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MICROWAVE NETWORKS AS A TOOL FOR INVESTIGATING ISOSCATTERING PHENOMENA*

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We show that microwave networks simulating quantum graphs are very useful in an experimental investigation of isoscattering phenomena in a broad frequency range from 0.01 to 5 GHz.

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1. Introduction

In 1966, Mark Kac [1] asked "Can one hear the shape of a drum?" The question of a great practical significance turned out to be also a great theoretical challenge and remained without any conclusive answer till 1992, when Gordon *et al.* [2, 3] show theoretically that it is possible to construct pairs different in shape but isospectral domains. Two years later this result was confirmed experimentally by Sridhar and Kudrolli [4] and later on by Dhar *et al.* [5], who obtained the same spectra for microwave cavities of different shapes.

It is natural to generalize Kac question to other physical systems. In 2001, almost ten years after Gordon *et al.* [2, 3], Gutkin and Smilansky [6] proved that spectra uniquely identify quantum graphs, which consist of vertices connected by one-dimensional bonds, only if they have bonds of incommensurate lengths. The graphs with commensurate bonds may have the same spectra. A method of construction of isospectral graphs was presented by Band *et al.* [7] and Parzanchevski and Band [8] in 2009. A procedure is similar to that in the case of two-dimensional systems and consists of cutting one system into subsystems and rearrangement, transplantation them into the second one.

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However, in 2005, Okada *et al.* [9] showed, using numerical simulation, that in scattering experiments, it is possible to distinguish shapes of the domains from their spectra. Thus, isospectrality of the systems does not prejudge that they are also isoscattering. In the case of the graphs, it was shown by Band *et al.* [10, 11] that any pair of isospectral quantum graphs obtained by the procedure described in [7, 8] is isoscattering provided that infinite leads are attached to the graphs in a way preserving the symmetry of their isospectral construction.

The first experimental confirmation of existing of isoscattering graphs was reported in [12]. The experiment was performed with a pair of different microwave networks simulating quantum graphs.

2. Microwave networks simulating quantum graphs

Quantum graphs are idealizations of physical systems whose elements have one dimension much bigger than the other two. A theoretical description of their properties and applications in modeling physical problems was given in [13]. There are several methods of experimental realization of systems which can simulate quantum graphs [14, 15]. In the paper of Hul *et al.* [16], it was shown that microwave networks can be used to simulate quantum graphs with and without times reversal symmetry. Microwave networks enabled for comprehensive investigation of a broad variety of quantum graphs features [17–28].

A microwave network consists of B bonds, which are coaxial cables, connecting n vertices. It may be described by a $n \times n$ connectivity matrix, whose elements C_{ij} equal 1 if the vertices i and j are connected and 0 otherwise. A valency v_i of the vertex i means that the vertex i is connected to other vertices by v_i bonds.

A coaxial cable is made up of an inner conductor of radius r_1 surrounded by a concentric conductor of inner radius r_2. A homogeneous material with dielectric constant ε is used to fill the space between conductors. Only a fundamental TEM mode, called Lecher wave, can propagate in the cable below the onset of the next TE_{11} mode [29]. The propagation of Lecher wave in the bond joining vertices i and j can be described by the continuity equation for a charge and current [16, 30]. In the case of an ideal lossless cable, it leads to the telegraph equation

$$\frac{d^2}{dx^2}U_{ij}(x) + \frac{\omega^2\varepsilon}{c^2}U_{ij}(x) = 0\,, \tag{1}$$

where $U_{ij}(x,t)$ is the potential difference between conductors, $\omega = 2\pi\nu$, ν is frequency, c is speed of light in a vacuum and ε is the dielectric constant.

On the other hand, the one-dimensional Schrödinger equation (with $\hbar = 2m = 1$) describing a quantum graph with time reversal symmetry is of the form [31]

$$\frac{d^2}{dx^2}\Psi_{ij}(x) + k^2\Psi_{ij}(x) = 0\,. \tag{2}$$

Equations (1) and (2) are formally equivalent due to the correspondence of $\Psi_{ij}(x) \Leftrightarrow U_{ij}(x)$ and $k^2 \Leftrightarrow \frac{\omega^2\varepsilon}{c^2}$.

The above situation is similar to the analogy between microwave cavities and quantum billiards which is based upon the equivalency of the Helmholtz equation and the Schrödinger one [32–35].

3. Experiment

An experimental setup consisting of a microwave network connected to the vector network analyzer (VNA) Agilent E8364B is shown in Fig. 1. Two isoscattering graphs obtained by attaching two infinite leads L_1^∞ and L_2^∞ to

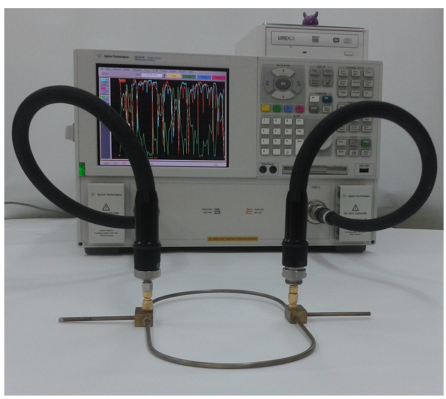

Fig. 1. Experimental setup. The Vector Network Analyzer (VNA) connected to the network of a shape "O" by means of the two microwave coaxial cables.

isospectral ones are presented in Fig. 2 (a) and Fig. 2 (b). The graph of a shape "H" (Fig. 2 (a)) consists of $n = 6$ vertices connected by $B = 5$ bonds. Four of vertices, with numbers 1, 2, 3, 5, satisfy Neumann boundary conditions, and the two others 4, 6, the Dirichlet one. The Neumann boundary condition imposes a continuity of waves propagating in bonds meeting in vertex i and vanishing of a sum of their derivatives calculated in this vertex. The Dirichlet boundary condition requires vanishing of the waves at the vertex. In the case of the graph of a shape "O", there are $n = 4$ vertices, only one, number 4, with the Dirichlet boundary condition, and vertices 1, 2, 3 with the Neumann boundary condition, connected by $B = 4$ bonds. The valency of vertices 1 and 2 including leads of both graphs is $v_{1,2} = 4$, while for the other ones $v_i = 1$. In Figs. 2 (c) and (d), we show the microwave networks simulating quantum graphs of shapes "H" and "O", respectively.

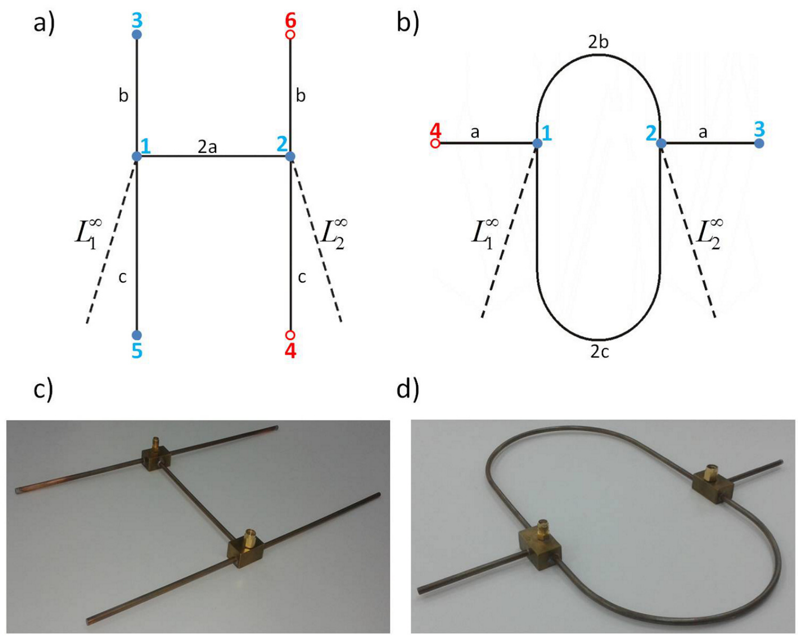

Fig. 2. A pair of isoscattering quantum graphs and the pictures of two isoscattering microwave networks are shown in the panels (a)–(b) and (c)–(d), respectively. Using the two isospectral graphs, (a) with $n = 6$ vertices (shape "H") and (b) with $n = 4$ vertices (shape "O"), isoscattering quantum graphs are formed by attaching the two infinite leads L_1^∞ and L_2^∞ (dashed lines). The vertices with Neumann boundary conditions are denoted by full circles, while the vertices with Dirichlet boundary conditions by the open ones. The two isoscattering microwave networks with $n = 6$ (shape "H") and $n = 4$ (shape "O") vertices which simulate quantum graphs (a) and (b), respectively, are shown in the panels (c)–(d).

The optical length of bonds, a physical length times $\sqrt{\varepsilon}$ ($\varepsilon \simeq 2.08$), of the microwave networks are the following:

$$\begin{aligned} a &= 0.0985 \pm 0.0005 \text{ m}\,, \\ b &= 0.1847 \pm 0.0005 \text{ m}\,, \\ c &= 0.2420 \pm 0.0005 \text{ m}\,, \\ 2a &= 0.1970 \pm 0.0005 \text{ m}\,, \\ 2b &= 0.3694 \pm 0.0005 \text{ m}\,, \\ 2c &= 0.4840 \pm 0.0005 \text{ m}\,. \end{aligned}$$

The main source of the uncertainties of the bond lengths was a process of preparation of the vertices with the Dirichlet and $v_{1,2} = 4$ vertices with Neumann boundary conditions. The Dirichlet vertices were prepared by shorting the internal and external conductors of the coaxial cable using brass caps. To get the Neumann boundary condition, it was necessary to solder internal conductors of the cables meeting at the vertex.

The vector network analyzer allowed us to preform two-port measurements to determine a two-dimensional scattering matrix $S(\nu)$ describing a relation between amplitudes of incoming to and outgoing from the networks electromagnetic waves

$$S(\nu) = \begin{pmatrix} S_{1,1}(\nu) & S_{1,2}(\nu) \\ S_{2,1}(\nu) & S_{2,2}(\nu) \end{pmatrix} . \tag{3}$$

In this paper, we present the results of the new measurements in the frequency range of 4–5 GHz. The results for the frequency range of 0.01–1.7 GHz and 0.01–3 GHz were shown in [12, 36]. Figure 3 (a) shows the comparison between the determinants of the $S^{(I)}(\nu)$ of the "H" network (solid line) and the $S^{(II)}(\nu)$ of the "O" network (open circles) in the frequency range of 4–5 GHz. In Fig. 3 (b), we present cumulative phases of the determinants of the scattering matrices $S^{(I)}(\nu)$ (solid line) and $S^{(II)}(\nu)$ (open circles). In panels (c) and (d) of Fig. 3, we remind the results for the same networks obtained in the lower frequency range of 0.5–1.5 GHz.

The inspection of Figs. 3 (a) and (b) shows that the networks remain isoscattering also for the higher frequency range of 4–5 GHz in spite of the limitation connected with the accuracy of their preparation. However, comparing these results to the ones in the frequency range of 0.5–1.5 GHz enables one to see bigger difference between spectra of the networks. The discrepancies increased for both, positions and amplitudes. It should be noticed that for both networks, a number of the observed resonances for lower and higher frequencies is in the agreement with the Weyl formula [31]

$$N(k) = \frac{2L}{2\pi}k + \frac{1}{2} , \tag{4}$$

where L is a total optical length of a graph and k is a wave vector.

The maximal relative shifts of the positions of the resonances connected with the networks length uncertainties are 0.01 and 0.04 for the frequency range of 0.5–1.5 GHz and 4–5 GHz, respectively. Figure 4 shows that the experimental shifts of the resonances are smaller than the ones expected due to the estimated accuracy of the networks preparation.

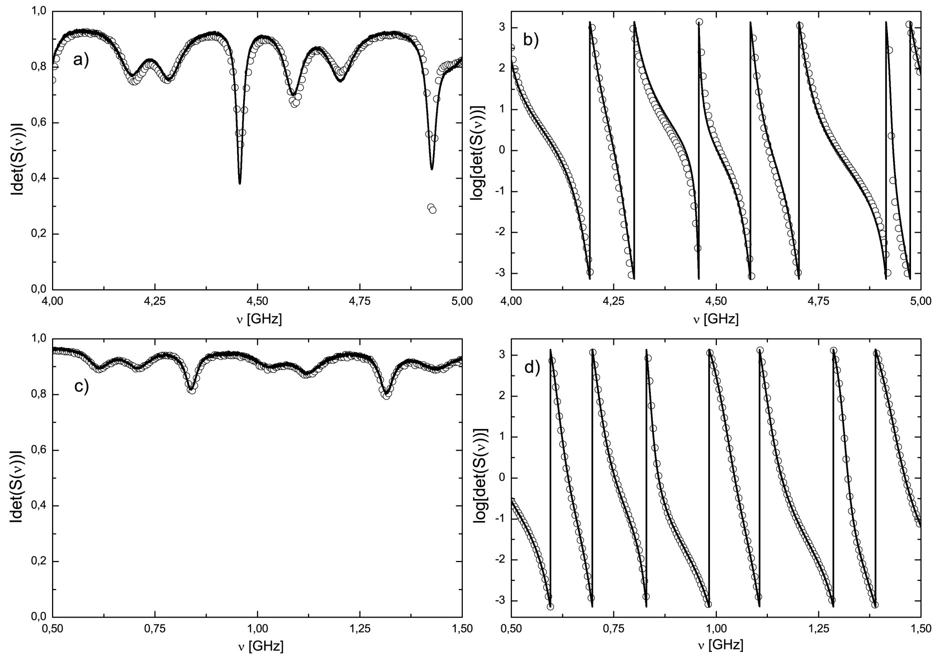

Fig. 3. The amplitude of the determinant of the scattering matrix obtained for the microwave networks "H" with $n=6$ (solid line) and "O" with $n=4$ (open circles) vertices in the frequency range of 4–5 GHz and 0.5–1.5 GHz in panels (a) and (c), respectively. The cumulative phase of the determinant of the scattering matrix obtained for the microwave networks "H" (solid line) and "O" (open circles) in the frequency range of 4–5 GHz and 0.5–1.5 GHz in the panels (b) and (d), respectively.

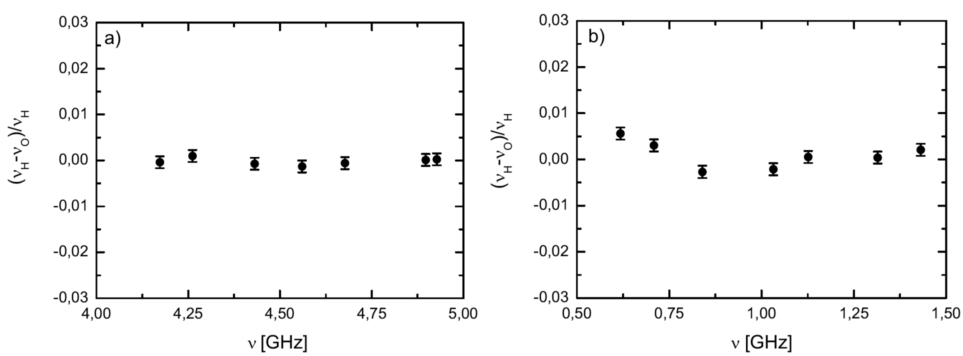

Fig. 4. The relative differences between the positions of the resonances of microwave networks "H" and "O" for frequency ranges of 4–5 GHz and 0.5–1.5 GHz in panels (a) and (b), respectively.

4. Conclusions

We report new results of the investigations of the isoscattering phenomena in the microwave networks simulating quantum graphs. The measurements were performed in the high frequency range of 4–5 GHz. We show that even for such a high frequency range, in spite of the limitations connected with the accuracy of the preparation of microwave networks, it was possible to obtain conclusive results that the networks of different topology may have the same scattering properties. Thus, we show that microwave networks can be used as a powerful tool to investigate isospectral and isoscattering problems.

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