COEXISTENCE OF MULTIPLE ATTRACTORS IN A NEW CHAOTIC SYSTEM

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In this paper, a new chaotic system with three nonlinear terms and six equilibria is presented. Of particular interest is that the new system has various types of multiple coexisting attractors with respect to different parameters and initial values. The existence of two butterfly chaotic attractors in the system is determined by bifurcation diagrams, Lyapunov exponents and phase portraits. Moreover, the system displays five attractors with either two strange attractors and three limit cycles or one strange attractor and four limit cycles for the same parameter values.

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1. Introduction

Chaos has been gradually attracting attention of the nonlinear science community with the discovery of the notable Lorenz attractor for atmospheric convection in 1963 [1]. After decades of research, scholars revealed many unique features of chaos which can be used for information encryption, weather forecast, fault diagnosis, pathological regulation, *etc.* The broad applications of chaos is just the major reason why chaos research has been proved everlasting.

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The generation of chaos is of great interest in academic field. It is universally acknowledged that first-order, autonomous, continuous ordinary differential equations in at least three dimensions can display chaos. Over the past few decades, a considerable number of three-dimensional chaotic systems have been proposed, such as: Rössler system [2], Sprott system [3], Chen system [4], Lü system [5], Liu system [6], no-equilibria chaotic system [7], multi-wing chaotic system [8], *etc.* With the further research of chaos, scholars found that some simple differential equations can display multiple chaotic attractors with respect to different initial conditions. Liu and Chen proposed a three-dimensional system described as $\dot{x} = ax - yz, \dot{y} = -by + xz, \dot{z} = -cz + xy$ (a, b, c are system parameters), which exhibits two double-scroll chaotic attractors simultaneously [9]. Sprott *et al.* claimed that the Sprott E system [3] with only one stable equilibrium yields point, periodic, chaotic attractors from different initial values [10]. Li *et al.* found a butterfly flow in which two point attractors, one limit cycle, two chaotic attractors coexists [11]. Kengne *et al.* analyzed a jerk system with cubic nonlinearity, and found that the system has two one-scroll chaotic attractors with their own domain of attraction [12]. More research results can be referred to the literature [13–19].

The coexistence of multiple attractors in a system refers to the non-uniqueness of steady states for a given set of system parameters. Each steady state of the system corresponds to an attractor with specific domain of attraction in phase space. An attractor can be a fixed point, a limit cycle or a strange attractor. Nonlinear systems are often unavoidable to generate multiple coexisting attractors. It is really very interesting to propose and analyze some new systems with multiple chaotic attractors. In this letter, we introduce a new three-dimensional autonomous system with three nonlinearities and six unstable equilibria. The dynamical behaviors of the system are closely related to system parameters and initial values. One remarkable feature of the system is that it exhibits multiple coexisting attractors. The attractors can be limit cycles or chaotic attractors. By using numerical simulation, the multiple attractors of the system are investigated.

2. A new chaotic system

Let us consider the following three-dimensional chaotic system [3]

$$\begin{cases} \dot{x} = a(y - x)\,, \\ \dot{y} = xz\,, \\ \dot{z} = k - xy\,. \end{cases} \tag{1}$$

By multiplying a constant b and a nonlinear term $(z^2 - c)$ to the second equation of system (1), a new chaotic system is established as follows:

$$\begin{cases} \dot{x} = a(y - x)\,, \\ \dot{y} = bxz\left(z^2 - c\right)\,, \\ \dot{z} = k - xy\,, \end{cases} \tag{2}$$

where a, b, c, k are real numbers. System (2) has three nonlinearities xy, xz, xz^3. It is obvious that system (2) has a symmetry about the z-axis under the transformation $(x, y, z) \to (-x, -y, z)$. Let $\dot{x} = \dot{y} = \dot{z} = 0$, the equilibria of system (2) are obtained as follows:

$$O_{1,2}\left(\pm\sqrt{k}, \pm\sqrt{k}, 0\right)\,, \qquad O_{3,4}\left(\pm\sqrt{k}, \pm\sqrt{k}, \sqrt{c}\right)\,,$$
$$O_{5,6}\left(\pm\sqrt{k}, \pm\sqrt{k}, -\sqrt{c}\right)\,.$$

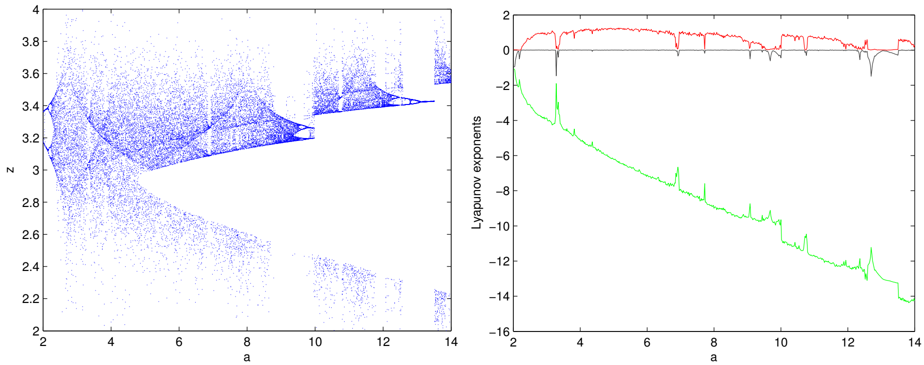
The eigenvalues λ at $O_i(i = 1, 2)$ satisfy $\lambda^3 + a\lambda^2 - bck\lambda - 2abck = 0$ and the eigenvalues λ at $O_i(i = 3, 4, 5, 6)$ satisfy $\lambda^3 + a\lambda^2 + 2bck\lambda + 4abck = 0$. If $a, b, c, k > 0$, then $O_i(i = 1, 2, 3, 4, 5, 6)$ is unstable since its eigenvalues have positive real parts according to the Routh–Hurwitz stability criterion. If $a, b, k > 0$ and $c = 0$, then the eigenvalues at $O_i(i = 1, 2)$ can be calculated as $\lambda_1 = \lambda_2 = 0, \lambda_3 = -a$. The equilibrium $O_i(i = 1, 2)$ is nonhyperbolic and its stability can be determined by using the center manifold theorem. Generally speaking, the number and type of equilibria have an important influence on the dynamic behaviors of a system. The existence of multiple equilibria usually provides a better flexibility to the system.

In the following part, we will investigate the complex dynamical behaviors of system (2), especially the multiple coexisting attractors. The bifurcation diagrams, Lyapunov exponents, phase portraits show that different types of multiple attractors are flash up on system (2) with respect to different parameters and initial values.

3. Multiple attractors

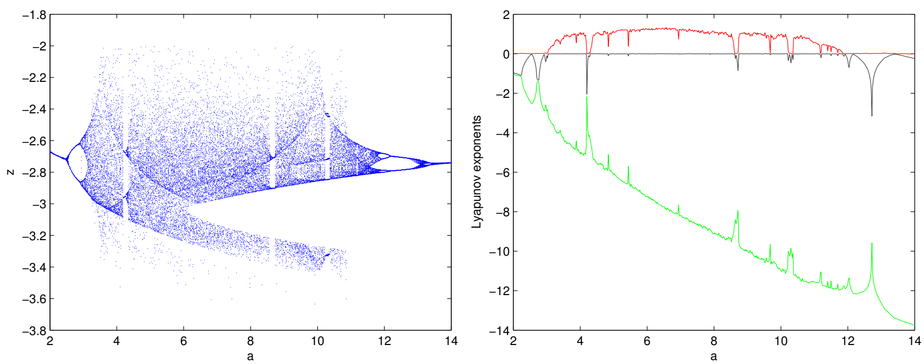
This section shows the multiple attractors of system (2) by numerical simulation on Matlab software platform. The classic fourth-fifth-order Runge–Kutta integrator is applied to get the numerical solutions of system (2). The corresponding step size and iteration time of the integrator are set at $\Delta t = 0.01$ and $t \in [0, 200]$.

Let $b = 2, c = 9, k = 1$, then the bifurcation diagrams and Lyapunov exponents of system (2) *versus* $a \in (2, 14)$ are shown in Figs. 1 and 2. Figure 1 is generated from the initial value $x_{01} = (0.9, 0.9, 2.9)$, while Fig. 2 is generated from the initial value $x_{02} = (0.9, 0.9, -2.9)$. Obviously, Fig. 1 (a) is within the range of $z \in (2, 4)$ and Fig. 2(a) is within the range of $z \in (-1.8, -3.8)$. It means that system (2) has an attractor on phase space $\Delta_1 = \{(x, y, z)\,|z > 0\}$ and an attractor on phase space $\Delta_2 = \{(x, y, z)\,|z < 0\}$. The attractors can be periodic or chaotic for different values of parameter a.



(a) (b)

Fig. 1. Bifurcation diagram and Lyapunov exponents of system (2) with $b = 2$, $c = 9$, $k = 1$ and initial value x_{01}.



(a) (b)

Fig. 2. Bifurcation diagram and Lyapunov exponents of system (2) with $b = 2$, $c = 9$, $k = 1$ and initial value x_{02}.

When $a = 5$, system (2) has six equilibria $S_{1,2}(\pm 1, \pm 1, 0)$, $S_{3,4}(\pm 1, \pm 1, 3)$, $S_{5,6}(\pm 1, \pm 1, -3)$. It is easy to verify that $S_{1,2}$ are saddle foci with unstable index-1 and $S_{3,4}$, $S_{5,6}$ are saddle foci with unstable index-2. Two butterfly chaotic attractors are numerically observed in system (2) with respect to the initial values x_{01} (gray/red color) and x_{02} (black/blue color), as shown in Fig. 3. The gray/red color attractor is located around $S_{3,4}$, while the black/blue color attractor is located around $S_{5,6}$. The Lyapunov exponents of the gray/red color attractor are $L_1 = 1.2199$, $L_2 = 0$, $L_3 = -6.2199$

and its Lyapunov dimension is $D_{\rm L} = 2 - L_1/L_3 = 2.1961$. The Lyapunov exponents of the black/blue color attractor are $L_1 = 1.1084, L_2 = 0, L_3 = -6.1084$ and its Lyapunov dimension is $D_{\rm L} = 2.1815$. The shape and structure of the attractors are very similar.

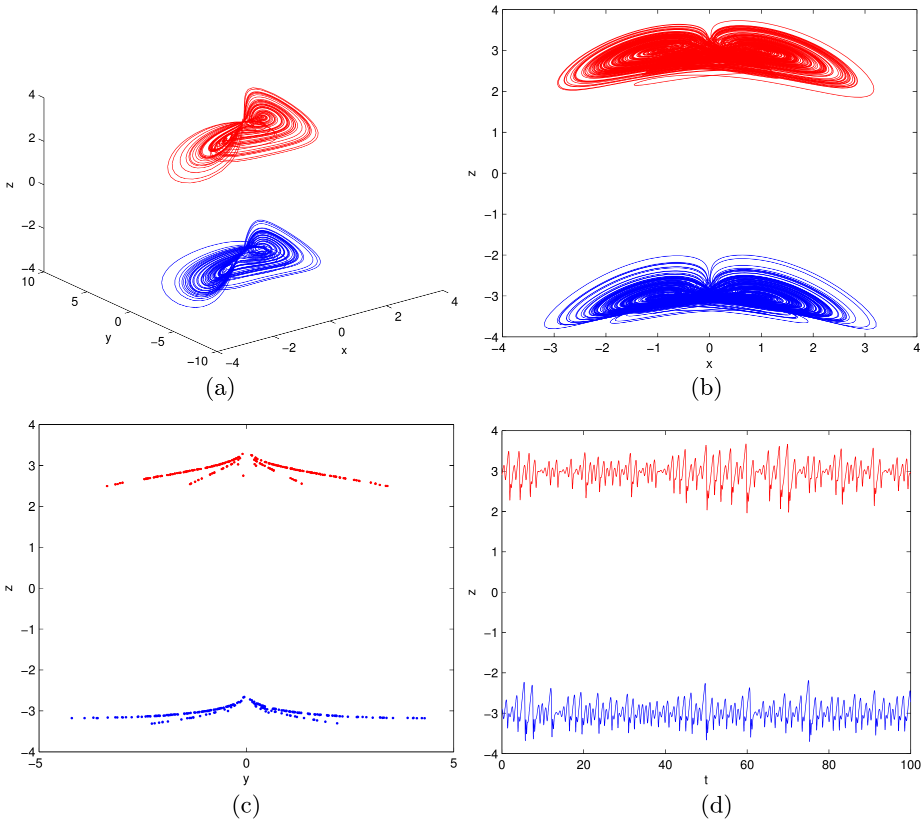
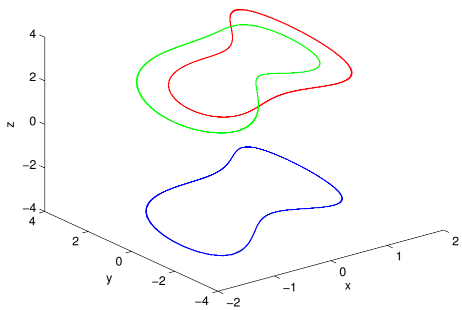


Fig. 3. (Color online) Coexisting two chaotic attractors of system (2) with $a = 5$: (a) $x - y - z$; (b) $x - z$; (c) Poincaré maps; (d) time series of z.

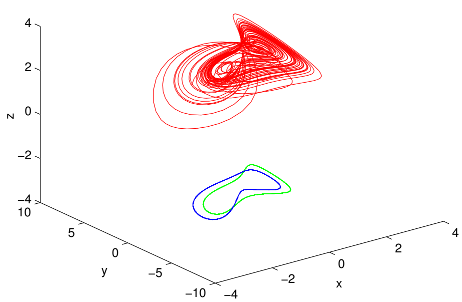
When $a = 2.1$, three limit cycles which yield from the initial values x_{01} (red color) and x_{02} (blue color), $x_{03} = (-0.9, -0.9, 2.9)$ (green color) of system (2) are shown in Fig. 4 (a). When $a = 2.3$, one limit cycle and two chaotic attractors appear in system (2) under initial values x_{01}, x_{02}, x_{03}, as shown in Fig. 4 (b).

When $a = 2.7$, system (2) displays one butterfly chaotic attractor and two limit cycles with respect to initial values x_{01} (red color) and x_{02} (blue color), $x_{04} = (-0.9, -0.9, -2.9)$ (green color), as shown in Fig. 5 (a). Interestingly, the trajectories that start from x_{01}, x_{02}, x_{04} can eventually settle onto three chaotic attractors if system parameter $a = 3.1$, as shown in Fig. 5 (b).



(a) (b)

Fig. 4. (Color online) Coexisting three attractors of system (2) with: (a) $a = 2.1$; (b) $a = 2.3$.

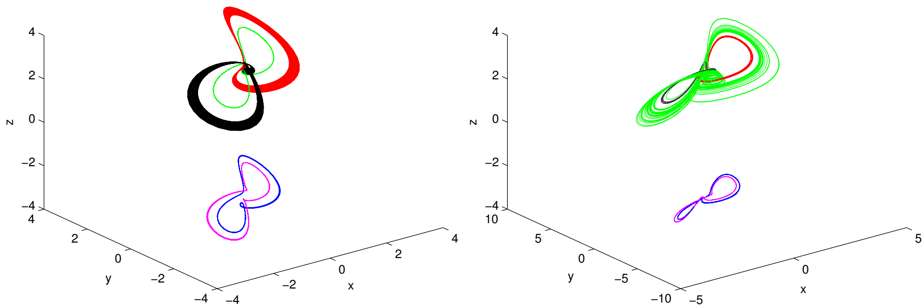


(a) (b)

Fig. 5. (Color online) Coexisting three attractors of system (2) with: (a) $a = 2.7$; (b) $a = 3.1$.

When $a = 12.1$, two chaotic attractors and three limit cycles are observed in system (2) corresponding to the initial values x_{01} (red color), x_{02} (blue color), x_{03} (black color), x_{04} (pink color), $x_{05} = (0.9, 0.9, 2.8)$ (green color), as shown in Fig. 6 (a). Under the same initial conditions, one chaotic attractors and four limit cycles are generated in system (2) with parameter $a = 12.9$, as shown in Fig. 6 (b).

By selecting appropriate parameters and initial values, more types of multiple attractors can be observed, as shown in Table I and Fig. 7. Figure 7 shows that system (2) respectively displays two attractors, three attractors, four attractors and five attractors for different parameter values.



(a) (b)

Fig. 6. (Color online) Coexisting five attractors of system (2) with: (a) $a = 12.1$; (b) $a = 12.9$.

TABLE I

Multiple coexisting attractors.

Types of attractor	Parameters	Initial values	Figure
Two chaotic attractors	$a = 2, b = 2$ $c = 9, k = 10$	$(\pm 0.9, \pm 0.9, -2.9)$	Fig. 7(a)
Two periodic attractors	$a = 2, b = 8$ $c = 4, k = 1$	$(0.1, 0.1, \pm 1.9)$	Fig. 7(b)
One periodic and one chaotic attractors	$a = 3, b = 8$ $c = 4, k = 1$	$(0.1, 0.1, \pm 1.9)$	Fig. 7(c)
One periodic and one chaotic attractors	$a = 20, b = 8$ $c = 4, k = 1$	$(0.1, 0.1, \pm 1.9)$	Fig. 7(d)
One periodic and two chaotic attractors	$a = 11, b = 2$ $c = 9, k = 1$	$(0.9, 0.9, \pm 2.9)$ $(0.9, 0.9, 2.8)$	Fig. 7(e)
Two periodic and one chaotic attractors	$a = 10, b = 4$ $c = 4, k = 1$	$(\pm 0.1, \pm 0.1, 1.9)$ $(0.1, 0.1, -1.9)$	Fig. 7(f)
Two periodic and two chaotic attractors	$a = 20, b = 3$ $c = 8, k = 1$	$(\pm 0.1, \pm 0.1, 3)$ $(0.1, 0.1, \pm 3)$	Fig. 7(g)
Four periodic and one chaotic attractors	$a = 15, b = 3$ $c = 8, k = 1$	$(\pm 0.1, \pm 0.1, 3)$ $(0.1, 0.1, \pm 3)$ $(0.1, 0.1, 2)$	Fig. 7(h)

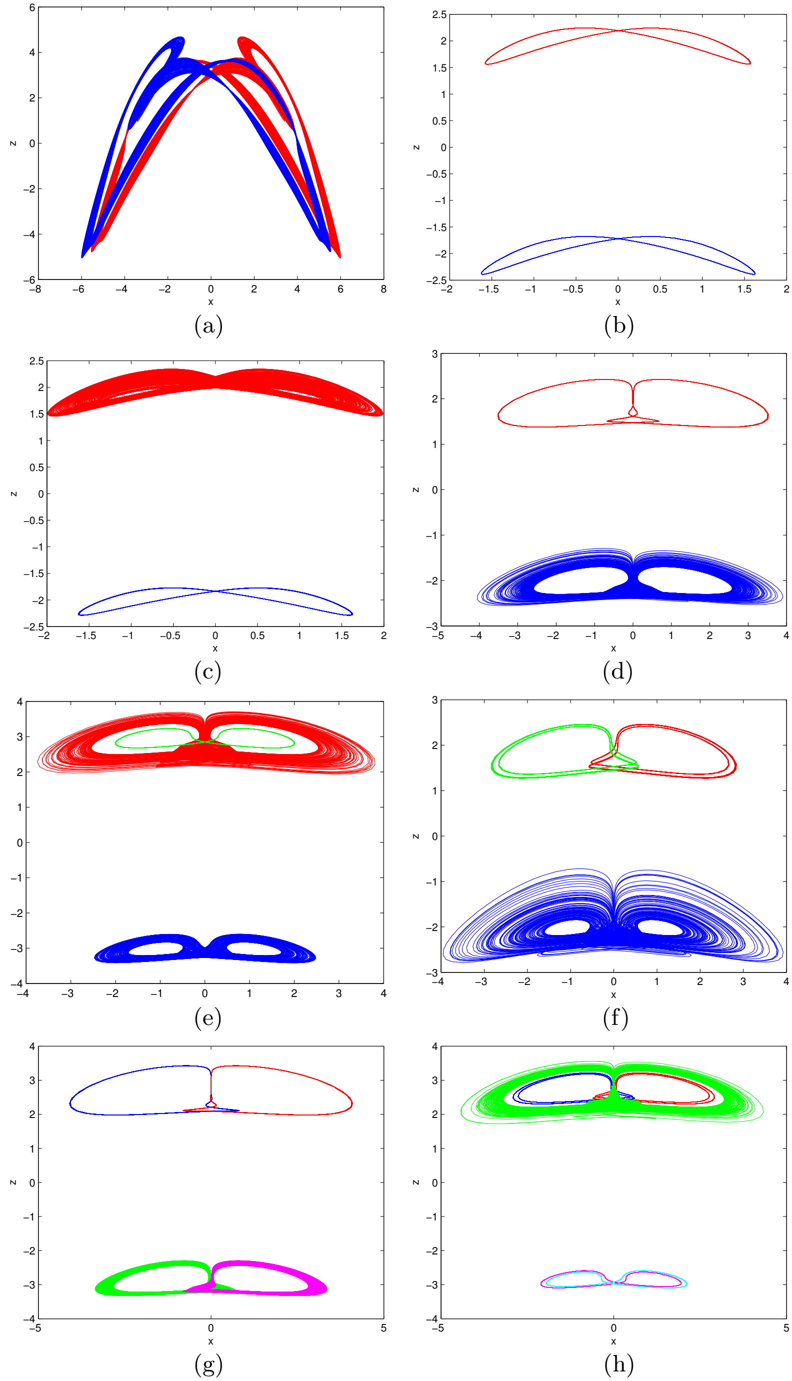


Fig. 7. Multiple coexisting attractors of system (2).

4. Conclusions

This paper proposed a novel three-dimensional continuous chaotic system with three nonlinear terms. The system has two index-1 saddle foci and four index-2 saddle foci. The coexistence of multiple attractors in the system is investigated by bifurcation diagrams, Lyapunov exponents and phase portraits. It is shown that the system displays two chaotic attractors, three chaotic attractors, four attractors with two chaotic attractors and two limit cycles, five attractors either with two chaotic attractors and three limit cycles or one chaotic attractor and four limit cycles, *etc.*

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