DEFORMED q^{-1}-LAGUERRE POLYNOMIALS, RECURRENCE COEFFICIENTS, AND NON-LINEAR DIFFERENCE EQUATIONS*

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In this paper, we study a one-parameter deformation of the q-Laguerre weight function. An investigation is made on the polynomials orthogonal with respect to such a weight. With the aid of the two compatibility conditions, previously obtained in Y. Chen, M.E.H. Ismail, *J. Math. Anal. Appl.* **345**, 1 (2008), and the q-analog of a sum rule obtained in this paper, we derive expressions for the recurrence coefficients in terms of certain auxiliary quantities, and show that these quantities satisfy a pair of first order non-linear difference equations. This manuscript is a shorter version of the full paper [*Indagat. Math.*, to appear] and has been modified to be suitable for the proceedings of Matrix 2014.

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1. Introduction

In this paper, we derive difference equations satisfied by the recurrence coefficients of a family of polynomials orthogonal with respect to the weight supported on $[0,\infty)$,

$$w(x,\alpha,t;q) = \frac{x^{\alpha}}{(-(1-q)x;q)_\infty(-(1-q)\frac{t}{x};q)_\infty}\,, \qquad t \geq 0\,, \quad \alpha > -1\,, \quad 0<q<1\,, \tag{1.1}$$

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where

$$(a;q)_\infty := \prod_{j=0}^{\infty} \left(1 - a\, q^j\right) .$$

In the special case of $q \to 1^-$, this reduces to

$$x^\alpha e^{-x} e^{-t/x} ,$$

first considered by Chen and Its [1], which is a singular deformation of the ordinary Laguerre weight. It was shown in that case that the log-derivative of the Hankel determinant (with respect to t) is the τ-function of a particular Painlevé III.

In the limit $t \to 0^+$, the weight (1.1) reduces to the q-Laguerre weight introduced by Moak [2]. Other deformations of the q-Laguerre weight have been studied in [3–5] and [6]. It transpires that for $t = q/(1-q)^2$ and $\alpha = 0$, the corresponding orthogonal polynomials are the Stieltjes–Wigert polynomials. For this value of t and $\alpha \neq 0$, the orthogonal polynomials were studied in [7] by Askey.

Let $\{P_n(x)\}$ be the monic polynomials orthogonal with respect to a weight w on the interval $[0,\infty)$. That is

$$\int_0^\infty P_n(x) P_m(x) w(x)\, dx = h_n\, \delta_{nm} , \tag{1.2}$$

where h_n is the square of the L^2 norm. It is well-known that the polynomials satisfy a three term recurrence relation

$$x P_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x) . \tag{1.3}$$

We take the initial conditions to be $P_{-1}(x) = 0$ and $P_0(x) = 1$. Our monic polynomial has a monomial expansion

$$P_n(x) = x^n + p(n) x^{n-1} + \dots \tag{1.4}$$

It is clear from the recurrence relation that

$$\sum_{j=0}^{n-1} \alpha_j = -p(n) . \tag{1.5}$$

In [8], Chen and Ismail showed that under suitable conditions on the weight w, the polynomials satisfied a first order structural relation with respect to the operator D_q, defined by

$$\left(D_q f\right)(x) = \frac{f(x) - f(qx)}{x(1-q)} .$$

Specifically, they proved the following theorem.

Theorem 1.1. *Let*

$$A_n(x) = \frac{1}{h_n} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_n(y/q) w(y)\, dy \tag{1.6}$$

and

$$B_n(x) = \frac{1}{h_{n-1}} \int_0^\infty \frac{u(qx) - u(y)}{qx - y} P_n(y) P_{n-1}(y/q) w(y)\, dy\,, \tag{1.7}$$

where

$$u(x) = -\frac{D_{q^{-1}} w(x)}{w(x)}\,. \tag{1.8}$$

Then, the orthogonal polynomials satisfy the q-difference relation

$$D_q P_n(x) = \beta_n A_n(x) P_{n-1}(x) - B_n(x) P_n(x)\,. \tag{1.9}$$

The above theorem is a q-analog of the structural relation appearing in [9]. Furthermore, it was shown in [8] that the functions $A_n(x)$ and $B_n(x)$ satisfy the supplementary conditions

$$B_{n+1}(x) + B_n(x) = (x - \alpha_n) A_n(x) + x(q-1) \sum_{j=0}^{n} A_j(x) - u(qx)\,, \tag{qS_1}$$

and

$$\beta_{n+1} A_{n+1}(x) - \beta_n A_{n-1}(x) = 1 + (x - \alpha_n) B_{n+1}(x) - (qx - \alpha_n) B_n(x)\,. \tag{qS_2}$$

Equations (qS_1) and (qS_2) are q-analogs of the supplementary conditions (S_1) and (S_2) appearing in [9]. Chen and Its also made use of the following equation that can be thought of as the first integral of (S_1) and (S_2)

$$B_n^2(x) + u(x) B_n(x) + \sum_{j=0}^{n-1} A_j(x) = \beta_n A_n(x) A_{n-1}(x)\,. \tag{S_2'}$$

A derivation of (S_2') is given in [1] and the equation first appeared in [10]. In [1], they found a pair of coupled non-linear difference equations whose solutions were related to the recurrence coefficients. In this context, they also found a particular Painlevé-III differential equation in the parameter t. Equation (S_2') appeared in [11] in connection with a Painlevé-V equation, in [12], in connection with a Painlevé-IV equation and in [13], in connection with a Painlevé-V equation.

For the weight appearing in (1.1), we make use of Theorem 1.1, as well as equations (qS_1) and (qS_2) in an attempt to find expressions for the recurrence coefficients in terms of solutions to a pair of non-linear difference equations. Observe that the quantity $\sum_j A_j(x)$ appears in (qS_1) and not in (qS_2). We will find that in order to deal with this sum effectively, we will require an additional equation involving this quantity. Therefore, instrumental in our approach is the derivation of a q-analog of the equation (S_2') which can be thought of as a first integral of (qS_1) and (qS_2).

The three main results of the paper are summarized below.

Theorem 1.2. *Let $A_n(x)$ and $B_n(x)$ be given by (1.6) and (1.7). Then,*

$$\beta_n A_n(x)A_{n-1}(x)=B_n^2(x)+u(qx)B_n(x)+(1+(1-vq)xB_n(x))\sum_{j=0}^{n-1}A_j(x)\,. \quad (qS_2')$$

Lemma 1.3. *Let $\{P_n\}$ be the monic polynomials orthogonal with respect to the weight (1.1) on the interval $[0,\infty)$. Furthermore, let*

$$R_n=\frac{1}{h_n}\int_0^\infty P_n(y)P_n(y/q)\frac{w(y,\alpha,t;q)}{y}\,dy$$

and

$$r_n=\frac{1}{h_{n-1}}\int_0^\infty P_n(y)P_{n-1}(y/q)\frac{w(y,\alpha,t;q)}{y}\,dy\,.$$

Then, the recurrence coefficients α_n and β_n have the following form

$$\begin{aligned} q^{2n+\alpha}\alpha_n &= \frac{(1-q^n)}{1-q}+\frac{1-q^{n+\alpha+1}}{q(1-q)}+q^{n-1}t\left(R_n+(1-q)S_{n-1}\right), \\ \beta_n q^{2n-1} &= \frac{1}{q^{2\alpha}q^{2n}}\frac{1-q^n}{1-q}\frac{1-q^{n+\alpha}}{1-q}+\frac{1-q^n}{q^{\alpha+1}}t+\frac{q^n}{q^{\alpha+1}}tr_n+\frac{1}{q^{2\alpha+1}q^n}tS_{n-1}, \end{aligned}$$

where $S_{n-1}:=\sum_{j=0}^{n-1}R_j$.

Remark 1. The sum S_n is computed in (3.8) entirely in terms of R_n and r_n. Therefore, the lemma above gives expressions for the recurrence coefficients in terms of R_n and r_n only.

Theorem 1.4. *Let*

$$x_n=\frac{q^{n+\alpha}(1-q)}{R_n}, \qquad y_n=q^n(1-r_n) \qquad \text{and} \qquad T=\frac{(1-q)^2}{q}t\,.$$

Then, the x_n and y_n satisfy the following coupled difference equations

$$\begin{aligned}(x_n y_n - 1)(x_{n-1} y_n - 1) &= q^{2n+\alpha} T \frac{(y_n - 1)(y_n - 1/T)}{(q^n - y_n)}\,, \\ (x_n y_n - 1)(x_n y_{n+1} - 1) &= -q^{2n+\alpha+1} \frac{(x_n - 1)(x_n - T)}{x_n}\,. \end{aligned} \tag{1.10}$$

This paper is organized as follows. In Section 2, we evaluate the rational functions A_n and B_n in terms of certain auxiliary quantities. In Section 3, we give a proof of Theorem 1.2 and Lemma 1.3. In Section 4, we give a proof of Theorem 1.4.

2. The structural relation

In this section, we compute the functions $A_n(x)$ and $B_n(x)$ appearing in relation (1.9) in terms of certain auxiliary quantities. Our first task is to compute the function u for the weight function (1.1). We have

$$u(x,\alpha,t;q) = \frac{x^2 + \left(\frac{1-q^{-\alpha}}{1-q}\right) qx - q^{1-\alpha} t}{x^2(1 + (q^{-1} - 1)x)}\,.$$

From this, it follows that

$$\begin{aligned}\frac{u(qx,\alpha,t;q) - u(y,\alpha,t;q)}{qx - y} &= \left(\frac{t}{q^{\alpha+1} y}\right)\frac{1}{x^2} + \left(\frac{q - t(1-q)^2}{q^{\alpha+2}(1 + (q^{-1}-1)y)}\right)\frac{1}{x} \\ - \left(\frac{q - t(1-q)^2}{q^{\alpha+2}(1 + (q^{-1}-1)y)}\right)&\frac{1-q}{1+(1-q)x} - \left(\frac{u(y,\alpha,t;q)}{q}\right)\frac{1}{x}\,. \end{aligned} \tag{2.1}$$

With the following definitions of the auxiliary quantities

$$\begin{aligned} R_n^{(1)} &= \frac{1}{h_n}\int_0^\infty P_n(y) P_n(y/q) \frac{w(y,\alpha,t;q)}{y}\, dy\,, \\ R_n^{(2)} &= \frac{1}{h_n}\int_0^\infty P_n(y) P_n(y/q) \frac{w(y,\alpha,t;q)}{1 + y(q^{-1}-1)}\, dy\,, \\ r_n^{(1)} &= \frac{1}{h_{n-1}}\int_0^\infty P_n(y) P_{n-1}(y/q) \frac{w(y,\alpha,t;q)}{y}\, dy\,, \\ r_n^{(2)} &= \frac{1}{h_{n-1}}\int_0^\infty P_n(y) P_{n-1}(y/q) \frac{w(y,\alpha,t;q)}{1 + y(q^{-1}-1)}\, dy\,, \end{aligned}$$

we find that $A_n(x)$ and $B_n(x)$ appearing in (1.9) are rational functions of x and read

$$
\begin{aligned}
A_n(x) &= \frac{R_n^{(1)}}{x^2}\left(\frac{t}{q^{\alpha+1}}\right) + \frac{R_n^{(2)}}{x}\left(\frac{q-t(1-q)^2}{q^{\alpha+2}}\right) \\
&\quad -(1-q)\frac{R_n^{(2)}}{1+x(1-q)}\left(\frac{q-t(1-q)^2}{q^{\alpha+2}}\right), \qquad (2.2)\\
B_n(x) &= \frac{r_n^{(1)}}{x^2}\left(\frac{t}{q^{\alpha+1}}\right) + \frac{r_n^{(2)}}{x}\left(\frac{q-t(1-q)^2}{q^{\alpha+2}}\right) \\
&\quad -(1-q)\frac{r_n^{(2)}}{1+x(1-q)}\left(\frac{q-t(1-q)^2}{q^{\alpha+2}}\right) - \frac{1}{x}\left(\frac{1-q^n}{1-q}\right). \qquad (2.3)
\end{aligned}
$$

In the derivation above we made use of the formulae

$$
\frac{1}{h_n}\int_0^\infty u(y,\alpha,t;q)P_n(y)P_n(y/q)w(y,\alpha,t;q)\,dy = 0
$$

and

$$
\frac{1}{h_{n-1}}\int_0^\infty u(y,\alpha,t;q)P_n(y)P_{n-1}(y/q)w(y,\alpha,t;q)\,dy = q\frac{1-q^n}{1-q},
$$

which follow easily from the q-product rule and the integration by parts formula for the D_q operator [8]. We now take note of the fact that the $R_n^{(2)}$ can be expressed in terms of $R_n^{(1)}$ likewise for $r_n^{(2)}$ in terms of $r_n^{(1)}$. To see this, observe that

$$
\begin{aligned}
&\left(\left(\frac{t(1-q)}{q}\right)\frac{1}{y} + \left(\frac{q-t(1-q)^2}{q^2}\right)\frac{1}{1+(q^{-1}-1)y}\right)w(y,\alpha,t;q) \\
&= q^{\alpha-1}\,w(y/q,\alpha,t;q). \qquad (2.4)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\frac{t(1-q)}{q}R_n^{(1)} + \left(\frac{q-t(1-q)^2}{q^2}\right)R_n^{(2)} &= q^{n+\alpha}, \qquad (2.5)\\
\frac{t(1-q)}{q}r_n^{(1)} + \left(\frac{q-t(1-q)^2}{q^2}\right)r_n^{(2)} &= -(1-q)q^{n+\alpha-1}\sum_{j=0}^{n-1}\alpha_j. \qquad (2.6)
\end{aligned}
$$

Using (2.5) and (2.6), note that $R_n^{(2)}$ and $r_n^{(2)}$ can be eliminated from (2.2) and (2.3).

3. The recurrence coefficients

In this section, we derive expressions for the recurrence coefficients in terms of the quantities $R_n^{(1)}$ and $r_n^{(1)}$. Because we no longer require $R_n^{(2)}$ and $r_n^{(2)}$, we will drop the superscript and use the notation

$$R_n = R_n^{(1)}\,, \qquad r_n = r_n^{(1)} \qquad \text{and} \qquad S_n = \sum_{j=0}^{n} R_n\,.$$

We begin with the derivation of (qS_2').

Proof of Theorem 1.2. First, we write (qS_2) in the form

$$\beta_{n+1}A_{n+1}(x)-\beta_n A_{n-1}(x) = 1+(x-\alpha_n)(B_{n+1}(x)-B_n(x))+(1-q)xB_n(x)\,.$$

If we multiply the above equations by $A_n(x)$ and use (qS_1) to substitute for $(x-\alpha_n)A_n(x)$, we obtain

$$\begin{aligned}&\beta_{n+1}A_{n+1}(x)A_n(x) - \beta_n A_n(x)A_{n-1}(x)\\ &= A_n(x) + (B_{n+1}^2(x) + u(qx)B_{n+1}(x)) - (B_n^2(x) + u(qx)B_n(x))\\ &+x(1-q)\left(B_{n+1}(x)\sum_{j=0}^{n} A_j(x) - B_n(x)\sum_{j=0}^{n-1} A_j(x)\right)\,.\end{aligned}$$

Observe that, up to $A_n(x)$ on the right-hand side, the above is a first order difference equation in n, hence, summing over n, we obtain the q-analog of (S_2')

$$\beta_n A_n(x)A_{n-1}(x) = B_n^2(x) + u(qx)B_n(x) + (1+(1-q)xB_n(x))\sum_{j=0}^{n-1} A_j(x)\,.$$

□

To proceed further, we obtain, equating the coefficients of x^{-2} in (qS_1),

$$r_{n+1} + r_n = -\alpha_n R_n + 1\,. \tag{3.1}$$

Equating the coefficients of x^{-1} in (qS_1), we obtain the equation

$$\begin{aligned}&-(1-q)\left(q^n\sum_{j=0}^{n}\alpha_j + q^{n-1}\sum_{j=0}^{n-1}\alpha_j - \frac{t}{q^{\alpha+1}}\,(r_{n+1}+r_n)\right) - \frac{1-q^{n+1}}{1-q} - \frac{1-q^n}{1-q}\\ &= -\alpha_n q^n - \frac{1-q^{-\alpha}}{1-q} + \left(\frac{t}{q^{\alpha+1}}\right)(R_n + \alpha_n(1-q)R_n - (1-q)S_n - (1-q))\,.\end{aligned}$$

The r_n terms can be eliminated by using (3.1) and after some simplification, we arrive at

$$\begin{aligned} q^{n+1}\sum_{j=0}^{n}\alpha_j - q^{n-1}\sum_{j=0}^{n-1}\alpha_j &= \frac{1-q^{n+1}}{1-q} + \frac{1-q^n}{1-q} - \frac{1-q^{-\alpha}}{1-q} \\ &\quad + \left(\frac{t}{q^{\alpha+1}}\right)(qS_n - S_{n-1})\,. \end{aligned}$$

We multiply both sides of this equation by the integrating factor q^{n-1} and sum to obtain

$$\begin{aligned} q^{2n}\sum_{j=0}^{n}\alpha_j &= \frac{1}{q}\left(\frac{1-q^{n+1}}{1-q}\right)^2 - \frac{1}{q}\left(\frac{1-q^{n+1}}{1-q}\right)\left(\frac{1-q^{-\alpha}}{1-q}\right) \\ &\quad + \left(\frac{t}{q^{\alpha+1}}\right) q^n S_n\,. \end{aligned} \tag{3.2}$$

From (3.2), we see that

$$\begin{aligned} q^{2n}\alpha_n &= 2\left(\frac{1-q^n}{1-q}\right) + \frac{1}{q} - \left(\frac{1}{q} + 1 - q^n\right)\left(\frac{1-q^{-\alpha}}{1-q}\right) + \left(\frac{tq^n}{q^{\alpha+1}}\right) \\ &\quad \times \left(R_n + (1-q)S_{n-1}\right)\,. \end{aligned} \tag{3.3}$$

Equating the coefficients of $(1+x(1-q))^{-1}$ in (qS_1) yields the same result. We go through the same process for (qS_2') and expect to find three more equations. Equating the coefficients of x^{-4} in (qS_2') gives

$$\beta_n R_n R_{n-1} = r_n^2 - r_n\,. \tag{3.4}$$

To proceed further, we equate the coefficients of x^{-2} and x^{-3} in (qS_2'), which are long formulas. First equating the coefficients of x^{-2} in (qS_2') produces

$$\begin{aligned} &\beta_n\left(q^{2n-1} - 2\frac{tq^{n-2}(1-q)}{q^{\alpha}}\left(R_n + qR_{n-1}\right)\right) \\ &= \left(\frac{1}{q^{2\alpha}q^{2n}}\frac{1-q^n}{1-q}\frac{1-q^{n+\alpha}}{1-q} + \left(\frac{t}{q^{2\alpha+1}}\right)\left(q^{\alpha} - \frac{2}{q^n}\right)(1-q^n)\right) \\ &+ \frac{t}{q^{2\alpha+1}q^n}(2-q^n)(2-q^{n+\alpha})r_n \\ &+ \left(\frac{1}{q^{2\alpha}q^n} - 2\frac{t(1-q)^2}{q^{2\alpha+1}}\right)\left(\frac{t}{q}\right)S_{n-1} + 2\frac{t^2(1-q)^2}{q^{2\alpha+2}}r_nS_{n-1}\,. \end{aligned} \tag{3.5}$$

Now, equating the coefficients of x^{-3} in (qS_2') gives

$$\beta_n q^{n-1}\left(R_n + qR_{n-1}\right) = \frac{1}{q^\alpha q^n}\left(\frac{1-q^n}{1-q} - \left(\frac{1-q^n}{1-q} + \frac{1-q^{n+\alpha}}{1-q}\right)r_n\right) + \frac{t(1-q)}{q^{\alpha+1}}(1-r_n)S_{n-1}\,. \tag{3.6}$$

We now combine (3.5) with (3.6) to obtain

$$\beta_n q^{2n-1} = \frac{1}{q^{2\alpha}q^{2n}}\frac{1-q^n}{1-q}\frac{1-q^{n+\alpha}}{1-q} + \frac{t(1-q^n)}{q^{\alpha+1}} + \frac{tq^n}{q^{\alpha+1}}r_n + \frac{t}{q^{2\alpha+1}q^n}S_{n-1}\,. \tag{3.7}$$

Note that both equations (3.7) and (3.6) give an expression for β_n in terms of the auxiliary quantities. Both of these equations are essential because they allow us to eliminate β_n and obtain an expression for the sum S_{n-1} in terms of R_n and r_n only. The sum S_{n-1} is given by the following lemma.

Lemma 3.1. *If $S_n = \sum_{j=0}^n R_j$, then*

$$\begin{aligned} S_{n-1}\left(\frac{t}{q}\right)\left(\frac{1}{q^{2\alpha}q^n} - \frac{q^n(1-q)(1-r_n)}{q^\alpha R_n}\right) &= -\frac{1}{q^{2n+2\alpha}}\left(\frac{1-q^n}{1-q}\right)\left(\frac{1-q^{n+\alpha}}{1-q}\right) \\ &- \frac{tq^n}{q^{\alpha+1}}r_n + \frac{1}{q^\alpha R_n}\left(\frac{1-q^n}{1-q} - \left(\frac{1-q^n}{1-q} + \frac{1-q^{n+\alpha}}{1-q}\right)r_n\right) \\ &- q^{2n}\frac{r_n^2 - r_n}{R_n^2} - \frac{t(1-q^n)}{q^{\alpha+1}}\,. \end{aligned} \tag{3.8}$$

Proof. First, multiply (3.6) by R_n and use (3.4) to eliminate R_{n-1}. Then substitute for β_n from (3.7) into (3.6). This gives an expression for the sum S_{n-1} in terms of r_n and R_n only. □

Note that this equation effectively eliminates the sum S_{n-1} from equations (3.3) and (3.7) and, consequently, we see that α_n and β_n are entirely determined by r_n and R_n.

4. Non-linear difference equations

In this section, we derive the coupled non-linear difference equations given in Theorem 1.4.

Proof of Theorem 1.4. Eliminating α_n from (3.3) and (3.1), we obtain

$$\begin{aligned} &q^{2n+\alpha}(1-r_{n+1}-r_n) \\ &= \left(\frac{(1-q^n)}{1-q} + \frac{1-q^{n+\alpha+1}}{q(1-q)} + tq^{n-1}\left(R_n + (1-q)S_{n-1}\right)\right)R_n\,. \end{aligned} \tag{4.1}$$

Eliminating β_n from (3.4) and (3.6), we receive

$$q^{2n+\alpha-1}\left(r_n^2-r_n\right)\left(R_n+qR_{n-1}\right)$$
$$=\left(\frac{1-q^n}{1-q}-\left(\frac{1-q^n}{1-q}+\frac{1-q^{n+\alpha}}{1-q}\right)r_n+tq^{n-1}(1-q)(1-r_n)S_{n-1}\right)R_nR_{n-1}\,. \tag{4.2}$$

We now replace S_{n-1} in (4.1) and (4.2) by the expression given in (3.8) and obtain respectively

$$q^{n-\alpha}\left((1-r_{n+1})+\frac{1}{q}(1-r_n)\right)R_n-q^{3n}(1-q)(1-r_{n+1})(1-r_n)$$
$$=\frac{t}{q^{2\alpha+1}}R_n^3+\left(\frac{1}{q^{2\alpha+n+1}}\frac{1-q^{2n+\alpha+1}}{1-q}-(1-q)t\frac{q^n}{q^{\alpha+1}}\right)R_n^2+q^{2n}R_n \tag{4.3}$$

and

$$q^{2n-1}(1-q)r_n(1-r_n)^2-\frac{R_n+qR_{n-1}}{q^{\alpha+1}}r_n(1-r_n)=R_nR_{n-1}$$
$$\times\left(\frac{1-q}{q^{\alpha+1}}t(1-r_n)(q^n(1-r_n)-1)+\frac{1-r_n}{q^{2n+2\alpha}}\left(\frac{1-q^{2n+\alpha}}{1-q}\right)-\frac{1}{q^{2n+2\alpha}}\frac{1-q^{n+\alpha}}{1-q}\right). \tag{4.4}$$

Note that (4.3) is a first order difference equation in r_n and a cubic in R_n, whilst (4.4) is the other way around. These equations admit the respective factorizations

$$\left(R_n-q^{n+\alpha}(1-q)(q^{n+1}-q^{n+1}r_{n+1})\right)\left(R_n-q^{n+\alpha}(1-q)(q^n-q^nr_n)\right)$$
$$=-(1-q)q^ntR_n\left(R_n-q^{n+\alpha}(1-q)\right)\left(R_n-\frac{q^{n+\alpha+1}}{t(1-q)}\right) \tag{4.5}$$

and

$$q^nr_n\left((q^n-q^nr_n)-\frac{R_n}{q^{n+\alpha}(1-q)}\right)\left((q^n-q^nr_n)-\frac{R_{n-1}}{q^{n+\alpha-1}(1-q)}\right)$$
$$=\frac{tR_nR_{n-1}}{q^\alpha}\left((q^n-q^nr_n)-\frac{q}{t(1-q)^2}\right)((q^n-q^nr_n)-1)\,. \tag{4.6}$$

Under the substitutions,

$$x_n=\frac{q^{n+\alpha}(1-q)}{R_n}\,,\qquad y_n=q^n(1-r_n)\qquad\text{and}\qquad T=\frac{(1-q)^2}{q}t$$

(4.5) and (4.6) are the coupled equations in Theorem 1.4. □

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