

ASYMPTOTIC FREEDOM OF GLUONS IN THE FOCK SPACE*

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Asymptotic freedom of gluons is described in terms of a family of scale-dependent renormalized Hamiltonian operators acting in the Fock space. The Hamiltonians are obtained by applying the renormalization group procedure for effective particles to the quantum Yang–Mills theory.

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1. Introduction

Asymptotic freedom (AF) as a property of interactions of gluons [1–4] is described below in terms of features of the Minkowski space-time Hamiltonian operators acting in the Fock space, in the front form of relativistic dynamics [5]. The Fock space representation of AF is obtained through application of the renormalization group procedure for effective particles (RGPEP) to QCD [6–8]. For the purpose of simplicity, quarks are removed from the presentation. They are not needed for understanding how the RGPEP is carried out and how it exhibits the asymptotic freedom of gluons. As a new approach to solving quantum field theory, the most recent version of the RGPEP employed here is considerably simpler than the one used in the front form Hamiltonian formulation of QCD [9], which used the similarity renormalization group (SRG) procedure [10].

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The issue we face is that, one way or the other, all approaches to QCD assume that there exists a corresponding Hamiltonian operator, \hat{H}_{QCD} , but when it comes to precise definitions, the problem of what exactly is meant by the operator \hat{H}_{QCD} in Minkowskian quantum mechanical sense is hardly solved. We tackle the issue in the context of pure Yang–Mills theory.

Canonical Hamiltonians for quantum YM fields are introduced in Sec. 2 and examples of singularities they generate in the dynamics of canonical field quanta are discussed using the concept of Fock space in Sec. 3. Regularization of these singularities is defined in Sec. 4 and the concept of effective particles of a finite size s is proposed in Sec. 6, for the purposes of renormalization and specification of a computable effective Hamiltonian eigenvalue problem, using the RGPEP. The perturbatively calculated effective Hamiltonians are shown in Sec. 7 to depend on the gluon size parameter s in an asymptotically free way. Section 8 concludes the article.

2. Canonical Hamiltonian

The classical Lagrangian density for the Yang–Mills theory,

$$\mathcal{L} = -\frac{1}{2} \text{tr} F^{\mu\nu} F_{\mu\nu}, \quad (1)$$

involves the vector field A^μ through the field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]. \quad (2)$$

The Nether theorem implies a conserved energy-momentum density tensor

$$\mathcal{T}^{\mu\nu} = -F^{a\mu\alpha} \partial^\nu A_\alpha^a + g^{\mu\nu} F^{a\alpha\beta} F_{\alpha\beta}^a / 4, \quad (3)$$

where the index a refers to color and is summed over. The associated expression for the four-momentum of a field configuration is

$$P^\nu = \int_{\Sigma} d\sigma_\mu \mathcal{T}^{\mu\nu}. \quad (4)$$

One has to make a choice of the space-time sub-manifold Σ over which the three-dimensional integration is carried out.

The most popular choice of Σ is the space of some inertial observer at her instant of time $t = 0$. The associated form of dynamics was called by Dirac [5] the instant form (IF). In the IF, the canonical Hamiltonian is obtained by integrating the energy-momentum tensor density component \mathcal{T}^{00} over the instant hyperplane.

The choice we make here for Σ is the hyperplane in Minkowskian space-time that is swept by the front of a plane wave of light moving against the z -axis. This form of dynamics was called by Dirac [5] the front form (FF).

In the FF, the variable $x^+ = t + z$ plays the role of evolution parameter, analogous to the role of time t in the IF, and the variables $x^- = t - z$ and $x^\perp = (x^1, x^2)$ are used to label points on the front hyperplane. This hyperplane plays in the FF the role analogous to the one played by space in the IF. The canonical Hamiltonian P^- is obtained by integrating the canonical energy-momentum tensor density component \mathcal{T}^{+-} over the front with $x^+ = 0$

$$P^- = \frac{1}{2} \int_{\text{front}} dx^- d^2x^\perp \mathcal{T}^{+-}. \quad (5)$$

The gluon fields are described in terms of components $A^\pm = A^0 \pm A^3$ and $A^\perp = (A^1, A^2)$, with the choice of gauge $A^+ = 0$. In this gauge, the FF constraint equation analogous to the IF Gauss law is solved explicitly for A^- and the Yang–Mills theory Hamiltonian density,

$$\mathcal{H}_{\text{YM}} = \mathcal{T}_{\text{YM}}^{+-}, \quad (6)$$

depends only on the fields A^\perp .

Note that the subgroup of the Poincaré group that preserves the FF hyperplane $x^+ = 0$ has *seven* generators, while the IF space is preserved by the subgroup that has only *six* generators. The latter are the familiar three momenta that generate translations in space and three angular momenta that generate rotations in space, all six generators being independent of interactions.

In the FF, the generators that preserve the front are expected to not depend on interactions. Therefore, the full FF Hamiltonian is invariant with respect to seven kinematical Poincaré transformations, instead of only six. Among these transformations, there is the boost along the z -axis. Therefore, the eigenstates of the FF Hamiltonian are described by boost-invariant wave functions in the associated Fock space. Thus, the FF approach to QCD is a natural framework to use for the purpose of connecting the spectroscopic classification and parton model of hadrons, gluons being their constituents.

When one uses the FF, physical systems are considered evolving from one value of x^+ to another. In operational terms, one describes the state of a system by collecting information about its features at a space-time point x when the point is reached by the front of a plane wave of light moving against the z -axis, with $z = -ct$, so that all the points in space-time that are inspected for collection of information satisfy the condition $x^+ = 0$. Such a front passes through the point $x = 0$ at time $t = 0$. At some later time, one collects information at space-time points reached by the front of another

plane wave of light moving against the z -axis so that $z = -c(t - t_0)$, which means that one collects data on the front defined by the condition $x^+ = ct_0$. The generator of evolution in x^+ from zero to ct_0 is P^- .

The canonical FF Hamiltonian density for gluons contains three types of terms,

$$\mathcal{H}_{\text{YM}} = \mathcal{H}_{A^2} + \mathcal{H}_{A^3} + \mathcal{H}_{A^4} . \quad (7)$$

The power of A in the subscripts indicates how many fields appear in the product that constitutes the term. The discussion of asymptotic freedom mainly concerns the term with three fields,

$$\mathcal{H}_{A^3} = g \, i \partial_\alpha A_\beta^a \left[A^\alpha, A^\beta \right]^a , \quad (8)$$

which is used for showing that in a properly defined effective quantum theory, the coupling constant that corresponds to the parameter g decreases when the momentum scale of the effective theory increases. The concepts of an effective theory and its scale will be defined precisely in what follows.

The quantum Hamiltonian \hat{H}_{YM} is obtained from Eqs. (5), (6) and (7) by replacing the classical field A^μ by the quantum field operator

$$\hat{A}^\mu = \sum_{\sigma c} \int_k \left[t^c \varepsilon_{k\sigma}^\mu a_{k\sigma c} e^{-ikx} + t^c \varepsilon_{k\sigma}^{\mu*} a_{k\sigma c}^\dagger e^{ikx} \right]_{\text{on } \Sigma} , \quad (9)$$

in which the momentum-dependent Fourier coefficients $a_{k\sigma c}$ and $a_{k\sigma c}^\dagger$ are the annihilation and creation operators that satisfy commutation relations

$$\left[a_{k\sigma c}, a_{k'\sigma'c'}^\dagger \right] = 2k^+ (2\pi)^3 \delta^3(k - k') \delta^{\sigma\sigma'} \delta^{cc'} \quad (10)$$

with all other commutators among them equal zero. Symbols t denote color matrices and ε polarization vectors with $\varepsilon_{k\sigma}^+ = 0$, $\varepsilon_{k\sigma}^- = 2k^\perp \varepsilon_\sigma^\perp / k^+$ and σ labeling the available two transverse polarizations. Finally, normal ordering, which amounts to putting all annihilation operators to the right of all creation operators, yields the canonical quantum operator for the YM theory

$$\hat{H}_{\text{YM}} = \frac{1}{2} \int dx^- d^2 x^\perp : \mathcal{H}_{\text{YM}}(\hat{A}) : . \quad (11)$$

However, this operator is too singular for using it in calculations of physical quantities.

3. Examples of canonical singularities

The singular nature of canonical Hamiltonians of the quantum YM theory can be exhibited using the term

$$\hat{H}_{A^3} = \int_{\Sigma} g : i\partial_{\alpha}\hat{A}_{\beta}^a [\hat{A}^{\alpha}, \hat{A}^{\beta}]^a : , \quad (12)$$

which is also the key term for discussing AF. First, we exhibit the divergences this term produces in the ground state problem of the theory.

3.1. IF vacuum problem

Since the quantum field is a linear combination of gluon annihilation and creation operators, so that symbolically $\hat{A} \sim a + a^{\dagger}$, the term containing a product of three fields involves four types of terms,

$$: \hat{A}^3 : \sim a^{\dagger 3} + a^{\dagger 2}a + a^{\dagger}a^2 + a^3 . \quad (13)$$

While all annihilation operators, by definition, annihilate the bare ground state, $|0\rangle$, which is a vacuum in a theory of the gluon quanta without any interaction,

$$a_k|0\rangle = 0 , \quad (14)$$

the terms with only creation operators produce three-gluon states out of the vacuum

$$a_{k_1}^{\dagger} a_{k_2}^{\dagger} a_{k_3}^{\dagger} |0\rangle . \quad (15)$$

In these states, the only limitation on the IF three momenta \vec{k}_1, \vec{k}_2 and \vec{k}_3 is that they sum to zero, as dictated by the translation invariance in space. The magnitude of an individual gluon momentum is not limited. This means that the three gluons may have arbitrarily large energy. Production of particles with unlimited energy is a generic feature of the canonical theories with local interactions.

Thus, the IF evolution operator acting on the vacuum,

$$e^{-i\hat{H}t/\hbar}|0\rangle , \quad (16)$$

creates states of infinite energy infinitely many times, each power of \hat{H} in the exponential series creating additional gluons of unlimited energies. Actually, the Hamiltonian can turn any basis state in the IF Fock space into an infinite-energy state by adding gluons.

The second-order perturbation theory produces an infinite result for the correction to vacuum energy

$$\Delta E_{|0\rangle}^{(2)} = \sum_{|3g\rangle} \frac{|\langle 3g | \hat{H}_{A^3} | 0 \rangle|^2}{-E_{3g}} = -\infty. \quad (17)$$

Analogous corrections are produced for all basis states in the Fock space of virtual gluons, *i.e.*, states built from $|0\rangle$ by applying a product of creation operators. In the FF, the situation is different.

3.2. FF vacuum problem

The Fourier expansion of the gluon field on the front, Eq. (9), is assumed only to extend over the plus-momentum components,

$$k^+ = E_k + k^3, \quad (18)$$

that are positive. The condition of positivity is introduced because for particles of mass $\mu > 0$, the energy $E_k = \sqrt{\vec{k}^2 + \mu^2} > k^3$, no matter how small is the mass or how large is the momentum, as long as they are finite. Gluons of canonical theory are considered massless and *a priori* may have $k^+ = 0$ when moving exactly against the z -axis, so that the condition of positivity of k^+ is equivalent for them to cutting out the domain of infinitesimal k^+ from the theory. In fact, the process of regularizing the theory removes this domain from the derivation of AF (see below).

Invariance under translations within the front hyperplane implies that the total plus-momentum of gluons created by a term in the FF Hamiltonian must be equal to the momenta of gluons annihilated by the same term. If a term were to contain only creation operators, the sum of plus-components of created gluons would have to be zero, which is impossible if the gluons can only have positive plus-momenta. Hence, the term \hat{H}_{A^3} in the FF Hamiltonian obeys the rule

$$: \hat{A}^3 : \sim a^{\dagger 2} a + a^{\dagger} a^2, \quad (19)$$

instead of Eq. (13). There are no terms allowed in the entire FF Hamiltonian for quantum YM theory that only create or only annihilate gluons. As a result, the FF free-gluon vacuum state $|0\rangle$ is an eigenstate of the FF Hamiltonian, with the eigenvalue zero. The explosive free-gluon vacuum behavior that in the IF prevents one from understanding the ground-state properties of quantum YM theory is removed in the regulated Hamiltonian for the same theory in the FF.

However, the states that contain gluons are still subject to diverging interactions. For example, the FF three-gluon term of Eq. (12) can change one gluon to a pair of gluons or *vice versa*, with an unlimited invariant mass of the pair. There also appear singularities due to small plus-momenta. In general, the dynamics of virtual quanta is singular and requires a method for resolving.

3.3. Divergences in dynamics of gluons

One may try to follow the physicists' habit in perturbative calculations of ignoring conceptual problems with the vacuum [11] and focusing only on interactions that occur in the states that already contain gluon-field quanta. In this case, using the FF of Hamiltonian dynamics, one is left with the task of calculating evolution of states that carry positive plus-momentum and always differ from the vacuum $|0\rangle$. Unfortunately, restriction to such states does not eliminate infinities that are inherent to the canonical Hamiltonians of local gauge-theories.

For example, due to the term in Eq. (12), a perturbative correction to the energy of one gluon is infinite

$$\Delta E_{|g\rangle}^{(2)} = \sum_{|2g\rangle} \frac{|\langle 2g | \hat{H}_{A^3} | g \rangle|^2}{E_g - E_{2g}} = -\infty. \quad (20)$$

The divergence has two origins. First of all, there are infinitely many different energy scales in the sum over gluon pairs and even if each scale contributed a little, together they would contribute infinity. In addition, both amplitudes of change, from one to two gluons and *vice versa*, increase with the increase of the relative transverse momentum of gluons or fraction of plus-momentum carried by one gluon in the pair. In other words, the larger the energy scale of the intermediate pair or more disparate momenta of gluons in it, the greater the amplitude for such pair to contribute. Local gauge theories lead to many interaction terms with the same property: the greater the change of scale, the stronger the interaction.

For systems more complex than one gluon and in orders higher than second, diverging terms pile up and one needs to make up one's mind about how to define the theory. So far, the lack of a clear *en bloc* solution to the divergence issue is the origin of numerous problems in quantum field theory, with a key example being the relativistic bound-state problem. The problem persists despite that an advanced perturbative Hamiltonian calculus for some quantities can be developed using ingenious recipes for handling divergences [12–17], and some intuitive pictures are developed for AF [18, 19].

3.4. Canonical eigenvalue problem

The difficulties in setting up and solving the bound-state problem are well-illustrated by the case of gluonium, which can be also called a glueball (G). The eigenvalue equation for a gluonium state of total momentum components P^+ and P^\perp , has the form

$$\hat{H}_{\text{YM}}|GP\rangle = \frac{M_G^2 + P^{\perp 2}}{P^+} |GP\rangle, \quad (21)$$

where $|GP\rangle$ denotes the bound state and M_G denotes its mass eigenvalue, according to the formula $P^2 = P^+P^- - P^{\perp 2} = M_G^2$. Since the Hamiltonian changes the number of quanta, its eigenstate is a superposition of states with different numbers of virtual gluons

$$|GP\rangle = |ggP\rangle + |gggP\rangle + |ggggP\rangle + \dots \quad (22)$$

The problem is that there are infinitely many Fock components in the expansion and in each of them all gluons are distributed over an infinitely large range of momentum.

Conservation of the plus-momentum implies that gluons must share P^+ in positive bits and if each and every one of these bits is greater than certain ϵ^+ , the number of gluons in the eigenstate cannot exceed $N(P, \epsilon^+) = P^+/\epsilon^+$. But the ratio $N(P, \epsilon^+)$ is *a priori* not limited and there can be unlimited numbers of gluons with arbitrarily small plus-momenta in a gluonium with an arbitrarily large P^+ . The transverse momenta of gluons are only subject to one condition of summing to P^\perp . Hence, for carrying out computations, the eigenvalue problem must be somehow made finite, which we shall try to accomplish by starting from regulating the Hamiltonian.

4. Regularization

The regularization adopted here for calculation of AF Hamiltonians is explained below using the example of term \hat{H}_{A^3} , with the result denoted by \hat{H}_{A^3R} . It will be made clear that all terms can be regulated using the same pattern.

Evaluation of Eq. (12) in terms of the FF creation and annihilation operators yields

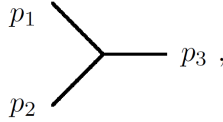
$$\hat{H}_{A^3} = \sum_{123} \int [123] \delta_{12.3} \left[g Y_{123} a_1^\dagger a_2^\dagger a_3 + g Y_{123}^* a_3^\dagger a_2 a_1 \right], \quad (23)$$

where the sum extends over color and spin quantum numbers of three gluons; below, c denotes color and ε spin. The integration extends over the gluon

momenta. It is constrained by the δ -function $\delta_{12,3}$ that enforces the condition $p_1 + p_2 = p_3$. The factor

$$Y_{123} = if^{c_1 c_2 c_3} \left(\varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3 \kappa - \varepsilon_1^* \varepsilon_3 \cdot \varepsilon_2^* \kappa \frac{1}{x_2} - \varepsilon_2^* \varepsilon_3 \cdot \varepsilon_1^* \kappa \frac{1}{x_1} \right) \quad (24)$$

follows from the structure of the YM Lagrangian density term with a product of three fields A . This factor depends on the relative transverse momentum κ^\perp and plus-momentum fractions that are defined in a way illustrated by the figure



which corresponds to the first term in the square bracket in Eq. (23); a gluon with quantum numbers denoted by 3 is annihilated and two gluons with quantum numbers denoted by 1 and 2 are created. The plus-momentum fractions and relative transverse momenta are defined by writing

$$x_1 = p_1^+ / p_3^+ = x, \quad (25)$$

$$k_1^\perp = p_1^\perp - x_1 p_3^\perp = \kappa^\perp, \quad (26)$$

$$x_2 = p_2^+ / p_3^+ = 1 - x, \quad (27)$$

$$k_2^\perp = p_2^\perp - x_2 p_3^\perp = -\kappa^\perp, \quad (28)$$

$$x_3 = p_3^+ / p_3^+ = 1, \quad (29)$$

$$k_3^\perp = p_3^\perp - x_3 p_3^\perp = 0^\perp, \quad (30)$$

where it happens that p_3 coincides with the total momentum carried by the annihilated quanta, in this case quantum 3, and p_3 is equal to the total momentum carried by the created quanta, in this case 1 and 2.

For every creation and annihilation operator in a term, labeled with the quantum numbers i , a factor r_i is defined,

$$r_i = x_i^\delta e^{-k_i^{\perp 2} / \Delta^2}, \quad (31)$$

where Δ is the ultraviolet regularization parameter that is meant to be sent to infinity, and δ is the small- x regularization parameter meant to be sent to zero, both only after regularization dependence is removed from physical predictions of the theory by adding to the regulated canonical Hamiltonian the counterterms to be found using the RGPEP (see below).

Regularization of the term \hat{H}_{A^3} is obtained by inserting in it the factor

$$R = \prod_{i=1}^3 r_i, \quad (32)$$

so that the regulated term is

$$\hat{H}_{A^3R} = \sum_{123} \int [123] \delta_{12,3} R \left[g Y_{123} a_1^\dagger a_2^\dagger a_3 + g Y_{123}^* a_3^\dagger a_2 a_1 \right]. \quad (33)$$

By regulating all terms in the Hamiltonian \hat{H}_{YM} in a similar way, one obtains the canonical regulated YM Hamiltonian \hat{H}_{YMR} .

4.1. Regulated eigenvalue problem

The gluonium eigenvalue problem for the regulated YM theory takes the form

$$\hat{H}_{\text{YMR}} |GP\rangle = \frac{M_G^2 + P^\perp{}^2}{P^+} |GP\rangle. \quad (34)$$

The expansion into the Fock-space components still involves infinitely many terms but gluons no longer can change momenta by arbitrarily large amounts, because of the regularization factors R in all interaction terms in \hat{H}_{YMR} . In principle, one can set a limit on the number of gluons and on their momenta and put the resulting limited eigenvalue problem on a computer to solve it using numerical methods. However, the results for eigenvalues and eigenstates have no *a priori* reason to come out independent of the regularization.

It is known that results of perturbative calculations heavily depend on the regularization. This dependence is not easy to remove from the perturbative calculations. There is no clear argument for why it should be easier to remove it from numerical eigenvalue problems. The problem of removing dependence on regularization is approached below using the concept of effective particles, and it is the interaction of effective particles that exhibits AF in the YM theory.

5. The concept of effective particles

One imagines the gluonium state in two different ways, meant to be mathematically equivalent. On the one hand, using the concept of canonical field quanta, for brevity called canonical gluons, one envisages a tower of Fock states built from the free vacuum $|0\rangle$ using the canonical creation operators. Each component has the wave function appropriate for the gluonium mass eigenvalue one considers. The canonical tower is on the left-hand side of the equality

$$\begin{bmatrix} \dots \\ |gggggg\rangle \\ |ggggg\rangle \\ |gggg\rangle \\ |ggg\rangle \\ |gg\rangle \end{bmatrix} = \begin{bmatrix} \dots \\ |ggggg\rangle \\ |gggg\rangle \\ |ggg\rangle \\ |gg\rangle \\ |g\rangle \end{bmatrix} \otimes \begin{bmatrix} \dots \\ |ggggg\rangle \\ |gggg\rangle \\ |ggg\rangle \\ |gg\rangle \\ |g\rangle \end{bmatrix} + \dots \quad (35)$$

On the other hand, the same tower of Fock components can be rewritten in terms of products of states built from entire towers of canonical gluons, as illustrated on the right-hand side of Eq. (35). The first term corresponds to a state built from two such effective gluons. The three dots indicate terms built from more than two effective gluons.

In a compact notation, Eq. (35) reads

$$|gg\rangle + |ggg\rangle + \dots = |g_s g_s\rangle + |g_s g_s g_s\rangle + \dots \quad (36)$$

Gluons on the left-hand side are the canonical ones and on the right-hand side the effective ones. The effective gluons are labeled with the parameter s , which denotes their *size* and also labels below the effective theory that will describe their dynamics. The canonical gluons of local YM theories are meant to be pointlike and they will be labeled below with subscript 0, corresponding to size $s = 0$. The effective theory of gluons of size s is calculated starting from the canonical YM theory and using the renormalization group procedure for effective particles (RGPEP).

6. RGPEP

To simplify our notation, we introduce the scale parameter $t = s^4$, drop hats indicating operators and suppress subscripts ‘YM’ and ‘R’. Creation and annihilation operators of canonical, pointlike gluons are denoted by a_0^\dagger and a_0 , and the corresponding operators for gluons of size s are denoted by a_t^\dagger and a_t . The interpretation of s as the size of effective gluons will be explained shortly.

We denote the YM theory canonical regulated Hamiltonian with counterterms in the limit of regularization being removed by $H_0(a_0)$. This means that this Hamiltonian is of the form

$$H_0(a_0) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \dots, i_n} c_0(i_1, \dots, i_n) a_{0i_1}^\dagger \dots a_{0i_n}, \quad (37)$$

where c_0 is used to denote the coefficients implied by the initial Lagrangian density, regularization and counterterms. The same Hamiltonian is written

in terms of operators for gluons of size s in the form

$$H_t(a_t) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \dots, i_n} c_t(i_1, \dots, i_n) a_{ti_1}^\dagger \dots a_{ti_n}, \quad (38)$$

where the coefficients c_t are found using the RGPEP.

One starts with the formula

$$H_t(a_t) = H_0(a_0), \quad (39)$$

which expresses the condition that the RGPEP does not change the Hamiltonian as an operator in the space of states. Instead, the RGPEP changes the gluon degrees of freedom from the canonical ones to the effective,

$$a_t = U_t a_0 U_t^\dagger, \quad (40)$$

where the operator U_t requires specification. It is meant to be unitary when the RGPEP procedure is completed. The initial condition implies $U_0 = 1$, but one has to remember that the initial canonical Lagrangian density is not sufficient to define a finite theory and in the process of calculating counterterms for specific regularization, one also establishes U_t .

By multiplying Eq. (39) on the left by U_t^\dagger and on the right by U_t , one obtains

$$H_t(a_0) = U_t^\dagger H_0(a_0) U_t. \quad (41)$$

For brevity of notation, it is useful to introduce

$$\mathcal{H}_t = H_t(a_0). \quad (42)$$

Differentiation with respect to t , denoted by prime, yields

$$\mathcal{H}'_t = \left[-U_t^\dagger U'_t, \mathcal{H}_t \right] = [G_t, \mathcal{H}_t], \quad (43)$$

where $G_t = -U_t^\dagger U'_t$ is called the generator. If one knew the generator, the transformation U_t would be given by the formula

$$U_t = T \exp \left(- \int_0^t d\tau G_\tau \right), \quad (44)$$

where the symbol T denotes ordering of operators in a product according to the value of τ .

6.1. RGPEP generator and non-perturbative QCD

The RGPEP generator used here is chosen in the form [7, 10, 20]

$$G_t = \left[\mathcal{H}_f, \tilde{\mathcal{H}}_t \right], \quad (45)$$

where

$$\mathcal{H}_f = \sum_i p_i^- a_{0i}^\dagger a_{0i} \quad (46)$$

is the free part of the Hamiltonian, *i.e.*, the term that is left of \mathcal{H}_t when one sets the coupling constant g to zero and all non-Abelian terms disappear. The gluon FF free energies are

$$p_i^- = \frac{p_i^{\perp 2}}{p_i^+}. \quad (47)$$

To define $\tilde{\mathcal{H}}_t$ that appears in the generator G_t , one uses Eq. (38) to write the operator \mathcal{H}_t as

$$\mathcal{H}_t = H_t(a_0) = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \dots, i_n} c_t(i_1, \dots, i_n) a_{0i_1}^\dagger \dots a_{0i_n}, \quad (48)$$

and then

$$\tilde{\mathcal{H}}_t = \sum_{n=2}^{\infty} \sum_{i_1, i_2, \dots, i_n} c_t(i_1, \dots, i_n) \left(\frac{1}{2} \sum_{k=1}^n p_{i_k}^+ \right)^2 a_{0i_1}^\dagger \dots a_{0i_n}. \quad (49)$$

The square of total plus-momentum of particles in interaction is inserted in $\tilde{\mathcal{H}}_t$ to obtain the effective dynamics that is invariant with respect to the boosts along z -axis and other kinematic symmetry operations of the FF.

With the generator of Eq. (45), the coefficients c_t in effective Hamiltonians for gluons of size s are obtained by solving the RGPEP equation,

$$\mathcal{H}'_t = \left[\left[\mathcal{H}_f, \tilde{\mathcal{H}}_t \right], \mathcal{H}_t \right], \quad (50)$$

in the constant operator basis of polynomials in a_0^\dagger and a_0 . Subsequently, one obtains $H_t(a_t)$ by replacing everywhere in \mathcal{H}_t the operators a_0^\dagger and a_0 by a_t^\dagger and a_t .

The RGPEP thus defines the theory of effective gluons as a function of their size. The same procedure provides a definition of effective QCD when one includes quarks in the dynamics. The size of effective particles

provides the scale parameter that distinguishes effective theories. They are all equivalent in their physical predictions but they are written in terms of different variables.

The arbitrariness in choice of the scale parameter is associated with the arbitrariness of choice of variables one can use. The equivalence of available choices of scales and associated variables is a consequence of equivalence of different choices of basis in a space of states in quantum mechanics.

The fact that different choices of variables differ in their utility in description of different physical phenomena corresponds to the fact that physical phenomena of different scale require variables of different scale for obtaining a simple description. Hence, one needs effective particles of the size adjusted to the scale of phenomena one wants to describe for the description in terms of particles to be simple. This is how the RGPEP explains the need for adjustment of scale in approximate calculations; *e.g.* see [17].

The above RGPEP formulation of the theory of strong interactions is not perturbative in its nature. However, Eq. (50) can be solved using an expansion in powers of the coupling constant g . This expansion shows how AF of gluons manifests itself in the Fock space.

7. Asymptotic freedom

In order to see how AF of gluons manifests itself in the Fock space, it is sufficient to expand the Hamiltonian up to third order in a series of powers of the bare coupling constant g

$$\mathcal{H}_t = \mathcal{H}_f + g\mathcal{H}_{1t} + g^2\mathcal{H}_{2t} + g^3\mathcal{H}_{3t} + \dots \quad (51)$$

The goal is to show how the effective coupling emerges and to observe its dependence on the effective gluon size s .

7.1. Terms of first order

Expanding both sides of Eq. (50) in powers of g and equating coefficients in front of the same powers on both sides, one obtains the first-order equation

$$\mathcal{H}'_{1t} = \left[\left[\mathcal{H}_f, \tilde{\mathcal{H}}_{1t} \right], \mathcal{H}_f \right]. \quad (52)$$

In terms of matrix elements between the canonical basis states of gluons of size $s = 0$ in the Fock space,

$$\mathcal{H}_{1t\,mn} = \langle m | \mathcal{H}_{1t} | n \rangle, \quad (53)$$

the first-order RGPEP equation reads

$$\mathcal{H}'_{1t\,mn} = -\left(\mathcal{M}_m^2 - \mathcal{M}_n^2 \right)^2 \mathcal{H}_{10\,mn}, \quad (54)$$

where \mathcal{M}_m and \mathcal{M}_n denote the total free invariant masses of the interacting particles. Hence, the solution is

$$\mathcal{H}_{1t} = f_t \mathcal{H}_{10}, \quad (55)$$

where the form factor f_t has the form that is universal for all physical quantum field theories. Namely,

$$f_t = e^{-t(\mathcal{M}_c^2 - \mathcal{M}_a^2)^2}, \quad (56)$$

where \mathcal{M}_c and \mathcal{M}_a denote the total invariant masses of the particles created and annihilated by the interaction, respectively.

In the case of three-gluon term discussed in Sec. 4, the invariant mass of the annihilated particles, actually one gluon labeled 3, is zero, and the invariant mass of created particles is the mass of gluons labeled 1 and 2,

$$\mathcal{M}_{12}^2 = (p_1 + p_2)^2 = \frac{\kappa^\perp{}^2}{x(1-x)}. \quad (57)$$

After replacing the canonical gluon operators by the effective ones, one obtains the first-order effective gluon interaction term

$$H_{A^3 1t} = \sum_{123} \int [123] \delta_{12,3} e^{-t\mathcal{M}_{12}^4} \left[Y_{123} a_{t1}^\dagger a_{t2}^\dagger a_{t3} + Y_{123}^* a_{3t}^\dagger a_{2t} a_{1t} \right]. \quad (58)$$

The exponential RGPEP factor appears in the role of a vertex form factor. Its width in momentum variables is given by

$$\lambda = 1/s. \quad (59)$$

As a result of associating the momentum width of the vertex form factor with an inverse of the size of the interacting particles with respect to the strong force, the RGPEP parameter s in the effective Hamiltonian is understood as referring to the size of gluons. This result explains the concept of scale in the RGPEP.

7.2. Terms of second and third order

Using Eq. (50), the calculation of terms of the order of g^3 in Eq. (51) involves a number of terms that are illustrated in Fig. 1 [8]. The calculation includes gluon mass squared counterterms of the order of g^2 and a vertex counterterm of the order of g^3 .

The ultraviolet divergent parts of the mass-squared and vertex counterterms are identified by demanding that the effective theory of finite momentum width $\lambda = 1/s$ does not depend on the parameter Δ introduced in

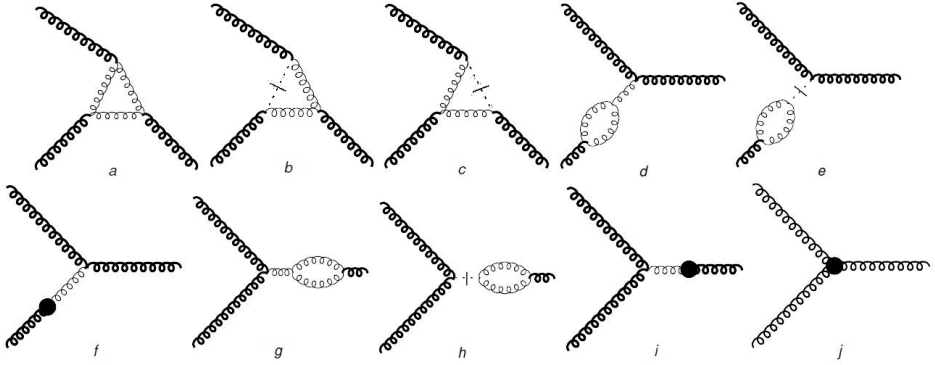


Fig. 1. Third-order contributions to the three-gluon vertex. Thick lines represent effective gluons and thin lines represent canonical gluons of the quantum YM theory. Dotted lines with bars on them in terms b , c , e and h correspond to contributions of the FF instantaneous interactions, analogous to the IF instantaneous Coulomb-like interactions. Black dots in terms f and i indicate contributions of the gluon mass-squared counterterm, and in term j , the contribution of the vertex counterterm.

Sec. 4. The ultraviolet finite part of the gluon mass-squared counterterm is fixed by demanding that the second-order perturbative result for the gluon mass eigenvalue is zero. The vertex counterterm is defined by subtracting the value of the effective vertex at an arbitrarily chosen value of $\lambda = \lambda_0$, corresponding to $s = s_0$, and demanding that the resulting value of the effective coupling constant is g_0 (see below). Divergences due to $\delta \rightarrow 0$ cancel out. Thus, using the regularization described in Sec. 4 and including the counterterms, one obtains the third-order terms that are added to the first-order terms in producing the three-gluon term in the effective theory.

7.3. Effective three-gluon term

The terms up to third order combine to the result for the effective three-gluon Hamiltonian interaction term

$$H_{A^3 t} = \sum_{123} \int [123] \delta_{12.3} e^{-t \mathcal{M}_{12}^4} \left[V_{123t} a_{t1}^\dagger a_{t2}^\dagger a_{t3} + V_{123t}^* a_{t3}^\dagger a_{t2} a_{t1} \right]. \quad (60)$$

The vertex function V_{123t} has the property

$$\lim_{\kappa^\perp \rightarrow 0} V_{123t} = g_\lambda Y_{123}, \quad (61)$$

where Y_{123} is the canonical YM factor of Eqs. (23) and (24). The constant g_λ with $\lambda = t^{-1/4}$ is given up to terms order g_0^3 by

$$g_\lambda = g_0 - \frac{g_0^3}{48\pi^2} N_c 11 \ln \frac{\lambda}{\lambda_0}. \quad (62)$$

The associated lowest-order Hamiltonian β -function is obtained by differentiating,

$$\lambda \frac{d}{d\lambda} g_\lambda = \beta_0 g_\lambda^3, \quad (63)$$

and the result for coefficient β_0 is

$$\beta_0 = -\frac{11N_c}{48\pi^2}. \quad (64)$$

By comparison with Refs. [1] and [2], one concludes that the effective coupling constant in the FF Hamiltonian for gluons of size $s = 1/\lambda$ depends on λ in the leading order in the same way as the running coupling in Green's functions obtained using Feynman diagrams for YM theories depends on the length λ of Euclidean momenta for gluons.

This result explains how AF appears in the Fock space of gluons. Namely, when an effective gluon of a small size s splits into two, or two such gluons combine into one as a result of action of their Hamiltonian in the Fock space, the coupling constant that determines the strength of this interaction is proportional to the inverse of the logarithm of s .

By the same token, the larger the size of gluons, the greater the strength of their interaction. As one increases the gluon size, the perturbative calculation of effective Hamiltonian eventually breaks down and one cannot obtain precise Hamiltonians for gluons of large size using a perturbative expansion of the RGPEP. However, it is not excluded that one can use Hamiltonians calculated in the perturbative RGPEP for gluons of size smaller than $1/\Lambda_{\text{YM}}$, where Λ_{YM} is the YM analog of Λ_{QCD} , to accurately describe glueballs. This hope is supported by the analogy with QED; the Coulomb potential is only of the formal order of e^2 , but nevertheless such low-order Hamiltonian is sufficient for describing a great variety of bound states and their interactions in atomic physics and chemistry. Models with asymptotic freedom and bound states support this hope [21].

8. Conclusion

Asymptotic freedom of gluons in the Fock space is of interest as a constructive representation in terms of quantum states and operators of a key feature of the theory of strong interactions. In particular, the size s of effec-

tive gluons in the Minkowski space-time, which is an argument of the Hamiltonian running coupling constant in the Fock space, appears to correspond to the Euclidean four-dimensional length $\lambda = 1/s$ of momenta ascribed to virtual gluons in Feynman diagrams, which is an argument of the running coupling constant obtained in the calculus of Euclidean Green's functions. The Minkowskian concept of size of quanta and the Euclidean concept of momentum scale are thus related to each other in terms of the RGPEP.

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