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SOLITON MODEL FOR BARYONS*

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We present a description of low-lying baryon states (below 2 GeV) based on the general soliton model, which follows from QCD in the limit of large number of colors. Relation to the quark model is discussed. The model describes the spectrum of baryons below 2 GeV with no extra nor missing states. Main properties of baryons (mass splitting and widths) are calculated and relations independent of the specific soliton model are derived. These relations agree rather well with the data.

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1. Baryons below 2 GeV

The spectrum of low-lying baryons is very rich. Even below 2 GeV, more than a hundred of resonances are known. Tables of Particle Data Group [1] list in this region at least 23 nucleons, 22 Δ-resonances, 18 Λ-hyperons, 26 Σs, 11 Ξs, and 4 Ω-hyperons. It seems that their masses (and other characteristics) give a huge amount of information about very intimate properties of QCD. In fact, this is not true: the good portion of this data is related to rather general symmetries of the theory and has nothing to do with specific dynamics of QCD. They are reproduced in any self-consistent theory.

The oldest [2] and the best-known example of general symmetry is SU(3) flavor symmetry. Current masses of all quarks which can enter light baryons (u, d, s) are small as compared to the scale of strong interaction ($\Lambda_{\mathrm{strong}} \approx$ 600 MeV). Additional SU(3) flavor symmetry arises in the limit of quark current masses equal to zero. This limit is working very well for u- and d quarks, the mass of the strange quark ($m_s \approx 120$ MeV) can be accounted for by the perturbation theory.

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Due to the SU(3) symmetry, baryons can be united into multiplets. For baryons consisting precisely of 3 quarks, only octets and decuplets are possible. Octets consists of N, Σ and Λ–hyperons, decuplets consist of Δ-resonances, Σ, Ξ, and Ω-hyperons. Known multiplets below 2 GeV are collected in two tables: Table I for positive parity baryons and Table II for negative parity baryons. At least 19 multiplets in this region of masses are known. There are 5 octets and 5 decuplets with positive parity with different spins, 5 octets and 3 decuplets have negative parity. Also two SU(3) singlets exist but they are present only in the negative parity sector.

The first column of Tables I and II lists SU(3) representations, the second is a mass of the center of multiplets (presumably it should be close to the masses of particles in the chiral limit). $6^{\rm th}$, $7^{\rm th}$, $8^{\rm th}$, $9^{\rm th}$ columns give the masses of the multiplet members. Not all members of SU(3) multiplets are known even in this, very well investigated range of energies. Experimentalists have to discover at least $9\,\Xi$s, $4\,\Omega$s, $2\,\Sigma$s and $1\,\Lambda$. We underline the masses of these, not yet known particles predicted by SU(3) symmetry. However, we have to note that even the identification of the SU(3) multiplets is still under discussion[1].

First of all, the QCD should explain the contents of SU(3) baryon multiplets, their spin and masses (first 3 columns in Tables I and II). The main tool used for this purpose is a quark model. In its simplest version, the quark model describes baryons as non-relativistic bound states of 3 massive (*constituent mass*) quarks in some phenomenological potential. The potential is *independent* of flavor, the main part of potential has spherical symmetry with relatively small relativistic corrections leading to the interaction of spin with angular momentum and hyperfine spin–spin interaction. Quark model picture leads to the idea of SU(6) spin–flavor symmetry for the baryon multiplets. The lowest SU(6) multiplet is **56**. Indeed, the lowest positive parity multiplets $(\mathbf{8}, \frac{\mathbf{1}}{\mathbf{2}})[1151]$ and $(\mathbf{10}, \frac{\mathbf{3}}{\mathbf{2}})[1382]$ agree very well with this symmetry. The 230 MeV splitting between these multiplets has to be attributed to the hyperfine splitting.

Next multiplet of SU(6) corresponds to angular momentum $L = 1$ of one quark inside a baryon. This multiplet should have negative parity and it is **70** according to SU(6) classification. This prediction [3] is a main triumph of the quark model as all low-lying negative parity baryons fit exactly to these $70 \times (2L + 1) = 210$ states. It can be seen directly from Table II that negative parity baryons, indeed, precisely correspond to the required expansion of SU(6) in SU(3) multiplets.

[1] We follow here mainly Ref. [5]. The advantage of this reference is that all SU(3) predictions (including particle widths) were checked in order to make an appropriate identification.

TABLE I

Positive parity baryon multiplets of spin S. M denotes the mass of the multiplet center. Splitting parameters $\mu_{1,2}^{(8)}$ are defined later in Table II. Columns labeled by baryon names correspond to the decay widths discussed in Sec. 6.

	S	M	$\mu_1^{(8)}$	$\mu_2^{(8)}$	N or Δ	Σ	Ξ	Λ or Ω	$\to 8$	F/D	$\to 10$
8	$\frac{1}{2}$	1151	40.0	139.5	939	1193	1318	1116	—	—	—
10	$\frac{3}{2}$	1382	—	146.6	1232	1385	1530	1672	142	—	—
8	$\frac{1}{2}$	1608	41.1	73.6	1440	1660	1690	1600	32.4	0.37	229
10	$\frac{3}{2}$	1732	—	156	1600	1690	1900	2050	38.2	—	129.3
8	$\frac{1}{2}$	1846	34.4	76.9	1710	1880	1950	1810	14.9	0.47	44.9
8	$\frac{3}{2}$	1865	−25.0	188.8	1720	1840	2035	1890	6.7	1.45	?
8	$\frac{5}{2}$	1872.5	45.0	118.8	1680	1915	2030	1820	52.0	0.64	19.2
10	$\frac{1}{2}$	2060	—	150.0	1910	2060	2210	2360	24.2	—	8.7
10	$\frac{5}{2}$	2071	—	160.5	1905	2070	2250	2380	45.9	—	39.7
10	$\frac{3}{2}$	2087	—	181	1920	2080	2240	2470	15.4	—	?
10	$\frac{7}{2}$	2038	—	99	1950	2030	2120	2250	60.7	−94.1	

TABLE II

Negative parity baryon multiplets. Description of columns as in Table I.

	S	M	$\mu_1^{(8)}$	$\mu_2^{(8)}$	N or Δ	Σ	Ξ	Λ or Ω	$\to 8$	F/D	$\to 10$
1	$\frac{1}{2}$	1405	—	—	—	—	—	—	13.1	—	—
									$\theta = -48^o$		
1	$\frac{3}{2}$	1520	—	—	—	—	—	—	115.4	—	—
									$\theta = 26^o$		
8	$\frac{1}{2}$	1592	−43.6	111	1535	1560	1625	1670	7.1	−0.34	93.4
8	$\frac{3}{2}$	1673	−6.7	158	1520	1670	1820	1690	43.6	2.90	12.7
8	$\frac{1}{2}$	1716	−93.1	235	1650	1620	1885	1800	8.3	4.05	30.4
10	$\frac{1}{2}$	1758	—	144	1620	1750	1900	2050	12.4	—	221.4
8	$\frac{5}{2}$	1801	−26.7	171	1675	1775	1950	1830	26.8	−0.19	158.7
10	$\frac{3}{2}$	1850	—	150	1700	1850	2000	2150	48.3	—	29.8
8	$\frac{3}{2}$	1895	45	116	1700	1940	2045	1850	8.3	−0.35	67.2

Unfortunately, the situation is much worse when we proceed to the excited positive parity baryons. Quark model predicts large number of multiplets as the next excited state, **56**, **70**, **20** are among them. Existing baryons do not fit this picture: their identification in terms of SU(6) multiplets is questionable, a large number of *missing states* is present. It is, of course, possible that required states have not been discovered yet but, in general, it seems that quark model fails for excited positive parity baryons.

The quark model[2] became, in fact, a standard language for discussing hadrons. Meanwhile, it is rather weakly motivated by QCD. First of all, it cannot be considered as a self-consistent field theory. The notion of exactly three interacting particles without additional quark–antiquark pairs or additional gluons contradicts the basic principles of quantum field theory. It can be realized only in the non-relativistic limit of the weakly interacting quark theory. This is definitely not the case for standard baryons. A concept of the constituent quarks is also not derived, strictly speaking, from QCD. In particular, this concept implies spontaneous breakdown of chiral symmetry. But spontaneous breakdown of chiral symmetry includes also another ingredient — massless Goldstone particles (pions). Quark model fails to recognize this essential QCD degrees of freedom.

We will present here another approach to the description of baryon properties. It is based on the limit of large number of colors $N_c \to \infty$ and in this case, it can be derived (to some extent!) directly from QCD. In some limit, it can be close to the quark model but, in general case, it can be quite different. We think that it works better than the quark model and, in any case, it is much better justified by quantum chromodynamics.

2. Baryons at large N_c

It is known that in the limit of large number of colors, QCD simplifies greatly but remains a non-trivial and very interesting theory. It is believed that confinement is present at any N_c and, in particular, at $N_c \to \infty$, so the spectrum of the theory consists of an infinite number of hadrons. Mesons (*i.e.* hadrons consisting of quark and antiquark, or better to say, made from a number of quarks independent of N_c) are *stable* (the width is $O(1/N_c)$) and *non-interacting* particles (vertex of the meson–meson interaction with k legs is $1/N_c^{k/2}$). The spectrum of mesons is infinite, the distances between meson masses are $O(1)$.

The baryons at $N_c \to \infty$ are very different from mesons. The reason is simple — they consist of N_c quarks. This means that at $N_c \to \infty$, baryons are heavy (mass is $O(N_c)$) and *semiclassical* objects. Witten proved [6] a rigorous theorem: at large N_c, baryons can be viewed as semi-classical solitons of some effective meson Lagrangian. In principle, all meson degrees of freedom should be included in this Lagrangian. At large N_c, they can be considered as independent (no interaction) and elementary (stable) degrees of freedom.

This idea was first realized in the Skyrme [7] model. It is natural to expect that the lightest mesons are the most important. This can be true for the ground state baryons if the size of the baryon is large. Skyrme

[2] For recent status of the quark model, see, *e.g.* reviews [4].

model accounts only for (pseudo) Goldstone mesons, *i.e.* the octet of pseudoscalar mesons. The interaction of pseudoscalar mesons at low energies is determined by effective chiral Lagrangian (ECL). This Lagrangian is an expansion in derivatives of the pion fields. Only lowest terms are known in the ECL (containing 2 and 4 derivatives) but, in fact, all terms are essential for constructing the soliton. Skyrme suggested to model all higher terms by a single 4-derivative term. Then, the soliton can be constructed (solving classical equations of motion following from Skyrme Lagrangian) and it possesses all expected qualitative properties of the nucleon [8]. Quantitatively, this model is not very successful. This is not surprising for such a crude model. To improve the quality, it was suggested to take into account other degrees of freedom such as vector mesons, modify Skyrme Lagrangian, *etc.* Nowadays, it is natural to use ideas of holographic QCD (see, *e.g.*, [9] or [10]). Indeed, holographic QCD can be formulated as a theory of infinite number of vector and pseudoscalar mesons with a specific interaction, which reproduces all known properties of QCD mesons at $N_{\rm c} \to \infty$. This program is already partly performed.

Soliton picture of baryons has evident advantages as compared to the quark model. First of all, the main ideas of this picture follow directly from first principles of QCD. Second, theory based on effective meson Lagrangian is a self-consistent quantum field theory. One can easily formulate the procedure to calculate quantum corrections using the standard semiclassical methods. Soliton–meson models effectively take care of $q\bar{q}$ pairs in baryons. The number of additional pairs is not suppressed in the limit $N_{\rm c} \to \infty$. On the other hand, information about quark degrees of freedom is lost in the soliton Skyrme-like models. For this reason, there is no direct way to describe baryon resonances in such models — these models can be applied only to the ground state baryons (lowest baryon octet and decuplet). One has to consider scattering amplitudes in order to discover baryon resonances as poles of amplitudes in the complex energy plane.

Soliton models and quark model look very different, in spite of the fact they describe the same object. A bridge between two approaches is provided by the so-called Chiral Quark–Soliton Model (χQSM) [11, 12]. In the framework of this model, it is assumed that at large $N_{\rm c}$, QCD is reduced to effective low-energy theory with degrees of freedom which are constituent quarks and colorless meson fields. This theory is essentially semiclassical: interaction of mesons is suppressed by $N_{\rm c}$. The spectrum of mesons can be read off from the effective Lagrangian, baryons appear as solitons which are the bound states of $N_{\rm c}$ quarks in the self-consistent meson field. Fluctuations of the meson mean field in baryons are also suppressed by $N_{\rm c}$.

The introduction of separate degrees of freedom for consituent quarks and mesons should be justified: indeed, as mesons ultimately consist of quarks, this could be a kind of double-counting. Let us outline the derivation of such a low energy effective theory from the QCD. At the first stage, one has to integrate out gluons, then the theory reduces to the theory of quarks only. Owing to the expected spontaneous chiral symmetry breaking, these constituent quarks are massive, their interaction is described by a number of many-fermion non-local terms. In the colorless channels and at large N_c, this interaction can be rewritten in the bosonized form by means of some auxiliary meson fields. Meson fields introduced this way do not lead to double counting and the corresponding effective theory of constituent quarks and mesons is self consistent.

This programme was carried out in the framework of the instanton liquid picture of the QCD vacuum [13]. Here, the integration over all gluon degrees of freedom is restricted to instantons and small fluctuations around them. One can derive the corresponding low-energy theory, it consists of the constituent quarks with a momentum-dependent mass $M(p)$ and massless pions interacting by means of a simple effective Lagrangian [14]. The model of baryons based on this idea is working rather well describing all known properties of low-lying baryons except one but, maybe, the most important feature — the model does not account for the quark confinement (it is not reproduced in the instanton liquid picture). However, this is only a particular realization of the effective theory under discussion, in fact, it is easy to suggest phenomenological generalizations which have confinement built-in.

Quark model at large N_c and the theory of baryon–soliton are extreme cases of the mean-field effective theory. Quark model represents a non-relativistic approximation to the theory. In this case, one can neglect quark–antiquark fluctuations and view baryons as consisting of precisely N_c quarks. Neglecting also the retarding of the quark–quark interaction, we arrive at the potential quark model, which one can treat in the mean-field approximation at large N_c. On the other hand, integrating over quark fields, we obtain a meson Lagrangian where baryons should appear as solitons. Meson Lagrangian is constructed as an expansion in gradients of the meson field, assuming that meson mean field is slowly varying. It can be seen that this corresponds to the baryon state in which the contribution of Fock components with large number of quark–antiquark pairs is dominating and valence quarks are ultra-relativistic [15]. Hence, this case is opposite to the case of the quark model.

Baryons in the large N_c limit were studied also using yet another approach. This approach uses the general rules of N_c counting together with the chiral symmetry and group-theoretical arguments (for reviews, see [16–19] and references therein). In this framework, many relations for baryon resonances can be derived, with no reference to the underlying dynamics.

The key new point of the approach presented here is that the large-N_c dynamics is essentially known and simple. We, thus, give the dynamical interpretation of the general large-N_c relations and find the physical meaning of the otherwise free numerical coefficients in those relations. We also derive new relations valid in the large-N_c limit.

3. Symmetry of the mean field

In the mean-field approximation, justified at large N_c, one looks for the solutions of the Dirac equation for single quark states in the background mean field. In the most general case, the background field couples to quarks through all five Fermi variants. The mean field is stationary in time, it leads to the Dirac eigenvalue equation for the u, d, s quarks in the background field, $H\psi = E\psi$, the Dirac Hamiltonian being schematically

$$H = \gamma^0 \left(-i\partial_i \gamma^i + S(\boldsymbol{x}) + P(\boldsymbol{x}) i\gamma^5 + V_\mu(\boldsymbol{x})\gamma^\mu + A_\mu(\boldsymbol{x})\gamma^\mu\gamma^5 + T_{\mu\nu}(\boldsymbol{x}) \frac{i}{2} [\gamma^\mu \gamma^\nu] \right), \tag{1}$$

where S, P, V, A, T are the scalar, pseudoscalar, vector, axial and tensor mean fields, respectively; all are matrices in flavor. In fact, the one-particle Dirac Hamiltonian (1) is generally non-local, however that does not destroy symmetries which we are primarily interested in. We include the current and the dynamically-generated quarks masses into the scalar term S.

The key issue is the symmetry of the mean field. At the moment, we assume the chiral limit for all quarks $m_u = m_d = m_s = 0$, which is an excellent approximation. A natural assumption, then, would be that the mean field is flavor-symmetric, and spherically symmetric. However, we know that baryons are strongly coupled to pseudoscalar mesons ($g_{\pi NN} \approx 13$). It means that there is a large pseudoscalar field inside baryons; at large N_c, it is a classical mean field. There is no way of writing down the pseudoscalar field (it must change sign under inversion of coordinates) that would be compatible with the $\mathrm{SU}(3)_{\mathrm{flav}} \times \mathrm{SO}(3)_{\mathrm{space}}$ symmetry. The minimal extension of spherical symmetry is to write the "hedgehog" Ansatz "marrying" the isotopic and space axes[3]

$$\pi^a(\boldsymbol{x}) = \begin{cases} n^a F(r), \quad n^a = \frac{x^a}{r}, & a = 1, 2, 3, \\ 0, & a = 4, 5, 6, 7, 8. \end{cases} \tag{2}$$

This Ansatz breaks the $\mathrm{SU}(3)_{\mathrm{flav}}$ symmetry. Moreover, it breaks the symmetry under independent space $\mathrm{SO}(3)_{\mathrm{space}}$ and isospin $\mathrm{SU}(2)_{\mathrm{iso}}$ rotations,

[3] A. Hosaka informed us that historically, this Ansatz for the pion field in a nucleon appears for the first time in a 1942 paper by Pauli and Dancoff [20]; it reappears in 1961 in the seminal papers by Skyrme [7].

and only a simultaneous rotation in both spaces remains a symmetry, since a rotation in the isospin space labeled a, can be compensated by the rotation of the space axes. The Ansatz (2) implies a spontaneous (as contrasted to explicit) breaking of the original $\mathrm{SU}(3)_{\mathrm{flav}} \times \mathrm{SO}(3)_{\mathrm{space}}$ symmetry down to the $\mathrm{SU}(2)_{\mathrm{iso+space}}$ symmetry. It is analogous to the spontaneous breaking of spherical symmetry by the ellipsoid form of many nuclei; there are many other examples in physics where the original symmetry is spontaneously broken in the ground state.

We list here all possible structures in the S, P, V, A, T fields, compatible with the $\mathrm{SU}(2)_{\mathrm{iso+space}}$ symmetry and with the C, P, T quantum numbers of the fields [21, 22]. The fields below are generalizations of the 'hedgehog' Ansatz (2) to mesonic fields with other quantum numbers.

Since SU(3) symmetry is broken, all fields can be divided into three categories:

I. Isovector fields acting on u, d quarks

$$\begin{aligned} \mathrm{pseudoscalar:}\;\; & P^a(\boldsymbol{x}) = n^a\, P_0(r)\,, \\ \mathrm{vector, space components:}\;\; & V_i^a(\boldsymbol{x}) = \epsilon_{aik}\, n_k\, P_1(r)\,, \\ \mathrm{axial, space components:}\;\; & A_i^a(\boldsymbol{x}) = \delta_{ai}\, P_2(r) + n_a n_i\, P_3(r)\,, \\ \mathrm{tensor, space components:}\;\; & T_{ij}^a(\boldsymbol{x}) = \epsilon_{aij}\, P_4(r) + \epsilon_{bij}\, n_a n_b\, P_5(r)\,. \end{aligned} \tag{3}$$

II. Isoscalar fields acting on u, d quarks

$$\begin{aligned} \mathrm{scalar:}\;\; & S(\boldsymbol{x}) = Q_0(r)\,, \\ \mathrm{vector, time component:}\;\; & V_0(\boldsymbol{x}) = Q_1(r)\,, \\ \mathrm{tensor, mixed components:}\;\; & T_{0i}(\boldsymbol{x}) = n_i\, Q_2(r)\,. \end{aligned} \tag{4}$$

III. Isoscalar fields acting on s quarks

$$\begin{aligned} \mathrm{scalar:}\;\; & S(\boldsymbol{x}) = R_0(r)\,, \\ \mathrm{vector, time components:}\;\; & V_0(\boldsymbol{x}) = R_1(r)\,, \\ \mathrm{tensor, mixed components:}\;\; & T_{0i}(\boldsymbol{x}) = n_i\, R_2(r)\,. \end{aligned} \tag{5}$$

All the remaining fields and components are zero as they do not satisfy the $\mathrm{SU}(2)_{\mathrm{iso+space}}$ symmetry and/or the needed discrete C, P, T symmetries. The 12 'profile' functions $P_{0,1,2,3,4,5}$, $Q_{0,1,2}$ and $R_{0,1,2}$ should be eventually found self-consistently from the minimization of the mass of the ground-state baryon. We shall call Eqs. (3)–(5) the hedgehog Ansatz. However, even if we do not know those profiles, there are important consequences of this Ansatz for the baryon spectrum.

Given the $\mathrm{SU}(2)_{\mathrm{iso+space}}$ symmetry of the mean field, the Dirac Hamiltonian for quarks actually splits into two: one for s quarks and the other one for u, d quarks. It should be stressed that the energy levels for u, d quarks, on the one hand, and for s quarks, on the other, are completely different, even in the chiral limit $m_s \to 0$.

The energy levels for s quarks are classified by half-integer J^P, where P is parity under space inversion, and $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ is quark angular momentum; all levels are $(2J+1)$-fold degenerate. The energy levels for u, d quarks are classified by integer K^P, where $\boldsymbol{K} = \boldsymbol{T} + \boldsymbol{J}$ is the 'grand spin' (T is isospin), and are $(2K+1)$-fold degenerate.

All energy levels, both positive and negative, are probably discrete owing to confinement. Indeed, a continuous spectrum would correspond to a situation when quarks are free at large distances from the center, which contradicts confinement. One can model confinement *e.g.* by forcing the effective quark masses to grow linearly at infinity, $S(\boldsymbol{x}) \to \sigma r$.

The Dirac equation (1) for s quarks in the background field (5) takes the form of a system of two ordinary differential equations for two functions $f(r)$, $g(r)$ depending only on the distance from the center. The system of equations depends on the (half-integer) angular momentum of level under considerations, and on its parity. For s-quark levels with parity $P = (-1)^{J-\frac{1}{2}}$, *e.g.* for the levels $J^P = \frac{1}{2}^+, \frac{3}{2}^-, \frac{5}{2}^+, \ldots$, the system takes the form

$$\left\{ \begin{array}{l} E\, f = -g' - \frac{J+\frac{3}{2}}{r}\, g + R_0\, f + R_1\, f + R_2\, g\,, \\ E\, g = f' + \frac{-J+\frac{1}{2}}{r}\, f - R_0\, g + R_1\, g + R_2\, f\,. \end{array} \right. \tag{6}$$

To find an s-quark energy level E with these quantum numbers, one has to solve Eq. (6) with the initial condition $f(r) \sim r^{J-\frac{1}{2}}$, $g(r) \sim r^{J+\frac{1}{2}}$, and both functions decreasing at infinity.

For levels with opposite parity $P = (-1)^{J+\frac{1}{2}}$, *e.g.* $J^P = \frac{1}{2}^-, \frac{3}{2}^+, \frac{5}{2}^-, \ldots$, one has to solve another system

$$\left\{ \begin{array}{l} E\, f = -g' - \frac{-J+\frac{1}{2}}{r}\, g + R_0\, f + R_1\, f + R_2\, g\,, \\ E\, g = f' + \frac{J+\frac{3}{2}}{r}\, f - R_0\, g + R_1\, g + R_2\, f\,. \end{array} \right. \tag{7}$$

We note that in the absence of the $R_{1,2}$ fields, the energy spectrum is symmetric under simultaneous change of parity and energy signs.

Dirac equation for u, d quarks in the background fields (3), (4) is more complicated: one has here a system of four ordinary differential equations. There are two different types of Dirac equations for the states with parity $(-1)^{K+1}$, (namely $K^P = 1^+, 2^-, \ldots$) and with parity $(-1)^K$. These equations are derived in [23].

The case of $K = 0$ is special, since the angular momentum is restricted to only one value $J = K + \frac{1}{2} = \frac{1}{2}$. At $K = 0$, a system of 4 equations for the $K^P = 0^-$ level reduces to two equations

$$E\,j = -h' + (Q_0 + Q_1 + P_2 - P_3 + P_4 - P_5)j + (P_0 - 2P_1 + Q_2)h\,,$$
$$E\,h = j' + \frac{2}{r}\,j + (-Q_0 + Q_1 - 3P_2 - P_3 + 3P_4 + P_5)h + (P_0 - 2P_1 + Q_2)j \tag{8}$$

with $h \sim r^0, j \sim r^1$. Similarly, to find the $K^P = 0^+$ levels, one has to solve only two equations

$$E\,j = -h' + (-Q_0 + Q_1 + P_2 - P_3 - P_4 + P_5)j - (P_0 + 2P_1 + Q_2)h\,,$$
$$E\,h = j' + \frac{2}{r}\,j + (Q_0 + Q_1 - 3P_2 - P_3 - 3P_4 - P_5)h - (P_0 + 2P_1 + Q_2)j\,. \tag{9}$$

In Fig. 1, we show an example of quark levels obtained from a 'natural' choice of external fields $Q_{0\text{–}2}, P_{0\text{–}5}$. We take a confining scalar field $S(r) = \sigma r$ with a standard string tension $\sigma = (0.44\ \text{GeV})^2$, and a topological chiral angle field $P(r) = 2\arctan(r_0^2/r^2)$ such that the profile functions introduced in Eqs. (3), (4) are $Q_0(r) = S(r)\cos P(r)$, $P_0(r) = S(r)\sin P(r)$; the other profile functions are exponentially decaying at large distances. The external fields are shown in Fig. 1 (left), and the resulting quark levels with various K^P are shown in Fig. 1 (right). These or similar levels dictate the masses of baryon resonances.

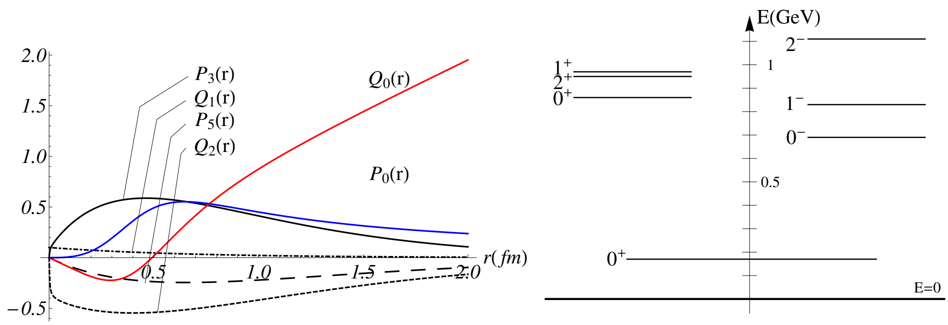

Fig. 1. An illustrative example of intrinsic quark levels with quantum numbers K^P (right) generated by the mean fields shown in the left panel.

According to the Dirac theory, all *negative*-energy levels, both for s and u, d quarks, have to be fully occupied, corresponding to the vacuum. It means that there must be exactly N_c quarks antisymmetric in color occupying all degenerate levels with J_3 from $-J$ to J, or K_3 from $-K$ to K;

they form closed shells. Filling in the lowest level with $E > 0$ by N_c quarks makes a ground state baryon, see Fig. 2. A similar picture arises in the chiral Bag Model [26]. Excited baryons can be related to different 1,2,3-quark excitations to the other levels. We will try to advocate the point of view that known baryon resonances below 2 GeV are related to one-quark excitations only.

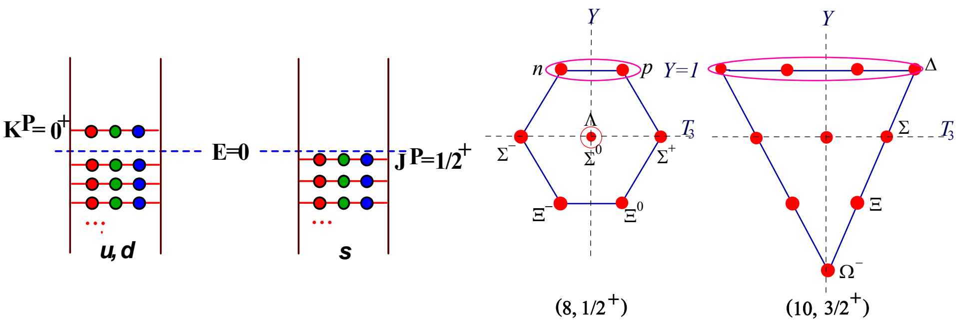

Fig. 2. Filling u, d and s shells for the ground-state baryon (left), and the two lowest baryon multiplets that follow from quantizing the rotations of this filling scheme (right).

The mass of a baryon is the aggregate energy of all filled states, and being a functional of the mesonic field, it is proportional to N_c since all quark levels are degenerate in color. Therefore, quantum fluctuations of mesonic field in baryons are suppressed as $1/N_c$ so that the mean field is indeed justified.

It is primarily the symmetry of the mean field which distinguishes the quark model at large N_c and the soliton model considered here. Quark model corresponds to the SU(6) symmetrical mean field. It appears in our approach in the case when only scalar field is present, while all other fields vanish. In particular, mean field corresponding to Goldstone pseudoscalar field is absent in the quark model. It is not surprising, as quark model does not take into account spontaneous breakdown of chiral symmetry in QCD.

4. Rotational bands about around quark levels

Hedgehog mean field breaks both the flavor and rotational symmetry. General semiclassical considerations tell us that breakdown of continuum symmetry implies that there are a number of *zero modes* around the semiclassical soliton mean field. Zero modes should be taken into account exactly. One has to introduce corresponding number of *collective* coordinates. Integration in this coordinates restores symmetries which were broken down and determine the quantum numbers of solitons.

Every intrinsic level is accompanied by the rotational band of the states. It appears as a result of the quantization of the slow rotations both in the flavor and ordinary space. The theory of rotational bands over the ground state was developed years ago [11], but for excited states and for general case of the mean field, it has some specifics [23].

4.1. Rotational Lagrangian and ground state baryons

Rotations slowly depending on time split the energy into the rotational bands. It is convenient to describe this effect by means of an effective Lagrangian depending on collective coordinates which are rotational matrices.

Let $R(t)$ be an SU(3) matrix for slow rotations in flavor space, and $\mathcal{S}(t)$ be an SU(2) matrix for slow space (and spin) rotations. They rotate quark wave functions $\phi^{\alpha i}(\boldsymbol{x})$ ($\alpha = 1 \ldots 3$ is flavor, $i = 1 \ldots 2$ — spin indices) in the given mean field as

$$\tilde{\phi}_n(x)^{\alpha i} = R^{\alpha}_{\alpha'}(t)\mathcal{S}^{i}_{i'}(t)\phi_n^{\alpha' i'}(O(t)\boldsymbol{x}) \,, \qquad O_{ik}(t) = \tfrac{1}{2}\mathrm{Tr}\left[\mathcal{S}^{+}(t)\sigma_k\mathcal{S}(t)\sigma_i\right] \,. \tag{10}$$

Then, it is easy to see that simultaneous transformation of the meson fields

$$\begin{aligned} \tilde{P}^a(\boldsymbol{x}) &= O_{ab}[R]P^b(O(\mathcal{S})\boldsymbol{x}) \,, \qquad \tilde{V}^{ai}(\boldsymbol{x}) = O_{ab}[R]O_{ij}[\mathcal{S}]V^{bj}(\boldsymbol{x}) \,, \\ \tilde{A}^{ai}(\boldsymbol{x}) &= O_{ab}[R]O_{ij}[\mathcal{S}]A^{bj}(\boldsymbol{x}) \,, \end{aligned} \tag{11}$$

etc. leaves Dirac equation in the mean-field invariant, provided that matrices R and $\mathcal{S}$ are constant in time.

Let us integrate out quarks. Then, the effective action of the theory is a sum of meson Lagrangian plus the contribution of constituent quarks which is the determinant of the Dirac equation in the mean field

$$S_{\mathrm{eff}} = \int dt\; \mathcal{L}(M) - i\sum_c \mathrm{Sp}_{\mathrm{occup}}\mathrm{Log}\left\{ i\frac{\partial}{\partial t} - \mathcal{H}[M] \right\} \,. \tag{12}$$

Here, sum denotes the summation over color indices and trace Sp is running over all *occupied* states. As the meson field M and the Hamiltonian $\mathcal{H}$ are color blind, usually the sum in color produces a factor of N_{c}. This is not true for one-particle excitations where one term out of N_{c} corresponds to some different filling of the levels.

Slow rotations $\mathcal{S}(t), R(t)$ are part of the quantum fluctuations of the general meson field M. These fluctuations are suppressed in the limit of large N_{c}. Rotations are not suppressed as they zero modes but their frequencies are small in N_{c}.

Let us parametrize the general meson field as $M = \bar{M} + \delta M$ (where $\bar{M}(\boldsymbol{x})$ is a time-independent mean field and $\delta M(\boldsymbol{x}, t)$ are quantum fluctuations) and calculate the effective action of (12) on the set of slowly rotated states (10), (11)

$$\begin{aligned} S_{\text{eff}} &= \int dt\, \mathcal{L}_{\text{meson}}\left(M + \delta M, \tilde{\Omega}, \tilde{\omega}\right) \\ &\quad - i \sum_c \mathrm{Sp}_{\text{occup}} \mathrm{Log}\left\{ i\frac{\partial}{\partial t} - \mathcal{H}[M + \delta M] - \tilde{\Omega}_a t_a - \tilde{\omega}_i j_i \right\}. \end{aligned} \tag{13}$$

Here, $\tilde{\Omega}_a$ and $\tilde{\omega}_i$ are flavor and angular frequencies in the body-fixed frame

$$\tilde{\Omega}_a = -i\,\mathrm{Tr}\left[R^+ \dot{R} \lambda_a\right], \qquad \tilde{\omega}_i = -i\,\mathrm{Tr}\left[\mathcal{S}^+ \dot{\mathcal{S}} \sigma_i\right] \tag{14}$$

(λ_a are Gell-Mann flavor and σ_i are Pauli spin matrices), t_a and j_i are one-particle operators of flavor and total angular momenta

$$t_a = \frac{1}{2}\lambda_a, \qquad j_i = s_i + l_i = \frac{1}{2}\sigma_i + i\varepsilon^{ikl} x_k \frac{\partial}{\partial x_l}. \tag{15}$$

Let us expand Eq. (13) in small $\delta M, \tilde{\Omega}, \tilde{\omega}$. The linear term should be absent, as mean field $\bar{M}(\boldsymbol{x})$ is a solution of equations of motion. There is a famous exclusion from this rule — Witten–Wess–Zumino term which is linear in Ω_8 and proportional to the baryon charge B of the state

$$\delta S^{(1)} = -\frac{N_{\mathrm{c}}}{2\sqrt{3}} \int dt \tilde{\Omega}_8. \tag{16}$$

The second order correction is in general

$$\begin{aligned} \delta S^{(2)} &= \frac{1}{2}\int d^4x \delta M W \delta M + \int d^4x \left(\delta M K^a_{\Omega} \tilde{\Omega}_a + \delta M K^i_{\omega} \tilde{\omega}_i\right) \\ &\quad - \frac{1}{2}\int dt \left[I^{\Omega\Omega}_{ab} \tilde{\Omega}_a \tilde{\Omega}_b + I^{(\omega\omega)}_{ab} \tilde{\omega}_i \tilde{\omega}_j + I^{(\omega\Omega)}_{ai} \tilde{\Omega}_a \tilde{\omega}_i \right]. \end{aligned} \tag{17}$$

Here, the first term is a quadratic form for the quantum fluctuations which are not rotations, second term describes mixing of rotations and other quantum fluctuations, and the third one is a generic quadratic form for space and flavor rotations. All terms are proportional to N_{c}. Thus, quantum fluctuations $\delta M = O\left(1/\sqrt{N_{\mathrm{c}}}\right)$. As to the frequencies $\tilde{\Omega}, \tilde{\omega}$, we will see that they are $\tilde{\Omega}, \tilde{\omega} = O(1/N_{\mathrm{c}})$.

We are interested in the collective rotational Lagrangian, *i.e.* the Lagrangian depending only on angular and flavor frequencies. We see that there are two sources for such a Lagrangian. First, it comes from the immediate expansion of the original action Eq. (13). Second, in presence of mixing it can arise after integration in other quantum fluctuations of the meson field δM. Indeed, in this case, the correction to the mean field

$$\delta M = W^{-1}\left[K^a_\Omega \tilde{\Omega}_a + K^i_\Omega \tilde{\omega}_i\right] \tag{18}$$

appears already in the first order in frequencies and should be accounted in the leading order rotational Lagrangian

$$\begin{aligned} S^{(2)}_{\rm rot} &= -\int dt\left[\tfrac{1}{2}\tilde{\Omega}_a \mathcal{I}^{(\Omega\Omega)}_{ab}\tilde{\Omega}_b + \tfrac{1}{2}\tilde{\omega}_i \mathcal{I}^{(\omega\omega)}_{ij}\tilde{\omega}_j + \tfrac{1}{2}\tilde{\Omega}_a \mathcal{I}^{(\omega\Omega)}_{ai}\tilde{\omega}_i\right], \\ \mathcal{I}^{(\Omega\Omega)}_{ab} &= I^{(\Omega\Omega)}_{ab} + K^a_\Omega W^{-1} K^b_\Omega\,, \qquad \mathcal{I}^{(\omega\omega)}_{ij} = I^{(\omega\omega)}_{ij} + K^i_\omega W^{-1} K^j_\omega\,, \\ \mathcal{I}^{(\omega\Omega)}_{ai} &= I^{(\omega\Omega)}_{ai} + K^a_\omega W^{-1} K^a_\Omega + K^a_\Omega W^{-1} K^a_\omega\,, \end{aligned} \tag{19}$$

i.e. the mixing leads to the renormalization of the moments of inertia. It is essential that the terms arising from mixing are of the same order in $N_{\rm c}$ (as $K \sim O(N_{\rm c})$ and $W \sim O(N_{\rm c})$) and contribute to the collective action.

This phenomenon is well-known from the nuclear physics. The approximation where we neglect mixing is called the *cranking* one [24]. The importance of the mixing was pointed out by Thouless–Valatin [25]. The mixing of the rotations and quantum fluctuations, however, is absent in many relativistic theories (at least this is true for models based only on pions). In such theories, the cranking approximation is exact [23].

Cranking moments of inertia $I^{(\Omega\Omega)}_{ab}$, $I^{(\omega\omega)}_{ij}$, $I^{(\omega\Omega)}_{ai}$ also consist of two parts, fermion and meson ones. The second one is the result of substitution of the rotated meson fields Eq. (11) into the meson Lagrangian (and substituting the meson fields by their mean-field approximations). If the meson Lagrangian $\mathcal{L}$ contains some time derivatives, this substitution produces terms quadratic in frequencies $\tilde{\Omega}$, $\tilde{\omega}$ (one should neglect the higher terms) and, therefore, contributes to the moments of inertia. The quark part of moments of inertia can be obtained directly expanding fermion determinant of Eq. (13) in $\tilde{\Omega}$, $\tilde{\omega}$. Corresponding part of the moments of inertia is given by well-known Inglis expression [23].

Hedgehog symmetry of the mean field leads to the following relations for tensor of moments of inertia:

$$\mathcal{I}^{(\Omega\Omega)}_{ab} = \begin{cases} I_1\delta_{ab}, & a,b = 1\ldots 3\,, \\ I_2\delta_{ab}, & a,b = 4\ldots 7\,, \\ 0, & a,b = 8\,, \end{cases} \qquad \mathcal{I}^{(\omega\Omega)}_{ai} = -2I_1\delta_{ai}\,, \qquad \mathcal{I}^{(\omega\omega)}_{ij} = I_1\delta_{ij}\,, \tag{20}$$

and hence the quadratic part of the rotational action reduces to

$$S^{(2)}_{\rm rot} = -\int dt \sum_{i=1}^{3} \frac{I_1}{2} \left(\tilde{\Omega}_i - \tilde{\omega}_i\right)^2 + \sum_{a=4}^{7} \frac{I_2}{2} \tilde{\Omega}_a^2 \,. \tag{21}$$

The complete rotational Lagrangian is

$$\mathcal{L}_{\rm rot} = \sum_{i=1}^{3} \frac{I_1}{2} \left(\tilde{\Omega}_i - \tilde{\omega}_i\right)^2 + \sum_{a=4}^{7} \frac{I_2}{2} \tilde{\Omega}_a^2 + \frac{BN_{\rm c}}{2\sqrt{3}} \tilde{\Omega}_8 \,. \tag{22}$$

This is a Lagrangian for spherical top both in the flavor and usual space. We calculate operators of angular momentum $\tilde{\boldsymbol{J}}$ and flavor momenta $\tilde{\boldsymbol{T}}$

$$\begin{aligned} \tilde{\boldsymbol{J}} &= -\tfrac{1}{2}\,\mathrm{Tr}\left[\mathcal{S}\boldsymbol{\sigma}\frac{\delta}{\delta\mathcal{S}}\right] = \frac{\partial\mathcal{L}_{\rm rot}}{\partial\boldsymbol{\omega}} = I_1(\boldsymbol{\omega} - \Omega)\,, \\ \tilde{T}_a &= -\tfrac{1}{2}\,\mathrm{Tr}\left[R\lambda_a\frac{\delta}{\delta R}\right] = \frac{\partial\mathcal{L}_{\rm rot}}{\partial\Omega_a} = \begin{cases} I_1(\Omega_a - \omega_a)\,, & a = 1\ldots 3\,, \\ I_2\Omega_a\,, & a = 4\ldots 7\,, \\ \frac{N_{\rm c}}{2\sqrt{3}}\,, & a = 8\,. \end{cases} \end{aligned} \tag{23}$$

The following quantization rules applied to the rotational bands of ground state baryons:

$$\tilde{\boldsymbol{J}} + \tilde{\boldsymbol{T}} = 0\,, \qquad \tilde{T}_8 = \frac{N_{\rm c}}{2\sqrt{3}}\,. \tag{24}$$

The second is celebrated Witten quantization rule [8] which claims that hypercharge in the body-fixed frame is $\tilde{Y} = \frac{2}{\sqrt{3}}\tilde{T}_8 = N_{\rm c}/3$. It is completely due to the hedgehog symmetry and the fact that $N_{\rm c}$ valence quarks with the hypercharge $\tilde{Y} = 1/3$ are put to some bound state in the sector of u, d quarks.

The Hamiltonian of rotations determined from Eq. (22) should be expressed in terms of momenta $\tilde{T}$, $\tilde{J}$

$$\mathcal{H}_{\rm rot} = \sum_{a=1}^{3} \frac{\tilde{T}_a^2}{2I_1} + \sum_{a=4}^{7} \frac{\tilde{T}_a^2}{2I_2} = \frac{c_2(r) - \tilde{T}\left(\tilde{T}+1\right) - \frac{3}{4}\tilde{Y}}{2I_2} + \frac{\tilde{T}\left(\tilde{T}+1\right)}{2I_1}\,. \tag{25}$$

Here, $c_2(r) = \sum_a \tilde{T}_a^2$ is Casimir operator in the given SU(3) representation r. It is easy to determine also the collective wave function which is an eigenfunction of the Hamiltonian and operators of momenta in the lab fixed frame

$$T_a = -\frac{1}{2}\,\mathrm{Tr}\left[\lambda_a R \frac{\delta}{\delta R}\right], \qquad \boldsymbol{J} = -\frac{1}{2}\,\mathrm{Tr}\left[\sigma\mathcal{S}\frac{\delta}{\delta\mathcal{S}}\right]. \tag{26}$$

Wave function is a product of two Wigner $\mathcal{D}$-functions, one for SU(3) and one for SU(2) group

$$\begin{aligned}\Psi_{\rm rot}\left(R,\mathcal{S}\right) &= \sqrt{\dim(r)(2J+1)}\sum_{\tilde T,\tilde T_3,} C^{00}_{\tilde T\tilde T_3\; J\tilde T_3}\mathcal{D}^{(r)}_{\tilde Y\tilde T\tilde T_3;YTt_3}\left(R^+\right)\mathcal{D}^{J}_{\tilde J_3;J_3}\left(\mathcal{S}^+\right)\\ &= \sqrt{\dim(r)}(-1)^{J+J_3}\mathcal{D}^{(r)}_{\tilde Y J,-J_3;YTT_3}\left(\mathcal{S}R^+\right)\,. \end{aligned}\tag{27}$$

This function is an eigenfunction of spin $\boldsymbol{J}^2=\tilde{\boldsymbol{J}}^2=\tilde{\boldsymbol{T}}^2$, J_3, isospin $\boldsymbol{T}^2$ and T_3 and hypercharge Y, index (r) labels the SU(3) representations with dimension $\dim(r)$. According to Eq. (24), the hypercharge $\tilde Y=N_{\rm c}/3$. At last, Clebsh–Gordan coefficients $C^{00}_{\tilde T\tilde T_3\; J\tilde J_3}$ sum isospin $\tilde{\boldsymbol{T}}$ and angular momentum $\tilde{\boldsymbol{J}}$ to zero in order to obey the other quantization rule Eq. (24). In fact, rotational wave function depends only on the combination RS^+. This is natural because owing to the hedgehog symmetry flavor isospin rotation can be compensated by space one.

4.2. Rotational bands of 1-particle excitations

Let us proceed now with 1-quark excitations, *i.e.* excitations where only one quark out of $N_{\rm c}$ is taken from the ground level and is put to some excited one. The effective Lagrangian Eq. (13) is only slightly changed: one term in the sum over the $N_{\rm c}$ quarks has different scheme of occupied levels. The other $N_{\rm c}-1$ terms, however, remain the same. This means that in the leading order in $N_{\rm c}$, the mean field does not change (the correction to the mean field is $O(1/N_{\rm c})$). This is also true for moments of inertia I_1 and I_2 — they acquire corrections $O(1)$ as compared to the leading order $O(N_{\rm c})$. Hence, the effective rotational Lagrangian Eq. (22) remains the same. However, additional *linear* terms in frequencies Ω and ω can appear. The reason is that the mean field is a solution of equations of motion for the ground state only, and not for excited ones. Hence, there is no theorem that linear terms in the perturbation (which is rotation in this case) should be absent. The corresponding linear terms are of the form of

$$\delta\mathcal{L}_{\rm rot}=\langle{\rm excited}|(\boldsymbol{\omega}\cdot\boldsymbol{j}+\boldsymbol{\Omega}\cdot\boldsymbol{t})|{\rm excited}\rangle+\langle{\rm excited}|\delta M|{\rm excited}\rangle\,,\tag{28}$$

where the second term accounts for possible change of the contribution of the correction Eq. (18) to the mean field as due to rotations. This correction should be also calculated only in the ground state (it is determined mainly by rotation of other $N_{\rm c}-1$ quarks) and assumed to be already known. It is also linear in frequencies $\tilde\omega,\tilde\Omega$.

Excited states are usually degenerate. Indeed, excitations in s-quark sector have degeneracy $2S+1$ (where $S = \frac{1}{2}, \ldots$ is a total momentum of the state), excitations in sector u, d quarks are degenerate $2K+1$ fold where K is the grand spin of the state. Any of degenerate states or their mixture can be taken as an excitation. We define

$$|\text{excited}\rangle = \sum \chi_{K_3} |KK_3JL\rangle \tag{29}$$

(we are dealing now, for definiteness, with the excitation to some $K \neq 0$) where $|KK_3JL\rangle$ is the wave function of some excited state with grand spin K, and projection K_3 and χ_{K_3} are amplitudes for different values of projection. Energy does not depend on χ_{K_3}. Hence, it is a new zero mode and should be considered as collective coordinate together with S and R. Effective rotational Lagrangian should be written for χ_{K_3} slowly changing with time, evidently, the complete Lagrangian is

$$\mathcal{L}_{\text{excited}}[\chi, R, S] = \sum_{K_3} \chi^+_{K_3} i \frac{\partial}{\partial t} \chi_{K_3} + \mathcal{L}_{\text{rot}} + \delta\mathcal{L}_{\text{rot}} \,, \tag{30}$$

where $\mathcal{L}_{\text{rot}}$ is the rotational Lagrangian for the ground state, Eq. (22).

Plugging Eq. (29) into Eq. (28), we obtain

$$\begin{aligned} \delta\mathcal{L}_{\text{rot}} \;=\; \sum_{K_3 K_3'} \chi^+_{K_3'} \chi_{K_3} \Big[&\langle KK_3JL|(\boldsymbol{\omega} \cdot \boldsymbol{j} + \boldsymbol{\Omega} \cdot \boldsymbol{t})|KK_3'JL\rangle \\ &+(\boldsymbol{\omega} - \boldsymbol{\Omega}) \langle KK_3'JL| \frac{\partial \delta M}{\partial \boldsymbol{\omega}} |KK_3JL\rangle \Big] \,. \end{aligned} \tag{31}$$

We used here a property that due to the hedgehog symmetry of the ground state, the rotational Lagrangian should depend only on the difference of flavor and space frequencies: $\delta M \sim \boldsymbol{\omega} - \boldsymbol{\Omega}$.

One-quark flavor momentum $\boldsymbol{t}$ and angular momentum $\boldsymbol{j}$ are not conserved in the hedgehog field. Nevertheless, as they transformed as vectors under simultaneous flavor and spin rotations, their matrix elements should be proportional to the matrix elements of the conserved quantity — grand spin $\boldsymbol{K}$

$$\begin{aligned} \langle KK_3jl|\, \boldsymbol{t} \,\big|KK_3'jl\big\rangle \;&=\; a_K \,\langle KK_3|\, \boldsymbol{K} \,\big|KK_3'jl\big\rangle \,, \\ \langle KK_3jl|\, \boldsymbol{j} \,\big|KK_3'jl\big\rangle \;&=\; (1 - a_K)\,\langle KK_3|\, \boldsymbol{K} \,\big|KK_3'jl\big\rangle \,, \\ \langle KK_3jl|\, \frac{\partial \delta M}{\partial \boldsymbol{\omega}} \,\big|KK_3'jl\big\rangle \;&=\; \zeta\,\langle KK_3jl|\, \boldsymbol{K} \,\big|KK_3'jl\big\rangle \,, \end{aligned} \tag{32}$$

where a_K and ζ_K are some constants specific for a given excited level. The expression for them can be found in [23]. Coefficient ζ depends on the form of meson Lagrangian. It renormalizes the coefficient a_K. Fortunately, the correction to the mean field δM is zero in the wide class of theories.

Collecting all terms, we obtain the collective Lagrangian for 1-quark excitations in the sector of u, d quarks

$$
\begin{aligned}
\mathcal{L}_K[\chi, R, S] &= \sum_{K_3} \chi^+_{K_3} i \frac{\partial \chi_{K_3}}{\partial t} + \frac{N_c}{2\sqrt{3}} \tilde{\Omega}_8 \\
&\quad + \left[(1-\tilde{a}_K)\boldsymbol{\omega} + \tilde{a}_K \Omega\right] \sum_{K_3 K_3'} \chi^+_{K_3'} \chi_{K_3} \left\langle K K_3 j l \right| \boldsymbol{K} \left| K K_3' j l \right\rangle \\
&\quad + \sum_{i=1}^{3} \frac{I_1}{2} \left(\tilde{\Omega}_i - \tilde{\omega}_i\right)^2 + \sum_{a=4}^{7} \frac{I_2}{2} \tilde{\Omega}_a^2
\end{aligned} \tag{33}
$$

($\tilde{a}_K = a_K - \zeta$). Quantization of χ_{K_3} with Lagrangian (33) is trivial. Due to the presence of collective variable χ_{K_3}, the quantity

$$
\sum_{K_3 K_3'} \chi^+_{K_3} \chi_{K_3'} \left\langle K K_3 j l \right| \boldsymbol{K} \left| K K_3' j l \right\rangle = \hat{\boldsymbol{K}} \tag{34}
$$

behaves as a quantum operator of angular momentum K. Differentiating over ω, Ω, we obtain momenta in the body fixed frame

$$
\begin{aligned}
\tilde{\boldsymbol{J}} &= I_1(\boldsymbol{\omega} - \Omega) + (1-\tilde{a}_K)\hat{\boldsymbol{K}}\,, \qquad & \tilde{\boldsymbol{T}} &= I_1(\boldsymbol{\Omega} - \boldsymbol{\omega}) + \tilde{a}_K \hat{\boldsymbol{K}}\,, \\
\tilde{T}_a &= I_2 \Omega \quad (a = 4\ldots 8)\,, & \tilde{T}_8 &= \frac{N_c}{2\sqrt{3}}\,.
\end{aligned} \tag{35}
$$

It leads to the following quantization conditions instead of Eq. (24):

$$
\tilde{\boldsymbol{T}} + \tilde{\boldsymbol{J}} = \hat{\boldsymbol{K}}\,, \qquad \tilde{Y} = \frac{N_c}{3}\,. \tag{36}
$$

Constructing now the Hamiltonian from the Lagrangian Eq. (33), we obtain

$$
\mathcal{H}_K = \frac{1}{2I_2} \sum_{a=4}^{7} \left(\tilde{T}_a\right)^2 + \frac{\left(\tilde{\boldsymbol{T}} - \tilde{a}_K \hat{\boldsymbol{K}}\right)^2}{2I_1} = \frac{1}{2I_2} \sum_{a=4}^{7} \left(\tilde{T}_a\right)^2 + \frac{\left(\tilde{\boldsymbol{T}} - \tilde{a}_K \left(\tilde{\boldsymbol{J}} + \tilde{\boldsymbol{T}}\right)\right)^2}{2I_1}\,. \tag{37}
$$

Energy levels are

$$
\begin{aligned}
\mathcal{E}_K &= \frac{c_2(r) - \tilde{T}\left(\tilde{T}+1\right) - \frac{3}{4}\tilde{Y}^2}{2I_2} \\
&\quad + \frac{1}{2I_1}\left[(1-\tilde{a}_K)\tilde{T}\left(\tilde{T}+1\right) + \tilde{a}_K J(J+1) - \tilde{a}_K(1-\tilde{a}_K) K(K+1)\right].
\end{aligned} \tag{38}
$$

We used here that $\tilde{J} = J$. Available spins are determined by the quantization rule Eq. (36): $J = |\tilde{T} - K| \ldots \tilde{T} + K$.

It is easy to construct the collective wave function. For this case, it depends on S, R and χ_{K_3}

$$\begin{aligned}\Psi_K(R,\mathcal{S},\chi) \;&=\; \sqrt{\frac{\dim(r)(2J+1)}{2K+1}}\\ &\times \sum_{\tilde{T},\tilde{T}_3,} C^{KK_3}_{\tilde{T}\tilde{T}_3\,J\tilde{J}_3}\mathcal{D}^{(r)}_{\tilde{Y}\tilde{T}\tilde{T}_3;YTT_3}\left(R^+\right)\mathcal{D}^{J}_{\tilde{J}_3;J_3}\left(\mathcal{S}^+\right)\chi_{K_3}\,. \quad (39)\end{aligned}$$

This wave function is an eigenfunction of hypercharge Y, isospin T and its projection T_3 as well as spin J and its projection J_3. In fact, it is completely fixed by the symmetry and quantization requirements (36).

At last, let us describe excitations in the sector of s quarks. Let us assume that we consider the 1-quark excitation where one quark is taken from ground state $K = 0$ and put to the level for s-quark with some total angular momentum $\mathcal{S}$. Excited state is $2\mathcal{S}+1$ fold degenerate, we take the mixture

$$|\text{excited}\rangle = \sum_{S_3}\chi_{S_3}|S_3\rangle\,, \quad (40)$$

where $|S_3\rangle$ are one-quark wave functions with different projections of $\boldsymbol{S}$. The calculation of matrix elements gives now instead of Eq. (32)

$$\langle S_3|\,\boldsymbol{j}\,\left|S_3'\right\rangle = \langle S_3|\,\boldsymbol{S}\,\left|S_3'\right\rangle\,, \qquad \langle S_3|\,\frac{\partial\delta M}{\partial\boldsymbol{\omega}}\,\left|S_3'\right\rangle = \zeta_S\,\langle S_3|\,\boldsymbol{S}\,\left|S_3'\right\rangle \quad (41)$$

and matrix elements of $\boldsymbol{t}$ are zero as s-quark does not carry isospin. Thus, these matrix elements are those of Eq. (32) with $a_K = 0$. It is straightforward to proceed now. The quantization rule of Eq. (36) changes now to

$$\tilde{\boldsymbol{T}}+\tilde{\boldsymbol{J}} = \hat{\boldsymbol{S}}\,, \qquad \tilde{Y} = \frac{N_{\rm c}-3}{3}\,. \quad (42)$$

The first rule is the one from Eq. (36) with evident substitution $\boldsymbol{K}\to\boldsymbol{S}$. The second rule appears because we substitute the quark with hypercharge $1/3$ (one of u, d quarks in the ground state) by one s-quark (on excited level) with hypercharge $-2/3$. This rule can be also derived by calculating directly a coefficient in front of the Wess–Zumino–Witten term.

Levels of energy for s-quark excitations can be obtained from Eq. (39) with substitution $K\to S$ and $a_K=0$. Available spins are $J=|\tilde{T}-S|\ldots\tilde{T}+S$; collective wave function is an analogue of Eq. (39)

$$\begin{aligned}\Psi_S(R,\mathcal{S},\chi) \;&=\; \sqrt{\frac{\dim(r)(2J+1)}{2S+1}}\\ &\times \sum_{\tilde{T},\tilde{T}_3,S_3} C^{SS_3}_{\tilde{T}\tilde{T}_3\,J\tilde{J}_3}\mathcal{D}^{(r)}_{\tilde{Y}\tilde{T}\tilde{T}_3;YTT_3}\left(R^+\right)\mathcal{D}^{J}_{\tilde{J}_3;J_3}\left(\mathcal{S}^+\right)\chi_{S_3}\,. \quad (43)\end{aligned}$$

To summarize: rotational bands around the given excited intrinsic energy should be constructed in the following way. One has to choose SU(3)-multiplets which contain states obeying quantization rule for $\tilde{Y}$, read off the value of $\tilde{T}$ corresponding to this $\tilde{Y}$, and use formulae (25), (39) for their rotational energy.

Quark wave functions in the mean-field approximation are the product of one-particle wave functions of the filled levels. One has to rotate them according to Eq. (10) and then project to collective wave functions obtained above (see Eqs. (27), (39), (43)). ("Projection" means that one has to multiply rotated quark wave function by conjugated collective wave function and integrate over matrices R and S.) This will produce quark wave functions of the excited baryons with given quantum numbers.

Multiplets we are dealing with are not completely arbitrary but obey Witten quantization rule. For these multiplets, one can rewrite formula (39) in the following simple form [29]

$$\begin{aligned}\Delta\mathcal{M} &= \Delta\mathcal{E} \\ &\quad+\frac{1}{2I_1}\left[\tilde{a}_K J(J+1)+(1-\tilde{a}_K)\,\tilde{T}\left(\tilde{T}+1\right)-\tilde{a}_K\,(1-\tilde{a}_K)\,K(K+1)\right] \\ &\quad+\frac{(1+X)\left(2+3\tilde{Y}\right)}{2I_2}+\frac{\tilde{T}_{\max}\left(\tilde{T}_{\max}+1\right)-\tilde{T}\left(\tilde{T}+1\right)}{2I_2}\,. \qquad (44)\end{aligned}$$

Here, $\Delta\mathcal{E}$ is the energy of excited level, $\tilde{Y}$ — hypercharge according to the Witten quantization rule, J — spin of the baryon, K — grand spin of the level. At last, X is *exoticness* of the excited baryon which is defined as a minimal number of quark–antiquark pairs which we have to add to usual N_c quarks. For non-exotic multiplets, $X=0$ and always $\tilde{T}_{\max}=\tilde{T}$.

We see that I_2 plays the role of the moment of inertia for exotic states; their spectrum is equidistant and distances between states are of the order of unity (we remind that $I_2 \sim N_c$ and $\tilde{Y} \sim N_c$). Moment of inertia I_1 governs ordinary excitations splittings [29]. We will not consider the *rotational* exotics here, they have some specifics related to the fact that their width is $\sim O(1)$ [30, 36]. Anyway, they are separated from the normal rotational band by the interval $\sim O(1)$.

We arrive at the following picture of excited states depicted in Fig. 3. Every excited state has a restricted number of *non-exotic* states entering the rotational band with definite $\tilde{T}$. They are determined from the condition that both $p>0$ and $q>0$. In particular, for excitations in sector of s quarks at $N_c=3, \tilde{Y}=0$, we get only one state — singlet with spins $J=S\pm 1/2$ (where S is the spin of excited states) and other multiplets are exotic. Excitations in sectors of u,d quarks have more reach structure. For non-exotic states ($X=0$) at $N_c=3$ and $\tilde{Y}=1$, we have only two possibilities: $\tilde{T} = \frac{1}{2}$ and $\tilde{T} = \frac{3}{2}$. In other words, they can come only as octets or decuplets. At larger N_c, other multiplets become non-exotic. At $K = 0$, we obtain the rotational band of $J = \tilde{T}$ with different spins changing in the limits of $\frac{1}{2} < J < \frac{N_c}{2}$. Their energies are given by general formula Eq. (44) with $X = 0$, $\tilde{a}_K = 0$.

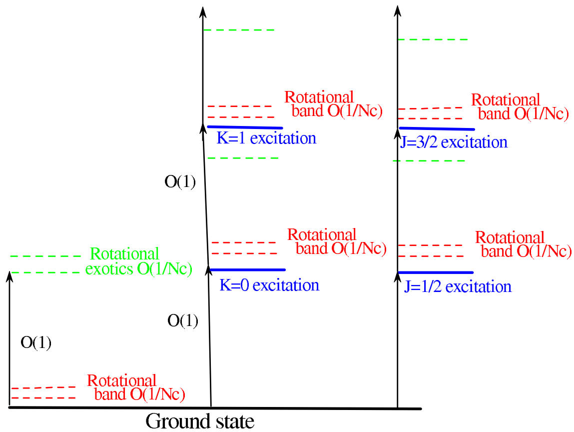

Fig. 3. Structure of excitations in sector of u, d quarks and s quarks.

4.3. Spurious states: Skyrme example

Not all states present in the mean-field approximation are realized in nature. Approximation of the mean field which is valid at $N_c \to \infty$ corresponds to the infinitely heavy nucleon with the center of mass located at the point $\boldsymbol{R} \to 0$. Only in this frame, the wave function of baryon can be presented as the product of one-particle wave functions.

The correct solution should satisfy translational invariance and hence should depend only on the differences of quark coordinates. One can consider the *projection* of the given state to the state with definite total momentum $\boldsymbol{P}$. However, the problem is that there is a *mixing* of the total momentum with *individual angular* momenta of the quarks. For this reason, the total scheme of the quantization should be reconsidered. Together with Pauli principle, this forbids a number of states. These states correspond to the pure movement of the center of mass and are *spurious*.

For the nuclear physics this problem is well-known, for the first time, it was considered in reference [27]. This paper considers the simplest quantum mechanical example: N particles interacting by pair oscillator potential. One can trace that some particular wave functions (after a shift of center-of-mass coordinate) representing excitations of the oscillator wave functions reduce to the ground state wave function multiplied by center-of-mass coordinate $\boldsymbol{r}_c$.

The systematic theory of the center-of-mass motion (in the limit of $N_c \to \infty$) is based on the observation that transformed wave function

$$\begin{aligned}
\tilde{\phi}_n(x)^{\alpha i} &= R^{\alpha}_{\alpha'}(t) S^{i}_{i'}(t) \phi_n^{\alpha' i'}(O(t)\boldsymbol{x} - \boldsymbol{r}_c(t)) \,, \\
O_{ik}(t) &= \tfrac{1}{2} \mathrm{Tr}\left[S^{+}(t)\sigma_k S(t)\sigma_i\right] \,, \\
H_I[\pi^{\mathrm{rot}}((O(t)\boldsymbol{x} &- \boldsymbol{r}_c(t))]\phi_n = \varepsilon_n \phi_n \,,
\end{aligned} \tag{45}$$

where $\boldsymbol{r}_c(\boldsymbol{t})$ is a *slowly* varying function of time. Substituting it into the Hamiltonian leads to the modification of our Hamiltonian

$$\mathcal{S}[C] = -\int dt \int_{C_\chi} \frac{d\epsilon}{2\pi i} \mathrm{Sp}_x \log \gamma^0 \left[\epsilon + i\frac{\partial}{\partial t} - \tilde{\Omega}_a t^a - \tilde{\omega}_i j^i - \boldsymbol{V}_c \cdot \nabla + H_I[\pi]\right] . \tag{46}$$

Here $\boldsymbol{V}_c = \partial_t \boldsymbol{r}_c$ is the speed of the center-of-mass motion.

Let us construct expansion in the center-of-mass velocity $\boldsymbol{V}$. Linear terms of the expansion of Eq. (46) in the soliton velocity

$$\mathcal{S}_V^{\mathrm{linear}} = \boldsymbol{V}_c \sum_{n=\mathrm{filled}} \langle n|\boldsymbol{\nabla}|n\rangle \tag{47}$$

are absent. The quadratic terms are

$$\mathcal{S}_V^{\mathrm{quad}} = \sum \frac{\langle n|\nabla_i|m\rangle\langle m|\nabla_j|n\rangle}{\varepsilon_n - \varepsilon_m} \boldsymbol{V}_i \boldsymbol{V}_j \equiv \tfrac{1}{2}\mathcal{M}_{\mathrm{in}} \boldsymbol{V}^2 \,, \tag{48}$$

where $\mathcal{M}_{\mathrm{in}}$ is an *inertial* mass of the soliton. It is the same in the leading order $O(N_c)$ for one-quark excited baryons as for ground state baryons. Moreover, it can be shown by means of equations of motion that this inertial mass coincides with the gravitational one: $\mathcal{M}_{\mathrm{in}} = \mathcal{M}$.

We return now to the discussion of spurious states, let us consider the same example of the spurious state that is considered in Ref. [27]. The example is α-particle — the nucleus of He^4. In the mean-field approximation, the ground state has a wave function which is a product

$$\Psi_{\mathrm{He}} = \varepsilon^{i_1 i_2 i_3 i_4} \varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\varphi(\boldsymbol{r}_3)\varphi(\boldsymbol{r}_4) \,. \tag{49}$$

Here, we used SU(4)-notation for spin, isospin of the nucleons. φ is a mean-field one-particle wave function which we assume to be the oscillator one. Let us consider now an excited wave function of He^4 with $L = 1$. We can look for such a wave function in the form of

$$\Psi'_{\mathrm{He}} = \varepsilon^{i_1 i_2 i_3 i_4} \mathrm{Sym}_{r_1 \ldots r_4} \left\{\varphi_{L=1}(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\varphi(\boldsymbol{r}_3)\varphi(\boldsymbol{r}_4)\right\} \,. \tag{50}$$

This is the correct wave function and it inevitably appears in the mean-field approximation. It has correct properties under the exchange of nucleons too.

Nevertheless, it is *spurious*. The reason is obvious in the case of oscillator wave functions, where

$$\varphi_{L=1}(\boldsymbol{r}_1) = \boldsymbol{r}_1\varphi(\boldsymbol{r}_1) \tag{51}$$

(different projections $m = -1, 0, 1$ correspond to different components of this vector). Hence,

$$\Psi'_{\mathrm{He}} = \boldsymbol{r}_c\Psi_{\mathrm{He}}\,. \tag{52}$$

In other words, this wave function corresponds to the movement of the center of inertia of He^4.

For oscillator potential $V(r) = \kappa^2 r^2$, it is easy to visualize the difference between mean field and exact solution. Indeed, the mean-field one-particle wave functions Ψ_{mf} (which are the product of one-particle oscillator wave functions) obey Schrödinger equation with the potential

$$V_{\mathrm{mf}} = \kappa^2 N \sum_i^N r_i^2\,, \tag{53}$$

while the correct potential is

$$\begin{aligned} V &= \tfrac{1}{2}\kappa^2 N \sum_{i\neq j}^N (r_i - r_j)^2 \\ &= \kappa^2(N-1)\sum_i^N r_i^2 - \kappa^2 \sum_{i\neq j} \boldsymbol{r}_i\cdot\boldsymbol{r}_j \\ &= \kappa^2(N-1)\sum_i^N r_i^2 - \sum_j^N \kappa^2(N\boldsymbol{r}_{\mathrm{c}} - \boldsymbol{r}_j)\cdot\boldsymbol{r}_j = V_{\mathrm{m.f.}} - N^2\kappa^2 r_{\mathrm{c}}^2\,, \end{aligned} \tag{54}$$

i.e. mean-field potential is

$$V_{\mathrm{mf}} = V + \kappa^2 r_{\mathrm{c}}^2\,. \tag{55}$$

Hence, it differs from the true potential by the some additional term which is, again, the potential of oscillator. Therefore, mean-field approximation produces oscillator wave function as the wave function of the center-of-mass motion. The general mean-field function is a product

$$\Psi_{\mathrm{mf}} = \Psi_{\mathrm{in}}\Psi_{\mathrm{cm}}(r_{\mathrm{c}})\,, \tag{56}$$

where Ψ_{in} is a wave function of the *internal* motion (it depends only on differences of the coordinates) and $\Psi_{\mathrm{cm}}(r_{\mathrm{c}})$ is some (oscillator) wave function of the center-of-mass motion. In the case of He^4, $\Psi_{\mathrm{cm}}(r_{\mathrm{c}})$ is a wave function with $L = 1$ (instead of $L = 0$ for the ground mean-field state), while Ψ_{in} *remains* the function of the ground state. This is why this state is, in fact, *spurious*, it is completely induced by the mean-field approximation.

4.4. Spurious states in rotational bands

When comparing the mean-field predictions (valid at $N_\mathrm{c} \to \infty$) with the data, it should be kept in mind that certain rotational states are in fact spurious, as they are artifacts of the mean-field approximation where the spatial wave function is a product of one-particle wave functions. When averaging over the center of mass is taken into account (which is an $\mathcal{O}(1/N_\mathrm{c})$ effect) the baryon wave functions depend only on the differences of quark coordinates, which for some states may contradict the Pauli principle. The simplest way to identify spurious states is to continuously deform the mean field to the non-relativistic oscillator potential where the wave functions are explicit. Again, they can be written directly projecting rotated mean-field quark wave functions with collective wave functions constructed here. If some state is absent in that limit, it cannot appear from a continuous deformation. An independent way to check for spurious states is to deform the problem at hand to the exactly solvable (0+1)-dimensional four-fermion interaction model [23, 34] where the large-N_c approximation is also possible and reveals extra states.

Specifically, in the parity-plus sector, the spurious state is $(\mathbf{10}, 1/2^+)$ arising from the rotational band about the $(0^+ \to 2^+)$ transition. Such state arises also from the $(0^+ \to 1^+)$ transition but then it is allowed.

In the parity-minus sector, there are more spurious states: the multiplets $(\mathbf{10}, 5/2^-)$ and $(\mathbf{10}, 7/2^-)$ stemming from the $(0^+ \to 2^-)$ transition are spurious, two out of three multiplets $(\mathbf{10}, 3/2^-)$ arising from $(0^+ \to 0^-, 1^-, 2^-)$ transitions are spurious, and one out of two multiplets $(\mathbf{10}, 1/2^-)$ stemming from $(0^+ \to 1^-, 2^-)$ transitions is spurious, too. As it was already said, remaining negative parity multiplets exactly coincide with octets and decuplets from $(\mathbf{70}, \mathbf{1})$ multiplet of SU(6)$\otimes$O(3) of the quark model.

Spurious rotational states should be deleted when comparing with the data.

4.5. Parity-plus resonances

The two lowest multiplets $(\mathbf{8}, 1/2^+, 1152)$ and $(\mathbf{10}, 3/2^+, 1382)$ (the last number in the parentheses is the center of the multiplet) form the rotational band about the ground-state filling scheme shown in Fig. 2. Fitting these masses by Eq. (44), we find $\mathcal{M}_0 + \frac{3}{4I_2} = 1090$ MeV, $1/I_1 = 153$ MeV.

Apart from the two lowest multiplets, there is another low-lying pair with the same quantum numbers, $(\mathbf{8}, 1/2^+, 1608)$ and $(\mathbf{10}, 3/2^+, 1732)$. Other parity-plus multiplets are essentially higher. Therefore, one needs a $0^+ \to 0^+$ transition to explain this pair. From the fit to the masses, one finds that the second $K^P = 0^+$ intrinsic quark level must be 482 MeV higher than the ground state 0^+ level, $\Delta\mathcal{E}(0^+ \to 0^+) = 482$ MeV. The moment of inertia appears to be considerably larger than for the ground-state multiplets, $1/I_1 = 83$ MeV. Although the difference is an $\mathcal{O}(1/N_\mathrm{c})$ effect, it may be enhanced if the radially excited 0^+ level has a much larger effective radius.

TABLE III

Interpretation of all baryon resonances below 2 GeV, as rotational excitations on top of intrinsic quark states.

Quark levels		Rotational bands		$(I_1)^{-1}$ [MeV]	$\tilde{a}_K$
$K^P = 0^+$	ground state	$(\mathbf{8}, 1/2^+, 1152)$	$(\mathbf{10}, 3/2^+, 1382)$	153	
$0^+ \to 0^+$	482 MeV	$(\mathbf{8}, 1/2^+, 1608)$	$(\mathbf{10}, 3/2^+, 1732)$	83	
$0^+ \to 2^+$	722 MeV	$(\mathbf{8}, 3/2^+, 1865)$	$(\mathbf{8}, 5/2^+, 1873)$	131	-0.050
		$(\mathbf{10}, 3/2^+, 2087)$	$(\mathbf{10}, 5/2^+, 2071)$		
		$(\mathbf{10}, 7/2^+, 2038)$			
$0^+ \to 1^+$	~ 780 MeV	$N(1/2^+, 1710)$	$N(3/2^+, 1900)$		
		$\Delta(1/2^+, 1910)$	$\Delta(3/2^+, \sim 1945)$?		
		$\Delta(5/2^+, 2000)$			
$0^+ \to 1^-$	468 MeV	$(\mathbf{8}, 1/2^-, 1592)$	$(\mathbf{8}, 3/2^-, 1673)$	171	0.336
		$(\mathbf{10}, 1/2^-, 1758)$	$(\mathbf{10}, 3/2^-, 1850)$		
$0^+ \to 0^-$	563 MeV	$(\mathbf{8}, 1/2^-, 1716)$		155(fit)	
$0^+ \to 2^-$	730 MeV	$(\mathbf{8}, 3/2^-, 1896)$	$(\mathbf{8}, 5/2^-, 1801)$	155(fit)	-0.244

There is a group of five multiplets, $(\mathbf{8}, 3/2^+, 1865)$, $(\mathbf{8}, 5/2^+, 1873)$, $(\mathbf{10}, 3/2^+, 2087)$, $(\mathbf{10}, 5/2^+, 2071)$, $(\mathbf{10}, 7/2^+, 2038)$, that are good candidates for the rotational band about the $0^+ \to 2^+$ transition. Indeed, this is precisely the content of the rotational band for this transition (the spurious multiplet $(\mathbf{10}, 1/2^+)$ excluded), and a fit to the masses according to Eq. (44) gives a small $\sqrt{\chi^2} = 15$ MeV. It should be kept in mind, though, that not all members of all multiplets are well-established [5], and those that are, have an experimental uncertainty in the masses. It means that the 'experimental' masses for the centers of multiplets are known at best to an accuracy of 20–40 MeV. We find from the fit $1/I_1 = 131$ MeV, $\Delta\mathcal{E}(0^+ \to 2^+) = 722\,\text{MeV}$. Therefore, the intrinsic 2^+ level must be higher than the 0^+ one.

The only relatively well-established multiplet that is left in the range below 2 GeV is $(\mathbf{8}, 1/2^+, 1846)$. It prompts that it can arise from the rotational band about the $0^+ \to 1^+$ transition, however, other parts of the band are poorly known. If one looks into non-strange baryons that are left, one finds $N(1/2^+, 1710^{***})$, $N(1/2^+, 1900^{**})$, $\Delta(1/2^+, 1910^{***})$ and $\Delta(5/2^+, 2000^{**})$, with $\Delta(3/2^+)$ missing. The quantum numbers and the masses of these supposed resonances fit rather well the hypothesis that they arise as a rotational band about the $0^+ \to 1^+$ transition, however, their low status prevents a definite conclusion. The intrinsic 1^+ level must be approximately 60 MeV higher than the 2^+ quark level.

4.6. Parity-minus resonances

The situation here is similar to the parity-plus sector: one needs intrinsic quark levels with $K^P = 0^-, 1^-, 2^-$ to explain the resonances as belonging to rotational bands about these transitions. Given that several rotational states in the parity-minus sector are spurious, one expects to find the following multiplets stemming from these transitions: $(\mathbf{8}, 1/2^-) \times 2$, $(\mathbf{8}, 3/2^-) \times 2$, $(\mathbf{8}, 5/2^-)$, $(\mathbf{10}, 1/2^-)$, $(\mathbf{10}, 3/2^-)$: these are precisely the observed multiplets.

We know that all remaining multiplets are spurious but we do not know the way to assign specific K to the observed one. We attribute them according to Eq. (44) requiring that no mixing can happen ($\zeta = 0$). There is only one way to do this.

We assign the four lowest multiplets $(\mathbf{8}, 1/2^-, 1592)$, $(\mathbf{8}, 3/2^-, 1673)$, $(\mathbf{10}, 1/2^-, 1758)$ and $(\mathbf{10}, 3/2^-, 1850)$ to the rotational band $K = 1^-$ level. The fit tells that corresponding moment of inertia is $1/I_1 = 171$ MeV and the energy of the level is close $\Delta\mathcal{E}(0^+ \to 1^-) = 468$ MeV to $\Delta\mathcal{E}(0^+ \to 0^+)$. This does not look impossible.

The multiplet $(\mathbf{8}, 1/2^-, 1716)$ should be ascribed as $0^+ \to 0^-$ transition and two remaining multiplets $(\mathbf{8}, 3/2^-, 1896)$ and $(\mathbf{8}, 5/2^-, 1801)$ to $0^+ \to 2^-$ transition.

These assignments produce reasonable values of mixing coefficients $\tilde{a}_K$ which can be explained without mixing of rotations and other degrees of freedom in the effective meson Lagrangian. Probably, some other information (mass splittings or resonance widths) should be used to fix finally the attribution of multiplets to the rotational bands. If the final scheme would be different from the assumed here, it will witness the large role of other than pion mesons in formation of negative parity baryons.

To summarize, all parity-plus and parity-minus baryons around 2 GeV and below can be accommodated by the scheme, assuming they all arise as rotational excitations about the $0^+ \to 0^+, 1^+, 2^+$ and $0^+ \to 0^-, 1^-, 2^-$ transitions, see Table I. There are no unexplained resonances left, but there appears an extra state $\Delta(3/2^+, \sim 1945)$ stemming from the $0^+ \to 1^+$ transition, which is so far unobserved, so this state is a prediction.

4.7. Strange quarks

Strange quarks are in a completely different external field than u, d quarks, even in the chiral limit. Only the confining forces which we model by a linear rising scalar field are the same for all quarks. The two excited levels for s quarks are shown in Fig. 3: they are needed to explain the singlet $\Lambda(1/2^-, 1405)$ and $\Lambda(3/2^-, 1520)$ resonances. No more singlet Λs are known below 2 GeV, therefore there should be no intrinsic s-quark levels either with positive or negative parity in this range.

5. Mass splittings

Non-zero mass of the strange quark m_s breaks down SU(3) flavor group and splits SU(3) multiplets. Let us calculate these splittings. Inserting quark mass m (a matrix in flavor) into the quark determinant Eq. (13) and expanding up to the first order in m, we obtain

$$\delta_m S = -i \sum_c \mathrm{Sp}_{\mathrm{occup}} \left\{ R^+ m R \gamma^0 \frac{1}{i\frac{\partial}{\partial t} - \mathcal{H}(M+\delta M) - \tilde{\Omega} t_a - \tilde{\omega}_i j_i} \right\} . \quad (57)$$

Strange quark mass has both singlet and octet part $m = m_0 \mathbf{1} + m_8 \lambda_8$ and $m_8 = ms/\sqrt{3}$ and splittings are determined only by the octet part m_8.

We want to calculate mass splitting in zero and first order in angular and flavor frequencies $\tilde{\omega}$ and $\tilde{\Omega}$. For ground-state baryons, this calculation was carried out in many papers (see, *e.g.* [15, 28]). Result reads

$$\begin{aligned} \delta_m S \;=\; & \frac{m_s}{\sqrt{3}} \Bigg[\mathcal{D}^{(8)}_{8a}(R) \sum_{\mathrm{occup}} \langle n|\lambda_a \gamma^0|n\rangle + 2K_1 \sum_{i=1}^{3} \mathcal{D}^{(8)}_{8i}(R) \left(\tilde{\Omega}_i - \tilde{\omega}_i\right) \\ & + 2K_2 \sum_{a=4}^{7} \mathcal{D}^{(8)}_{8a}(R) \tilde{\Omega}_a \Bigg] . \end{aligned} \quad (58)$$

Here, the first term is of zero order in frequencies, second and third ones represent first order corrections. K_1 and K_2 are some constants analogous to the moments of inertia (see [15]). (If there is a mixing of rotations with δM, expressions from [15] should be modified.) SU(3) Wigner $\mathcal{D}$-functions belong to the adjoint representation of SU(3).

Expression (58) is valid for rotational bands above the ground state and one-particle excitations. The first term for ground state baryons is non-zero only at $a = 8$, it can be expressed through the experimentally known quantity — a so-called Σ-term

$$\sum_c \sum_{\mathrm{occup}} \langle n|\lambda_8 \gamma^0|n\rangle = \frac{1}{3} \frac{m_s}{m_u + m_d} \Sigma \,, \qquad \Sigma = (m_u + m_d) \frac{\partial \mathcal{M}}{\partial (m_u + m_d)} \,. \quad (59)$$

In this relation, it was used that all valence levels are located in the sector of u, d quarks. We will imply below only this case. Indeed, we have seen that at $N_c = 3$, one-particle excitations in the sector of s quarks (from the ground state) are singlets, so there is no mass splitting present for this type of excitations.

In the sector of u, d quarks, there is also another possibility $a = 1, 2, 3$ for an excited level

$$\langle \mathrm{excited}| \frac{\lambda_i}{2} \gamma^0 |\mathrm{excited}\rangle = d_K \sum_{K_3, K_3'} \chi^+_{K_3'} \chi_{K_3} \left\langle K_3' | K_i | K_3 \right\rangle \,, \quad (60)$$

where d_K is some constant which is determined by the wave function of the level (for its calculation, see [23]).

First order terms in frequencies in Eq. (58) can be simplified as well. We substitute frequencies by operators $\tilde{T}_a$ according to Eq. (35). Using relation

$$T_8 = \sum_{a=1}^{8} \mathcal{D}^{(8)}_{8a}(R)\tilde{T}_a \,,$$

one can express the last term in Eq. (58) in terms of the sum with $a = 1, 2, 3$ and hypercharge $Y = 2T_8/\sqrt{3}$. Proceeding to the Hamiltonian

$$\mathcal{H}_m = \alpha\mathcal{D}^{(8)}_{88}(R) + \beta Y + \sqrt{3}\gamma \sum_{i=1}^{3} \mathcal{D}^{(8)}_{8i}(R)\tilde{T}_i + \sqrt{3}\delta \sum_{i=1}^{3} \mathcal{D}^{(8)}_{8i}(R)\hat{K}_i \,, \tag{61}$$

where

$$\begin{aligned} \alpha &= -\frac{2}{3}\frac{m_s}{m_u+m_d}\Sigma + m_s\frac{K_2}{I_2}\,, \qquad && \beta = -m_s\frac{K_2}{I_2}\,, \\ \gamma &= \frac{2m_s}{3}\left(\frac{K_1}{I_1} - \frac{K_2}{I_2}\right), && \delta = \frac{2m_s}{3}\left(d_K - \frac{K_1}{I_1}\tilde{a}_K\right), \end{aligned} \tag{62}$$

we see that the mass splittings are determined by four possible structures. Only the last term is novel, other three are known for ground-state baryons. Moreover, constants α, β, γ up to corrections of the order of $1/N_{\rm c}$ are the same for all levels. As to δ, it is determined by the properties of the excited level and is individual for a given level. Nevertheless, δ is the same for all rotational bands of the given level. Note also that $\alpha \sim O(N_{\rm c})$, while $\beta, \gamma, \delta \sim O(1)$.

Mass splittings are determined by the average of the Hamiltonian (61) over the collective wave functions. Resulting expressions, of course, respect Gell-Mann–Okubo formula. We parametrize the masses of particles in the octet as

$$\begin{aligned} \mathcal{M}_N &= M_8 - \frac{7}{4}\mu_1^{(8)} - \mu_2^{(8)}\,, \qquad && \mathcal{M}_\Lambda = M_8 - \mu_1^{(8)}\,, \\ \mathcal{M}_\Sigma &= M_8 + \mu_1^{(8)}\,, && \mathcal{M}_\Xi = M_8 + \frac{3}{4}\mu_1^{(8)} + \mu_2^{(8)} \end{aligned} \tag{63}$$

and masses of decuplet particles as

$$\begin{aligned} \mathcal{M}_\Delta &= M_{10} - \mu^{(10)}\,, \qquad && \mathcal{M}_\Sigma = M_{10}\,, \\ \mathcal{M}_\Xi &= M_{10} + \mu^{(10)}\,, && \mathcal{M}_\Omega = M_{10} + 2\mu^{(10)}\,. \end{aligned} \tag{64}$$

This parametrization obeys Gell-Mann–Okubo formula automatically. In Table IV, we give the values of μ in terms of α, β, γ, and δ for different values of K and the spin of the multiplet J.

TABLE IV

Mass splittings for octet and decuplet particles for different K.

K	Rep.	J	$\mu_1^{(8)}$	$\mu_2^{(8)}$	$\mu^{(10)}$
0	8	$\frac{1}{2}$	$-\frac{\alpha}{10}-\frac{3\gamma}{20}$	$-\frac{\alpha}{8}-\beta+\frac{5\gamma}{16}$	
	10	$\frac{3}{2}$			$-\frac{\alpha}{8}-\beta+\frac{5\gamma}{16}$
1	8	$\frac{1}{2}$	$-\frac{\alpha}{10}-\frac{11\gamma}{20}-\frac{3\delta}{5}$	$-\frac{\alpha}{8}-\beta+\frac{55\gamma}{48}-\frac{5\delta}{4}$	
		$\frac{3}{2}$	$-\frac{\alpha}{10}+\frac{\gamma}{20}+\frac{3\delta}{10}$	$-\frac{\alpha}{8}-\beta-\frac{5\gamma}{48}-\frac{5\delta}{8}$	
	10	$\frac{1}{2}$			$-\frac{\alpha}{8}-\beta+\frac{35\gamma}{48}+\frac{5\delta}{8}$
		$\frac{3}{2}$			$-\frac{\alpha}{8}-\beta+\frac{23\gamma}{48}+\frac{\delta}{4}$
		$\frac{5}{2}$			$-\frac{\alpha}{8}-\beta+\frac{\gamma}{16}-\frac{3\delta}{8}$
2	8	$\frac{3}{2}$	$-\frac{\alpha}{10}-\frac{3\gamma}{4}-\frac{9\delta}{10}$	$-\frac{\alpha}{8}-\beta+\frac{25\gamma}{16}+\frac{15\delta}{8}$	
		$\frac{5}{2}$	$-\frac{\alpha}{10}+\frac{\gamma}{4}+\frac{3\delta}{5}$	$-\frac{\alpha}{8}-\beta-\frac{25\gamma}{48}-\frac{5\delta}{4}$	
	10	$\frac{1}{2}$			$-\frac{\alpha}{8}-\beta+\frac{17\gamma}{16}+\frac{9\delta}{8}$
		$\frac{3}{2}$			$-\frac{\alpha}{8}-\beta+\frac{13\gamma}{16}+\frac{3\delta}{4}$
		$\frac{5}{2}$			$-\frac{\alpha}{8}-\beta+\frac{19\gamma}{48}+\frac{\delta}{8}$
		$\frac{7}{2}$			$-\frac{\alpha}{8}-\beta-\frac{3\gamma}{16}-\frac{3\delta}{4}$

Experimental values for 19 multiplets below 2 GeV are given in Tables I and II in columns 4 and 5. Accepting some model for soliton Lagrangian, one can immediately calculate α, β, γ, and δ and compare the splitting with experimental data. On the other hand, expressions for mass splittings give rise to the number of relations between masses of the particles entering the same rotational band. These relations are similar to the well-known Guadagnini [33] relation which is valid for the ground state octet and decuplet (see, *e.g.*, [35]). These relation are *model independent*. Let us list them here for $K=2$ rotational band of the baryons with positive parity, for octets and decuplets

$$\begin{aligned} 5\mu_2^{(8)}\left(\tfrac{3}{2}\right)+9\mu^{(10)}\left(\tfrac{5}{2}\right) &= 14\mu^{(10)}\left(\tfrac{3}{2}\right)\,, \\ 5\mu_2^{(8)}\left(\tfrac{5}{2}\right)+11\mu^{(10)}\left(\tfrac{3}{2}\right) &= 16\mu^{(10)}\left(\tfrac{5}{2}\right)\,, \end{aligned} \tag{65}$$

and for decuplets only

$$\begin{aligned} 5\mu^{(10)}\left(\tfrac{7}{2}\right)+7\mu^{(10)}\left(\tfrac{3}{2}\right) &= 12\mu^{(10)}\left(\tfrac{5}{2}\right)\,, \\ 3\mu^{(10)}\left(\tfrac{5}{2}\right)+5\mu^{(10)}\left(\tfrac{1}{2}\right) &= 8\mu^{(10)}\left(\tfrac{3}{2}\right) \end{aligned} \tag{66}$$

(we put in parenthesis the spin of the particles). All these relations work with accuracy better than 10% and some even with accuracy 1–2%.

For $K=1$ (negative parity), we get two relations

$$\begin{aligned} 7\mu^{(10)}\left(\tfrac{1}{2}\right)+3\mu_2^{(8)}\left(\tfrac{3}{2}\right) &= 10\mu^{(10)}\left(\tfrac{3}{2}\right) , \\ 5\mu^{(10)}\left(\tfrac{3}{2}\right)+3\mu_2^{(8)}\left(\tfrac{1}{2}\right) &= 8\mu^{(10)}\left(\tfrac{1}{2}\right) . \end{aligned} \tag{67}$$

While the first is fulfilled with accuracy of 2%, the second one is broken at the level of 10%.

The last relation for $K=0$ (which is precisely Guadagnini's one but for excited baryons) reads: $\mu^{(10)}\left(\frac{3}{2}\right)=\mu_2^{(8)}\left(\frac{1}{2}\right)$. This relation which is satisfied rather well for the ground state octet and decuplet is surprisingly strongly broken for $K=0^+$ excited state.

The situation changes in the strict limit $N_{\mathrm{c}}\to\infty$, in the approach advocated in [16]. The difference is that we have to consider Clebsch–Gordan coefficients in the same limit. They can be extracted, *e.g.*, from [23].

Results demonstrate different N_{c} counting than in previous logic. It appears that mass splittings are not $O(m_s N_{\mathrm{c}})$ but only $O(m_s)$. Both constants α and β enter the leading term, while γ and δ appear in corrections $O(m_s/N_{\mathrm{c}})$. Probably, this picture is more satisfactory from the purely theoretical point of view. Let us note that it coincides with N_{c} counting developed in [16, 17] (all mass relations derived there are also automatically fulfilled here).

Gell-Mann–Okubo relations appear to be still valid. This is not trivial, especially for "decuplets", where not one but two final states at arbitrary N_{c} are available (so it is possible to talk about F- and D-scheme for "decuplets"). However, at large N_{c} Gell-Mann–Okubo relations are restored, up to the order $O(1/N_{\mathrm{c}})$ inclusive (they are not exact in N_{c}!). To save space, we will not fill up the complete table of masses analogous to the table at $N_{\mathrm{c}}=3$. Instead, we write down only mass relations which are independent of the concrete model (some of them were already known). For $K=0$,

$$\mu^{(10)}\left(\tfrac{3}{2}\right)=\mu_2^{(8)}\left(\tfrac{1}{2}\right)-\tfrac{1}{4}\mu_1^{(8)}\left(\tfrac{1}{2}\right) , \tag{68}$$

which substitutes Guadagnini's relation derived at $N_{\mathrm{c}}=3$ (see above). We see that the accuracy of this relation is less than of original one. It is not surprising, as the continuation of the Clebsch–Gordan coefficient introduces a new source of inaccuracy. At $K=1$, there are the following relations:

$$\begin{aligned} 12\mu_2^{(8)}\left(\tfrac{3}{2}\right)-3\mu_1^{(8)}\left(\tfrac{3}{2}\right)+14\mu^{(10)}\left(\tfrac{3}{2}\right) &= 26\mu^{(10)}\left(\tfrac{1}{2}\right) , \\ 12\mu_2^{(8)}\left(\tfrac{1}{2}\right)-3\mu_1^{(8)}\left(\tfrac{1}{2}\right)+20\mu^{(10)}\left(\tfrac{3}{2}\right) &= 32\mu^{(10)}\left(\tfrac{1}{2}\right) . \end{aligned} \tag{69}$$

And at last, for $K=2$,

$$\begin{aligned} 20\mu_2^{(8)}\left(\tfrac{5}{2}\right)-5\mu_1^{(8)}\left(\tfrac{5}{2}\right)+44\mu^{(10)}\left(\tfrac{3}{2}\right) &= 64\mu^{(10)}\left(\tfrac{5}{2}\right) , \\ 20\mu_2^{(8)}\left(\tfrac{3}{2}\right)-5\mu_1^{(8)}\left(\tfrac{3}{2}\right)+34\mu^{(10)}\left(\tfrac{3}{2}\right) &= 54\mu^{(10)}\left(\tfrac{5}{2}\right) \end{aligned} \tag{70}$$

(relation for decuplets is the same as at $N_{\rm c} = 3$). In general, these relations are less accurate than the original ones at $N_{\rm c} = 3$.

6. Decays of excited baryons

Typical decays of excited baryons below 2 GeV are of the type $B_{\rm i} \to B_{\rm f} M$ with one emitted meson, at least such decays always give essential part of the width. To be specific, we will talk about decays into Goldstone octet of π mesons. Let us estimate the width in the limit of large $N_{\rm c}$ (see [18, 23, 37]). One-pion decays of the excited baryons are described by the effective Lagrangian of the type

$$\mathcal{L}_{\rm eff} = \frac{g_a}{F_\pi} \int d^3x \, \bar{\Psi}_B^{(\rm f)} \gamma_\mu \gamma_5 \frac{\lambda^a}{2} \Psi_B^{(\rm i)} \partial_\mu \pi \, . \tag{71}$$

Here, $\Psi_B^{(\rm i)}$ and $\Psi_B^{(\rm f)}$ are fields of initial and final baryon, π is the π-meson field with flavor a, λ_a is the corresponding Gell-Mann matrix. At last, g_a is a transitional axial coupling constant. The width $\Gamma_{\rm fi}$ of partial decay to $B_{\rm f}\pi$ is proportional to this coupling constant squared and the phase volume

$$\Gamma_{\rm fi} \sim \frac{g_a^2}{8\pi F_\pi^2} \Delta^3 \tag{72}$$

(see, *e.g.* [36]), where $\Delta = M_{\rm i} - M_{\rm f}$ is the difference of mass of the initial and final baryon.

The coupling constant can be calculated as a matrix element of the corresponding quark operator between mean-field initial and final state

$$g_a(k) \sim \int d^3x \, \langle {\rm fin} | \bar{\psi} \gamma_5 \gamma_\mu \psi(x) | {\rm in} \rangle e^{ikx} \, . \tag{73}$$

The role of quark operator is played by axial current for decays with π mesons, vector current for decays into ρ mesons, *etc.* Expression (73) already implies the $N_{\rm c} \to \infty$ limit, as baryons are considered to be heavy (mass $O(N_{\rm c})$) non-relativistic objects. (Expression (72) is also written in this limit.) Plane wave e^{ikx} represents the wave function of emitted meson, k being its momentum. At last $|{\rm in}\rangle$ and $|{\rm fin}\rangle$ are mean-field approximations for initial and final baryon quark wave functions. They are product of all 1-quark wave functions — solutions of the Dirac equation in the mean field — for all filled levels. In general, one has to write here wave functions rotated by matrices R and S in order to take into account degeneracy of the mean field. After the calculation of matrix element (73), we obtain some operator depending on collective coordinates. Averaging this operator with collective wave functions of initial and final baryon, we obtain the coupling constant for some specific decay.

In fact, Eq. (73) is only the first term of an expansion in the time derivatives of the collective coordinates. Next terms can be obtained in the same manner as it was done for corrections in m_s. Due to the limit $N_{\rm c} \to \infty$, all collective coordinates are slowly varying functions of coordinates, so this expansion is the expansion in $1/N_{\rm c}$, Eq. (73) being its leading term.

Decays of excited baryons are possible either to baryons belonging to the same rotational band or to the baryons which have the different filling of intrinsic quark levels (*e.g.* to ground state baryons). In the first case, the coupling constant is large, $O(N_{\rm c})$. An example is a transitional axial constant [8]. In the second case, the coupling constant is always smaller. This difference is clearly seen from Eq. (73).

Indeed, since the configuration of levels is the same in the initial and final state, the coupling constant is a sum of $N_{\rm c}$ 1-particle matrix elements corresponding to all $N_{\rm c}$ quarks. If excited quark changes its intrinsic state, then only one of $N_{\rm c}$ contributions survives which is the overlap of one-particle matrix element between initial and final states of this quark (all other are zero due to orthogonality of wave functions). However, if the final state is the ground state, an additional factor $\sqrt{N_{\rm c}}$ appears, which is due to the different normalization of initial and final wave functions

$$g_a(R,S) \sim \int d^3x \phi^*_{\rm f}(\boldsymbol{x}) S^+ \gamma_3\gamma_5 S R^+ \frac{\lambda^a}{2} R \phi_{\rm i}(\boldsymbol{x}) j_l(k|\boldsymbol{x}|) \mathcal{D}^l_{m_1,m_2}(S) Y_{lm_2}\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \times \begin{cases} N_{\rm c} & {\rm i,f} = \text{the same band}\,, \\ \sqrt{N_{\rm c}} & {\rm i} = \text{excited}\,,\ {\rm f} = \text{ground}\,, \\ 1 & {\rm i} = \text{excited}\,,\ {\rm f} = \text{excited}'\,. \end{cases} \tag{74}$$

This expression is written a bit schematically. Wave functions $\psi_{\rm i}$ and $\psi_{\rm f}$ are initial and final wave functions of excited quark, R is a rotational matrix in flavor and S in ordinary space, $\mathcal{D}^l_{m_1 m_2}(S)$ is Wigner function and Y_{lm} is an ordinary spherical harmonics (summation over all possible m_2 is implied). At last, $j_l(kr)$ is a spherical Bessel function. It appears (together with spherical harmonics) as a result of expansion of a plane wave in Eq. (73) into a set of spherical waves. If the momentum of emitted meson is small, $ka \ll 1$ (a is the scale of wave functions $\psi_{\rm i,f}$ which coincides with the characteristic size of the baryon), it is sufficient to account for the lowest angular momentum $l=0$ (angular momentum of emitted pion is 1).

Axial constant (74) is an operator in the space of collective coordinates (derived in the leading order in $N_{\rm c}$). To obtain coupling constant responsible for the decay of a concrete baryon to some other one, we have to average expression (74) over collective wave functions

$$g_a({\rm i} \to {\rm f}) = \int dR dS\, \psi^{(\rm rot)*}_{\rm f}(R,S) g_a(R,S) \psi^{(\rm rot)}_{\rm i}(R,S)\,. \tag{75}$$

Despite of the fact that coupling constant is smaller, the widths of decays to the different quark levels are typically larger in N_c. The reason is that phase volume in this case is always larger. The mass differences are $O(1/N_\mathrm{c})$ for decays inside the same rotational band but $O(1)$ for transitions with change intrinsic state of excited quark. As a result, the width of decays inside the rotational band is suppressed as $O(1/N_\mathrm{c}^2)$, while decays of excited baryons with discharge of the excitation are always $O(1)$ (and decays to the other levels are suppressed). In particular, the total width of ground state baryons (decuplet with spin $\frac{3}{2}$) is only $O(1/N_\mathrm{c}^2)$, while all remaining baryons have the total width of $O(1)$.

In practical terms, only decays to the ground octet or decuplet are observable. For all baryons, they have partial widths independent of N_c up to corrections in $1/N_\mathrm{c}$ which can be still essential at $N_\mathrm{c} = 3$. Let us prove a theorem: widths of all baryons belonging to the same rotational band are the same in the leading order in N_c.

Indeed, mass differences of all baryons entering the same rotational band are the same in the leading order in N_c. Hence

$$\mathit{\Gamma}_{\mathrm{tot}}^{(\mathrm{i})} = \sum_{\mathrm{f}} \mathit{\Gamma}(\mathrm{i} \to \mathrm{f}) = \frac{\mathit{\Delta}^3}{8\pi} \sum_{\mathrm{f}} g_a^2(\mathrm{i} \to \mathrm{f}) = \mathit{\Gamma}_{\mathrm{level}} \,. \tag{76}$$

However, the sum of axial constants squared over all possible final states does not depend on the initial state of the band. According to Eq. (75), axial constant squared contains two integrals over R, S and R', S'. The completeness of final baryon rotational functions,

$$\sum_{\mathrm{f}} \psi_{\mathrm{f}}^{(\mathrm{rot})*}(R,S)\psi_{\mathrm{f}}^{(\mathrm{rot})}\left(R',S'\right) = \delta\left(R - R'\right)\delta\left(S - S'\right) \,,$$

leads to $R = R'$ and $S = S'$. Then, the sum over all possible flavors of pseudoscalar mesons and directions of axial current gives the expression which does not depend on R and S due to Fiertz identities. Dependence on matrices remains only in initial wave function. Integral over R and S is becoming the normalizing integral for initial collective wave function and the dependence on initial state disappears completely. A total width obtained in this way has a sense of the complete width of the intrinsic quark level and is universal for the whole rotational band around it.

The proved theorem is strongly broken in nature. There are many reasons for that: corrections in N_c and mass of the strange quark m_s to the coupling constants, mixing of multiplets, *etc.* Perhaps, the strongest source of the deviations is simply the difference in the phase volumes (which is $O(1/N_\mathrm{c}$ effect) for different baryons entering the same rotational band.

The fact that widths of excited baryons are not suppressed in the large N_c limit makes these baryons not well-defined. One can count only on numerical smallness of the width not related to N_c. In such a situation, baryon resonances can be defined only as poles in the complex plane of meson–nucleon scattering amplitude. This approach was applied in [36] to the problem of pentaquark (which also has width independent of N_c) for the Skyrme model but, in general case, it looks too complicated. If the width is small, one returns to the self-consistent field description presented here.

It seems that width of the baryon which is not suppressed at $N_c \to \infty$ possesses the danger to our approach in general. Indeed, due to unitarity non-zero width implies not only imaginary part of the pole but also a shift in the real part, *i.e.* leads to the change of the baryon mass. It can be small numerically but it is $O(1)$ in N_c. If it is different for the baryons entering the same rotational band, our formulae for mass splittings inside the rotational band become pointless. Fortunately, it is not the case.

Corrections to the mass which are due to decays into the π mesons are presented by the self-energy diagram of Fig. 4. The imaginary part of this diagram gives the width of $B_i \to B_f M$ decay, real part gives the shift of mass. The point is that the mass shift being again the sum over all rotational states does not depend on the baryon B_i from the same rotational band

$$\Delta M \sim \sum_{f} \int \frac{d^4k}{(2\pi)^4} \frac{g_a^2(\mathrm{i} \to \mathrm{f})}{k^2(\Delta + k_0)}$$

as it was proved above. Hence, we arrive at conclusion that the mass shift is universal for all baryons inside the rotational bands. It contributes to the general shift of the given intrinsic level and does not break the mass relations in the $O(1/N_c)$ order. Next order corrections in N_c due to the finite width of the resonance also do not destroy these relations. However, they can renormalize the moment of inertia I_1. An example of such a situation is given by pentaquark in the Skyrme model [36].

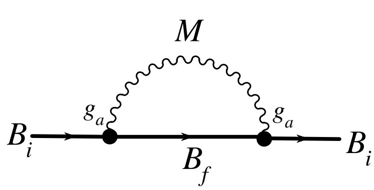

Fig. 4. Self energy correction to the excited baryon mass.

The experimental values for widths to one meson plus ground state baryons are listed in columns 8–10 of Tables I and II. Decays to the ground state octet and decuplet are possible. The first decay is determined by two coupling constants, their ratio F/D is presented in column 9.

There is a number of model-independent, Guadagnini-type relations between coupling constants following from the general expression (74). Consideration of these relations is outside the scope of this paper.

7. Conclusions

We presented here an approach to low-lying baryon resonances which is valid at large number of colors. In this approach, baryons can be described as bound states of quarks in the mean meson field. We believe that it works much better than the usual quark model. The quark model fails to incorporate spontaneous breakdown of chiral symmetry which is the most essential feature of QCD at small energy. The main difference between suggested approach and the quark model is a symmetry of mean field: in the quark model, we deal with mean field which keeps SU(6) symmetry, while the soliton approach starts from the hedgehog symmetry which breaks SU(6) strongly.

All low-lying baryons below 2 GeV can be described as 1-particle excitations in the mean field. There are no extra and no missing states: rotational bands around one-particle excitations fit exactly to the data.

A soliton picture of the baryons gives a self consistent relativistic quantum field description of the properties of baryon resonances. There is a systematic way to calculate quantum corrections (which include creation of additional $q\bar{q}$ pairs). These corrections are small in $1/N_{\mathrm{c}}$ parameter. This parameter introduces a definite hierarchy in the splitting of multiplets.

In reality, N_{c} is only 3 and the above idealistic hierarchy of scales is somewhat blurred. Nevertheless, an inspection of the spectrum of baryon resonances reveals certain hierarchy schematically summarized as follows:

— Baryon mass: $\mathcal{O}(N_{\mathrm{c}})$, numerically 1200 MeV, the average mass of the ground-state octet.

— One-quark and particle-hole excitations in the intrinsic spectrum: $\mathcal{O}(1)$, typically 400 MeV, for example, the excitation of the Roper resonance.

— Splitting between the centers of SU(3) multiplets arising as rotational excitations of a given intrinsic state: $\mathcal{O}(1/N_{\mathrm{c}})$, typically 133 MeV.

— Splitting between the centers of rotational multiplets differing by spin, that are degenerate in the leading order: $\mathcal{O}(1/N_{\mathrm{c}}^2)$, typically 44 MeV.

— Splitting inside a given multiplet owing to the nonzero strange quark mass: $\mathcal{O}(m_s N_{\mathrm{c}})$, typically 140 MeV.

In practical terms, we have shown that all baryon resonances up to 2 GeV, which are made of light quarks, can be understood as rotational excitations about certain transitions between intrinsic quark levels. The quantum numbers of the resonances and the splittings between multiplets belonging to the same rotational band are dictated by the quantum numbers of the intrinsic quark levels, and appear to be in a good agreement with the data.

Of course, the concrete form of the effective meson Lagrangian is unknown. One can try different ideas about this Lagrangian. However, as we have shown, there is a number of relations which are model independent. All these relations agree very well with the data.

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