 1/n)$ -Gaussian if $i = j$, and the family $\{Y_{i,j}(u, n) : 1 \leq i \leq j \leq n\}$ is independent for any u , then*

$$\lim_{n \rightarrow \infty} Y(u, n) = \omega(u),$$

which should be understood as the convergence of mixed moments of matrices under $\tau(n)$ to the mixed moments of the corresponding free standard semicircular operators $\omega(u)$ under the vacuum state Φ in the free Fock space.

Remark 3.1. Let us make some remarks which will enable us to formulate our main results.

1. We stated the above theorem in a simplified form, which will also be used when we state our results on the asymptotic distributions of random blocks. A more explicit formulation says that

$$\lim_{n \rightarrow \infty} \tau(n)(Y(u_1, n) \dots Y(u_m, n)) = \Phi(\omega(u_1) \dots \omega(u_m))$$

for any $u_1, \dots, u_m \in \mathcal{U}$, where $\{\omega(u) : u \in \mathcal{U}\}$ is a standard free semicircular family, which means in particular that $\Phi(\omega(u)^2) = 1$ for any u . Alternatively, one could say that the family $\{Y(u, n) : u \in \mathcal{U}\}$ is asymptotically free under $\tau(n)$ and that the limit distribution of each $Y(u, n)$ is the standard semicircular Wigner distribution $W(0, 1)$ with density $(2\pi)^{-1}\sqrt{4 - x^2}$ on $[-2, 2]$.

2. In order to define a family of partial traces, take the decomposition of the set $[n] := \{1, 2, \dots, n\}$ into r disjoint intervals

$$[n] := N_1 \cup \dots \cup N_r$$

and denote by n_q the cardinality of N_q . Next, let

$$I(n) = D_1 + \dots + D_q$$

be the corresponding decomposition of the $n \times n$ identity matrix $I(n)$, that is $(D_k)_{i,j} = 1$ whenever $i = j \in N_k$, with the remaining entries equal to zero, where $1 \leq k \leq r$. Of course, the objects N_q, n_q, D_q depend on n , but this is suppressed in the notation.

3. By *partial traces* we then understand states of the form

$$\tau_q(n) = \mathbb{E} \circ \text{Tr}_q(n),$$

where

$$\text{Tr}_q(n)(A) = \frac{1}{n_q} \text{Tr}(D_q A D_q)$$

and $1 \leq q \leq r$.

4. By *random blocks* of a random matrix $Y(u, n)$, we shall understand $n \times n$ matrices of the form

$$S_{p,q}(u, n) = D_p Y(u, n) D_q,$$

for any $1 \leq p, q \leq r$ and $u \in \mathcal{U}$, where $r, n \in \mathbb{N}$. In particular, if $Y(u, n)$ is Hermitian, then $S_{q,p}(u, n) = S_{p,q}(u, n)^*$. Clearly, we have the decomposition

$$Y(u, n) = \sum_{1 \leq p, q \leq r} S_{p,q}(u, n)$$

for any u, n .

5. We assume that all variables $Y_{i,j}(u, n)$ which belong to the same block $S_{p,q}(u, n)$ have the same covariance $\mathbb{E}(\overline{Y_{i,j}(u, n)} Y_{i,j}(u, n)) = v_{p,q}(u)/n$ whenever $(i, j) \in N_p \times N_q$. We denote by $V(u) = (v_{p,q}(u))$ the corresponding matrices of block covariances. Apart from the dimension matrix

$$D = \text{diag}(d_1, \dots, d_r),$$

these matrices are additional parameters of the ensemble. In fact, the limit distributions will be expressed in terms of matrices $B(u) = DV(u)$.

6. By *random symmetric blocks* of a random matrix $Y(u, n)$, we shall understand $n \times n$ matrices of the form

$$T_{p,q}(u, n) = \begin{cases} S_{p,q}(u, n) + S_{q,p}(u, n) & \text{if } p < q \\ S_{q,q}(u, n) & \text{if } p = q \end{cases}$$

for any $1 \leq p \leq q \leq r$ and $u \in \mathcal{U}$, where $r, n \in \mathbb{N}$. We have the decomposition

$$Y(u, n) = \sum_{1 \leq p \leq q \leq r} T_{p,q}(u, n)$$

for any u, n . Clearly, if $Y(u, n)$ is Hermitian, then $T_{p,q}(u, n) = T_{p,q}(u, n)^*$.

7. When speaking of limit distributions of mixed moments of symmetric blocks, we will use a simplified formulation, similar to that in the free case. For instance,

$$\lim_{n \rightarrow \infty} T_{p,q}(u, n) = \widehat{\omega}_{p,q}(u)$$

in the Hermitian case will mean that

$$\begin{aligned} \lim_{n \rightarrow \infty} \tau_q(n)(T_{p_1,q_1}(u_1, n) \dots T_{p_m,q_m}(u_m, n)) \\ = \Psi_q(\widehat{\omega}_{p_1,q_1}(u_1) \dots \widehat{\omega}_{p_m,q_m}(u_m)) \end{aligned}$$

for any $1 \leq p_1 \leq q_1 \leq r, \dots, 1 \leq p_m \leq q_m \leq r, 1 \leq q \leq r$ and $u_1, \dots, u_m \in \mathcal{U}$. The operators will always belong to one of the families defined in Section 2. A similar formulation will be used for blocks $S_{p,q}(u, n)$.

We can describe limit joint distributions of blocks and symmetric blocks of Hermitian random matrices under rather general assumptions. It suffices to assume that the family $\{Y(u, n) : u \in \mathcal{U}\}$ is asymptotically free and asymptotically free from $\{D_1, \dots, D_r\}$ under $\tau(n)$ and that their norms are uniformly bounded almost surely. This class includes unitarily invariant random matrices whose limit moments are compactly supported probability measures on the real line. This general version was proved in [5, Theorem 6.1]. For the sake of simplicity, we restrict our attention here to the case of Hermitian Gaussian random matrices.

Theorem 3.2. *If we are given an ensemble $\{Y(u, n) : u \in \mathcal{U}\}$ of independent Hermitian $n \times n$ random matrices whose entries $Y_{i,j}(u, n)$ satisfy $Y_{i,j}(u, n) = \overline{Y_{j,i}(u, n)}$, are complex $(0, v_{p,q}(u)/n)$ -Gaussian if $i \neq j$ and $(i, j) \in N_p \times N_q$, and real $(0, v_{q,q}(u)/n)$ -Gaussian if $i = j \in N_q$ and the family $\{Y_{i,j}(u, n) : 1 \leq i \leq j \leq n\}$ is independent for any u , then*

$$\lim_{n \rightarrow \infty} T_{p,q}(u, n) = \widehat{\omega}_{p,q}(u)$$

for any $p \leq q$ and $u \in \mathcal{U}$, where convergence is in the sense of mixed moments under partial traces and the arrays $(\widehat{\omega}_{p,q}(u))$ are associated with the symmetric covariance matrices $B(u) = DV(u)$, respectively.

Sketch of the proof. First, let us assume that $v_{p,q}(u) = 1$ for any p, q, u . In that case, we can use asymptotic freeness of the family $\{Y(u, n) : u \in \mathcal{U}\}$ and its asymptotic freeness with respect to the family $\{D_1, \dots, D_r\}$ (these are deterministic diagonal matrices) to describe the limit joint distributions of the blocks $S_{p,q}(u, n)$ under $\tau(n)$ since

$$S_{p,q}(u, n) = D_p Y(u, n) D_q$$

for any p, q, u . Moreover, we know that the limit distribution of each $Y(u, n)$ is the standard semicircle Wigner law $W(0, 1)$ and a direct computation gives $\tau(n)(D_q) = n_q/n \rightarrow d_q$ for any q . Now, it is not hard to show that the family of operators

$$\omega(u) := \sum_{p,q} \omega_{p,q}(u),$$

where $\omega_{p,q}(u) = \wp_{p,q}(u) + \wp_{p,q}(u)^*$ for any u and $\wp_{p,q}(u)$ has covariance d_p , is free with respect to $\Psi = \sum_q d_q \Psi_q$. We abuse the notation a little since by $\omega(u)$ we denoted a standard semicircular operator on the free Fock space, but this is justified by the fact that each $\omega(u)$ has the standard semicircle distribution under Ψ . Therefore, we have, in fact, a realization of the standard free semicircular family $\{\omega(u) : u \in \mathcal{U}\}$ as operators living in the matricially free Fock space \mathcal{M} . Now, it suffices to find an appropriate limit realization for $\{D_1, \dots, D_r\}$. We have shown in [5] that it is given by the family $\{P_1, \dots, P_r\}$, where

$$P_q = 1 \otimes e(q, q)$$

using the tensor product realization described in Section 2, since the family $\{\omega(u) : u \in \mathcal{U}\}$ is free from $\{P_1, \dots, P_r\}$ and $\Psi(P_q) = d_q$. Therefore, any limit mixed $*$ -moment of the random blocks $S_{p,q}(u)$ under $\tau(n)$ can be expressed as a mixed $*$ -moment of the corresponding operators $P_p \omega(u) P_q$ under Ψ . Symbolically, since $\lim_{n \rightarrow \infty} Y(u, n) = \omega(u)$, we have

$$\lim_{n \rightarrow \infty} S_{p,q}(u, n) = P_p \omega(u) P_q = \wp_{p,q}(u) + \wp_{q,p}(u)^* := s_{p,q}(u)$$

except that the moments of blocks are computed under $\tau(n)$ and those of the operators $P_p \omega(u) P_q$ are computed under Ψ . In order to pass from $\tau(n)$ to $\tau_q(n)$, notice that

$$\tau_q(n)(A) = \frac{n}{n_q} \tau(n)(D_q A D_q)$$

so the limit mixed $*$ -moments of random blocks under $\tau_q(n)$ are given by the mixed $*$ -moments of the above operators under Ψ_q , where $1 \leq q \leq r$. All this holds provided $d_q \neq 0$. The case when $d_q = 0$ is slightly more complicated and is omitted here (we refer the reader to [5]). It easily follows that

$$\lim_{n \rightarrow \infty} T_{p,q}(u, n) = \widehat{\omega}_{p,q}(u)$$

for any $1 \leq p \leq q \leq r$ and $u \in \mathcal{U}$. It can also be seen that one can rescale blocks $S_{p,q}(u, n)$, which means that one can rescale the block covariances and take arbitrary non-negative $v_{p,q}(u)$ (except that we must have the symmetry

$v_{p,q}(u) = v_{q,p}(u)$ since $Y(u, n)$ is Hermitian). This proves that in the case when the covariances are equal to $v_{p,q}(u)$ within block $S_{p,q}(u)$, the limit operator $\varsigma_{p,q}(u)$ gets rescaled by $v_{p,q}(u)$ and thus the covariance of $\wp_{p,q}(u)$ becomes $b_{p,q}(u) = d_p v_{p,q}(u)$. This completes the proof. \square

Corollary 3.1. *In particular,*

1. if $d_p = 1$ and $d_q = 0$, then $\lim_{n \rightarrow \infty} T_{p,q}(u, n) = \omega_{p,q}(u)$,
2. if $d_p = 0$ and $d_q = 1$, then $\lim_{n \rightarrow \infty} T_{p,q}(u, n) = \omega_{q,p}(u)$,
3. if $d_p = 0$ and $d_q = 0$, then $\lim_{n \rightarrow \infty} T_{p,q}(u, n) = 0$.

Proof. If $d_p = 1$ and $d_q = 0$, then $\omega_{q,p}(u) = 0$ since $b_{q,p}(u) = d_q v_{q,p}(u) = 0$ and thus $\widehat{\omega}_{p,q}(u)$ reduces to $\omega_{p,q}(u)$. In turn, if $T_{p,q}(u, n)$ is evanescent, then $\omega_{p,q}(u) = \omega_{q,p}(u) = 0$ and thus $\widehat{\omega}_{p,q}(u) = 0$, which completes the proof. \square

Remark 3.2. It should be remarked that the matricially free Gaussian operators are not operatorial realizations of the usual (non-symmetric) blocks $S_{p,q}(u)$ of Gaussian Hermitian matrices. In fact, it follows from the proof of Theorem 3.2 that

$$\lim_{n \rightarrow \infty} S_{p,q}(u) = g(p, q, u) \otimes e(p, q)$$

for any $1 \leq p, q \leq r$ and $u \in \mathcal{U}$, where convergence is understood in the usual sense (mixed *-moments of blocks under partial traces $\tau_q(n)$ converge to mixed *-moments of the corresponding operators under Ψ_q). Here, $\{g(p, q, u) : 1 \leq p \leq q \leq r, u \in \mathcal{U}\}$ is a family of circular operators and $\{s(q, u) : 1 \leq q \leq r, u \in \mathcal{U}\}$ is a family of semicircular operators and we assume that $g(q, p, u) = g(p, q, u)^*$ for $p < q$.

Example 3.1. Fix $u \in \mathcal{U}$ (omitted in our notations) and $p \neq q$. Let $g(p, q) = \ell_1 + \ell_2^*$ and thus $g(q, p) = \ell_1^* + \ell_2$ for some *-free creation operators ℓ_1, ℓ_2 . In view of Remark 3.2, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \tau_q(n)(S_{q,p}(n)S_{p,q}(n)S_{q,p}(n)S_{p,q}(n)) &= \Psi_q(\varsigma_{q,p}\varsigma_{p,q}\varsigma_{q,p}\varsigma_{p,q}) \\ &= \varphi((\ell_1^* + \ell_2)(\ell_1 + \ell_2^*)(\ell_1^* + \ell_2)(\ell_1 + \ell_2^*)) \\ &= (d_p^2 + d_p d_q) v_{p,q}^2, \end{aligned}$$

where $\varphi(\ell_1^* \ell_1) = \Psi_q(\wp_{p,q}^* \wp_{p,q}) = d_p v_{p,q}$ and $\varphi(\ell_2^* \ell_2) = \Psi_p(\wp_{q,p}^* \wp_{q,p}) = d_p v_{q,p}$ and we use the symmetry $v_{p,q} = v_{q,p}$.

4. Ginibre Block Ensemble

Blocks of non-Hermitian Gaussian random matrices can be treated in a similar way. The associated ensemble of blocks can be called the *Ginibre Block Ensemble*. The main difference is that the limit joint $*$ -distributions of blocks are described by matricial R -circular operators $\zeta_{p,q}(u)$ and their adjoints and those of the symmetric blocks by matricial circular operators $\eta_{p,q}(u)$ and their adjoints.

Let us recall Voiculescu's theorem on the asymptotic freeness of the ensemble of independent Gaussian random matrices with i.i.d. entries (Ginibre Ensemble) [14].

Theorem 4.1. *If we are given an ensemble of independent $n \times n$ random matrices $\{Y(u, n) : u \in \mathcal{U}\}$, whose entries $Y_{i,j}(u, n)$ are complex $(0, 1/n)$ -Gaussian for any i, j and the family $\{Y_{i,j}(u, n) : 1 \leq i, j \leq n\}$ is independent, then*

$$\lim_{n \rightarrow \infty} Y(u, n) = \eta(u),$$

where convergence is in the sense of mixed $*$ -moments of matrices under $\tau(n)$ to the mixed $*$ -moments of the corresponding free standard circular operators $\eta(u)$ under the vacuum state Φ in the free Fock space.

Let us now formulate an analogous theorem for blocks and symmetric blocks of Gaussian random matrices with i.b.i.d. entries.

Theorem 4.2. *If we are given an ensemble of independent $n \times n$ random matrices $\{Y(u, n) : u \in \mathcal{U}\}$, whose entries $Y_{i,j}(u, n)$ are complex $(0, v_{p,q}(u)/n)$ -Gaussian for any $(i, j) \in N_p \times N_q$ and the family $\{Y_{i,j}(u, n) : 1 \leq i, j \leq n\}$ is independent, then*

$$\lim_{n \rightarrow \infty} S_{p,q}(u, n) = \zeta_{p,q}(u),$$

where convergence is in the sense of mixed $*$ -moments of blocks under partial traces $\tau_q(n)$ to the mixed $*$ -moments of the corresponding matricial systems of operators under the vacuum states Ψ_q , respectively, in the matricially free Fock space.

Sketch of the proof. The proof is similar to that of Theorem 3.1. In the case when all variables are i.i.d., we can use the asymptotic $*$ -freeness of the family $\{Y(u, n) : u \in \mathcal{U}\}$ under $\tau(n)$ as $n \rightarrow \infty$ as well as their asymptotic $*$ -freeness from the family of diagonal matrices. One can check that the family $\{\eta(u) : u \in \mathcal{U}\}$, where

$$\eta(u) = \sum_{p,q} \zeta_{p,q}(u) = \sum_{p,q} g(p, q, u) \otimes e(p, q)$$

is $*$ -free under Ψ as well as $*$ -free with respect to $\{P_1, \dots, P_r\}$ under Ψ , where, again, $P_q = 1 \otimes e(q, q)$. Moreover, each $\eta(u)$ has the standard circular distribution under $\Psi = \sum_q \Psi_q$. Since the asymptotic joint distribution of $\{D_1, \dots, D_r\}$ under $\tau(n)$ agrees with that of $\{P_1, \dots, P_r\}$ under Ψ , we must have

$$\lim_{n \rightarrow \infty} S_{p,q}(u, n) = \lim_{n \rightarrow \infty} D_p Y(u, n) D_q = P_p \eta(u) P_q = \zeta_{p,q}(u),$$

where convergence is understood as described in Remark 3.1, which completes the proof. \square

Corollary 4.1. *Under the assumptions of Theorem 4.2, it holds that*

$$\lim_{n \rightarrow \infty} T_{p,q}(u, n) = \eta_{p,q}(u)$$

for any p, q, u .

Proof. This is an easy consequence of Theorem 4.2. \square

Example 4.1. Computations of limit mixed $*$ -moments of blocks reduce to the computation of mixed $*$ -moments involving matricial R -circular systems of operators. For instance,

$$\begin{aligned} \lim_{n \rightarrow \infty} \tau_q(n) (S_{p,q}(n)^* S_{q,p}(n)^* S_{q,p}(n) S_{p,q}(n)) &= \Psi_q (\zeta_{p,q}^* \zeta_{q,p}^* \zeta_{q,p} \zeta_{p,q}) \\ &= \varphi(\ell_1^* \ell_1) \varphi(\ell_3^* \ell_3) \\ &= d_p d_q v_{p,q} v_{q,p}, \end{aligned}$$

where $p \neq q$, since

$$\zeta_{p,q} = (\ell_1 + \ell_2^*) \otimes e(p, q), \quad \zeta_{q,p} = (\ell_3 + \ell_4^*) \otimes e(q, p),$$

where $\{\ell_1, \ell_2, \ell_3, \ell_4\}$ is a $*$ -free system of free creation operators with covariances $\varphi(\ell_1^* \ell_1) = d_p v_{p,q}$ and $\varphi(\ell_3^* \ell_3) = d_q v_{q,p}$. In fact, it is well-known that any pair of free circular operators can be written in the form $\ell_1 + \ell_2^*$ and $\ell_3 + \ell_4^*$, respectively.

5. Products of independent Gaussian random matrices

The first concrete application of our method concerns products of independent rectangular Gaussian random matrices [10]. For any given $p \in \mathbb{N}$ and any $n \in \mathbb{N}$, consider the product of independent rectangular Gaussian random matrices

$$B(n) = X_1(n) X_2(n) \dots X_p(n),$$

where all entries are assumed to be independent $(0, 1/n)$ -Gaussian variables. If $X_j(n)$ is an $n_{j-1} \times n_j$ matrix for any $1 \leq j \leq p$, we assume that

$$\lim_{n \rightarrow \infty} \frac{n_j}{n} = d_j$$

for any $j \in \{0, \dots, p\}$ (it is convenient to start with n_0 rather than with n_1). Let $\tau_0(n)$ be the trace over the set of first n_0 basis vectors composed with classical expectation.

Theorem 5.1. *Under the above assumptions, it holds that*

$$\lim_{n \rightarrow \infty} \tau_0(n) \left((B(n)B^*(n))^k \right) = P_k(d_0, d_1, \dots, d_p),$$

where

$$P_k(d_0, d_1, \dots, d_p) = \sum_{j_0 + \dots + j_p = pk+1} \frac{1}{k} \binom{k}{j_0} \binom{k}{j_1} \dots \binom{k}{j_p} d_0^{j_0-1} d_1^{j_1} \dots d_p^{j_p}$$

for any natural k and j_0, j_1, \dots, j_p . These polynomials are called *multivariate Fuss–Narayana polynomials*.

Sketch of the proof. The proof is based on embedding the matrices $X_1(n), \dots, X_p(n)$ in symmetric blocks $T_{1,2}(n), \dots, T_{p,p+1}(n)$, respectively, of a large square Gaussian random matrix $Y(n)$ of dimension $N \times N$, where $N = n_0 + \dots + n_p$, with $(0, 1/n)$ -Gaussian entries for any n (we use only one matrix and thus we omit u in our notations). Computing the moments of $B(n)B^*(n)$ under $\tau_0(n)$ becomes now the normalized partial trace over the first subset of basis vectors. We then use Corollary 4.1 to realize the limit moments in terms of $\eta_{1,2}, \dots, \eta_{p,p+1}$ and their adjoints. These limit moments can be computed explicitly, which was done in [10], which completes the proof. \square

Let us make some additional remarks on the above result:

1. The special case of $p = 1$ corresponds to Wishart matrices. If we set $d_0 = 1$ and $d_1 = t$, we obtain

$$P_k(1, t) = \sum_{i+j=k+1} \frac{1}{k} \binom{k}{i} \binom{k}{j} t^j,$$

the moments of the Marchenko–Pastur distribution with shape parameter $t > 0$, namely

$$\varrho_t = \max\{1 - t, 0\} \delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \mathbb{1}_{[a,b]}(x) dx,$$

where $a = (1 - \sqrt{t})^2$ and $b = (1 + \sqrt{t})^2$.

2. The moments of the Marchenko–Pastur distributions are known to have the form of *Narayana polynomials*

$$N_k(t) = \sum_{j=1}^k \frac{1}{j} \binom{k-1}{j-1} \binom{k}{j-1} t^j$$

for any $k \in \mathbb{N}$. It is easy to show that $P_k(1, t) = N_k(t)$, but our formula is more suitable for multivariate generalizations.

3. If $d_0 = 1$, the multivariate Fuss–Narayana polynomials become the moments of the p -fold free multiplicative convolution of the Marchenko–Pastur distributions with arbitrary shape parameters. Namely,

$$P_k(1, t_1, \dots, t_p) = m_k(\varrho_{t_1} \boxtimes \varrho_{t_2} \boxtimes \dots \boxtimes \varrho_{t_p}),$$

where $m_k(\mu)$ stands for the k^{th} moment of μ and \boxtimes denotes the free multiplicative convolution.

4. A less general class of polynomials was studied in the context of *free Bessel laws*

$$\pi_{p,t} = \varrho_1^{\boxtimes(p-1)} \boxtimes \varrho_t,$$

where $p \in \mathbb{N}$, defined by Banica, Belinschi, Capitaine and Collins [15]. The moments of free Bessel laws are polynomials in one variable t ,

$$Q_k(t) = \sum_{j=1}^k \frac{1}{j} \binom{k-1}{j-1} \binom{pk}{j-1} t^j,$$

called *Fuss–Narayana polynomials*. Our polynomials are natural multivariate generalizations of these polynomials. Clearly, $Q_k(t) = P_k(1, \dots, 1, t)$, where 1 appears p times.

6. New random matrix models

The results on the asymptotics of random blocks can be applied to the construction of some new random matrix models, as we showed in [9]. In this paper, we choose to present a simple version of such a model for monotone independence [16]. In a similar way, one can construct random matrix models for Boolean independence and s-free independence (freeness with subordination). In fact, more general versions, not restricted to Gaussian random matrices, have been constructed in [5].

Monotone independence will appear in the study of the asymptotic joint distributions of two independent Hermitian Gaussian random matrices of the same block form. Our assumptions are the following:

(A1) we have a family of independent Gaussian random matrices

$$Y(u, n) = \begin{pmatrix} A(u, n) & B(u, n) \\ C(u, n) & D(u, n) \end{pmatrix},$$

where $u \in \mathcal{U}$, and blocks $(A(u, n))$ are evanescent, symmetric blocks built from $(B(u, n))$ and $(C(u, n))$ are unbalanced, and blocks $(D(u, n))$ are balanced,

(A2) the matrices are Hermitian, thus the off-diagonal blocks are Hermitian conjugate and the diagonal blocks are Hermitian,

(A3) the complex Gaussian variables $Y_{i,j}(u, n)$ have zero mean and have identical covariances within blocks, namely

$$\mathbb{E}(\overline{Y_{i,j}(u, n)} Y_{i,j}(u, n)) = \frac{v_{p,q}(u)}{n}$$

whenever the pair (i, j) belongs to the block indexed by (p, q) ,

(A4) the decomposition of the identity matrix corresponding to the block decomposition is given by

$$I(n) = D_1 + D_2$$

for any $n \in \mathbb{N}$.

Theorem 6.1. *Under assumptions (A1)–(A4), if $\mathcal{U} = \{1, 2\}$ and $v_{p,q}(u) = 1$ for any p, q, u , the pair $\{B(1, n) + C(1, n), Y(2, n)\}$ is asymptotically monotone independent with respect to the partial trace $\tau_1(n)$.*

Proof. By Theorem 3.2, the proof reduces to showing that the pair

$$\{\omega_{2,1}(1), \omega_{2,1}(2) + \omega_{2,2}(2)\}$$

is monotone independent with respect to Ψ_1 since, by assumption, the asymptotic dimensions are $d_1 = 0$ and $d_2 = 1$, which means that the remaining operators can be neglected. Denote $a = \omega_{2,1}(1)$ and $b = \omega_{2,1}(2) + \omega_{2,2}(2)$. We need to show that

$$\Psi_1(w_1 a_1 b_1 a_2 w_2) = \Psi_1(b_1) \Psi_1(w_1 a_1 a_2 w_2)$$

for any $a_1, a_2 \in \mathbb{C}[a, 1_1]$, $b_1 \in \mathbb{C}[b, 1_2]$, where $1_1 = 1_{2,1}$ and $1_2 = 1$ and w_1, w_2 are arbitrary elements of $\mathbb{C}\langle a, b, 1_1, 1_2 \rangle$. It suffices to consider the action of a and b onto their invariant subspace in \mathcal{M} of the form

$$\mathcal{M}' = \mathbb{C}\Omega_1 \oplus (\mathcal{F}(2) \otimes \mathcal{H}(1)) \oplus (\mathcal{F}(2) \otimes \mathcal{H}(2)),$$

where $\mathcal{F}(2) = \mathcal{F}(\mathbb{C}e_{2,2}(2))$ with the vacuum vector Ω and $\mathcal{H}(u) = \mathbb{C}e_{2,1}(u)$ for $u \in \{1, 2\}$, where we identify $\Omega \otimes e_{2,1}(u)$ with $e_{2,1}(u)$. Now, the range of any polynomial in a is $\mathbb{C}\Omega_1 \oplus \mathcal{H}(1)$ since

$$a^k \Omega_1 = \begin{cases} \Omega_1 & \text{if } k \text{ is even} \\ e_{2,1}(1) & \text{if } k \text{ is odd} \end{cases}$$

and $1_1 \Omega_1 = \Omega_1$, $1_1 e_{2,1}(1) = e_{2,1}(1)$. Therefore, it suffices to compute the action of any polynomial in b onto Ω_1 and $e_{2,1}(1)$. Now, the action of powers of b onto Ω_1 and onto $e_{2,1}(1)$ is the same as the action of the free Gaussian operator onto the vacuum vector in the free Fock space. Namely, we have

$$b^{2k} \Omega_1 = C_k \Omega_1 \bmod (\mathcal{M}' \ominus \mathbb{C}\Omega_1)$$

and

$$b^{2k} e_{2,1}(1) = C_k e_{2,1}(1) \bmod (\mathcal{M}' \ominus (\mathbb{C}\Omega_1 \oplus \mathcal{H}(1))),$$

where C_k is the k^{th} Catalan number and

$$b^{2k-1} \Omega_1 = b^{2k-1} e_{2,1}(1) = 0 \bmod (\mathcal{M}' \ominus (\mathbb{C}\Omega_1 \oplus \mathcal{H}(1)))$$

for any $k \in \mathbb{N}$. Thus $\Psi_1(b^{2k}) = C_k$ and, moreover, since $\mathcal{M}' \ominus (\mathbb{C}\Omega_1 \oplus \mathcal{H}(1)) \subset \text{Ker } a$, the required condition for monotone independence holds if b_1 is a positive power of b . It is easy to see that it also holds if $b_1 = 1_2$, which completes the proof. \square

Interestingly enough, unbalanced symmetric blocks were also used in the construction of a simple random matrix model for free Meixner laws [7, 8]. These are probability measures on the real line associated with the sequences of Jacobi parameters of the form

$$(\alpha_1, \alpha_2, \alpha_2, \dots) \text{ and } (\beta_1, \beta_2, \beta_2, \dots)$$

and thus we can say that they are associated with quadruples $(\alpha_1, \alpha_2, \beta_1, \beta_2)$. Let us formulate the theorem in the most interesting case when β_1 and β_2 are positive.

Theorem 6.2. *Under assumptions (A1)–(A4), let $\beta_1(u) = v_{2,1}(u) > 0$ and $\beta_2(u) = v_{2,2} > 0$, for any $u \in \mathcal{U}$. Then*

1. *the asymptotic distributions of the matrices*

$$M(u, n) = Y(u, n) + \alpha_1(u)D_1(n) + \alpha_2(u)D_2(n)$$

under the partial trace $\tau_1(n)$ are the free Meixner distributions associated with the parameters $(\alpha_1(u), \alpha_2(u), \beta_1(u), \beta_2(u))$, respectively,

2. the family $\{M(u, n) : u \in \mathfrak{U}\}$ is asymptotically conditionally free with respect to the pair of partial traces $(\tau_1(n), \tau_2(n))$ as $n \rightarrow \infty$.

Sketch of the proof. It follows from Theorem 3.2 that

$$\lim_{n \rightarrow \infty} Y(u, n) = \omega_{2,1}(u) + \omega_{2,2}(u),$$

where the variances of $\omega_{2,1}(u)$ and $\omega_{2,2}(u)$ are $b_{2,1}(u) = d_2 v_{2,1}(u) = \beta_1(u)$ and $b_{2,2}(u) = d_2 v_{2,2}(u) = \beta_2(u)$ since $d_1 = 0$ and $d_2 = 1$. The fact that $d_1 = 0$ is the reason why $\omega_{1,1}(u)$ and $\omega_{1,2}(u)$ become trivial operators and that is why they do not show up in the limit realization. Therefore,

$$\lim_{n \rightarrow \infty} M(u, n) = \gamma(u) := \omega_{2,1}(u) + \omega_{2,2}(u) + \alpha_1(u)P_1 + \alpha_2(u)P_2,$$

where P_1 and P_2 are as in the proof of Theorem 3.2. This implies, in particular, that the asymptotic distribution of $M(u, n)$ under $\tau_1(n)$ agrees with that of $\gamma(u)$ under Ψ_1 . The proof that the moments of $\gamma(u)$ are the moments of the free Meixner law associated with the parameters $(\alpha_1(u), \alpha_2(u), \beta_1(u), \beta_2(u))$ is purely combinatorial and can be found in [9]. We also refer the reader to [9] for the proof of asymptotic conditional independence of the family of such matrices. \square

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