

THE HENON–HEILES SYSTEM DEFINED ON
CANONICALLY DEFORMED SPACE-TIME

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In this article, we provide the canonically deformed classical Henon–Heiles system. Further we demonstrate that for proper value of deformation parameter θ , there appears chaos in the model.

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1. Introduction

There exist a lot of papers dealing with physical models of which dynamics remains chaotic; the most popular of them are: Lorentz system [1], Henon–Heiles system [2], Rayleigh–Bernard system [3], Duffing equation [4], double pendulum [5, 6], forced damped pendulum [5, 6] and quantum forced damped oscillator model [7]. The especially interesting seems to be Henon–Heiles system defined by the following Hamiltonian function:

$$H(p, x) = \frac{1}{2} \sum_{i=1}^2 (p_i^2 + x_i^2) + x_1^2 x_2 - \frac{1}{3} x_2^3, \quad (1)$$

which in Cartesian coordinates x_1 and x_2 describes the set of two nonlinearly coupled harmonic oscillators. In polar coordinates r and θ , it corresponds to the particle moving in noncentral potential of the form of

$$V(r, \varphi) = \frac{r^2}{2} + \frac{r^3}{3} \sin(3\varphi), \quad (2)$$

with $x_1 = r \cos \varphi$ and $x_2 = r \sin \varphi$. The above model has been inspired by the observational data indicating that star moving in a weakly perturbed central potential should have apart of constant in time total energy E_{tot} , the

second conserved physical quantity I . It has been demonstrated with the use of the so-called Poincaré section method that such a situation appears in the case of Henon–Heiles system only for small values of control parameter E_{tot} . For high energies, the trajectories in phase space become chaotic and the quantity I does not exist (see *e.g.*, [8, 9]).

In this article, we investigate the impact of the well-known (simplest) canonically deformed Galilei space-time [10–12]^{1,2}

$$[t, \hat{x}_i] = 0, \quad [\hat{x}_i, \hat{x}_j] = i\theta_{ij} \quad (3)$$

on the mentioned above Henon–Heiles system. Particularly, we provide the corresponding canonical equations of motion as well as we find the Poincaré sections of the phase space trajectories of the model. In such a way, we demonstrate that for proper value of deformation parameter θ and for proper values of control parameter E_{tot} , there appears chaos.

The paper is organized as follows. In the second section, we briefly remind the basic properties of classical Henon–Heiles system. In Section 3, we recall canonical noncommutative quantum Galilei space-time proposed in article [12]. Section 4 is devoted to the new canonically deformed Henon–Heiles model, while the conclusions are discussed in Section 5.

2. Classical Henon–Heiles model

As it was already mentioned, the Henon–Heiles system is defined by the following Hamiltonian function

$$H(p, x) = \frac{1}{2} \sum_{i=1}^2 (p_i^2 + x_i^2) + x_1^2 x_2 - \frac{1}{3} x_2^3, \quad (4)$$

with canonical variables (p_i, x_i) satisfying

$$\{x_i, x_j\} = 0 = \{p_i, p_j\}, \quad \{x_i, p_j\} = \delta_{ij}, \quad (5)$$

i.e., it describes the system of two nonlinearly coupled one-dimensional harmonic oscillator models. One can check that the corresponding canonical

¹ The canonically noncommutative space-times have been defined as the quantum representation spaces, so-called Hopf modules (see, *e.g.*, [10, 11]), for the canonically deformed quantum Galilei Hopf algebras $\mathcal{U}_\theta(\mathcal{G})$.

² It should be noted that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [13, 14]), apart of canonical [10–12] space-time noncommutativity, there also exists Lie-algebraic [12–17] and quadratic [12, 17–19] type of quantum spaces.

equations of motion take the form:

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -x_1 - 2x_1x_2, \quad \dot{x}_i = \frac{\partial H}{\partial p_i} = p_i, \quad (6)$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = -x_2 - x_1^2 + x_2^2, \quad (7)$$

while the proper Newton equations look as follows:

$$\begin{cases} \ddot{x}_1 &= -x_1 - 2x_1x_2, \\ \ddot{x}_2 &= -x_2 - x_1^2 + x_2^2. \end{cases} \quad (8)$$

Besides, it is easy to see that the conserved in time total energy of the model is given by

$$E_{\text{tot}} = \frac{1}{2} \sum_{i=1}^2 (\dot{x}_i^2 + x_i^2) + x_1^2x_2 - \frac{1}{3}x_2^3. \quad (9)$$

In order to analyze the discussed system, we find numerically the Poincaré maps in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for five fixed values of total energy: $E_{\text{tot}} = 0.03125$, $E_{\text{tot}} = 0.06125$, $E_{\text{tot}} = 0.10125$, $E_{\text{tot}} = 0.125$, $E_{\text{tot}} = 0.15125$ and $E_{\text{tot}} = 0.16245$, respectively; the obtained results are summarized in figures 1–6. We see that for $E_{\text{tot}} = 0.03125$, the trajectories remain completely regular. However, for increasing values of control parameter E_{tot} , they gradually become disordered until to the almost completely chaotic behavior of the system at $E_{\text{tot}} = 0.16245$ ³.

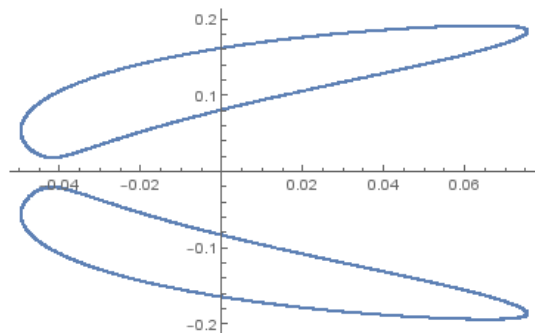


Fig. 1. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for total energy $E_{\text{tot}} = 0.03125$. The trajectory is completely regular — there is no chaos in the system.

³ The calculations are performed for single trajectory with initial condition $x_1(0) = (2E_{\text{tot}})^{\frac{1}{2}}$ and $x_2(0) = p_1(0) = p_2(0) = 0$.

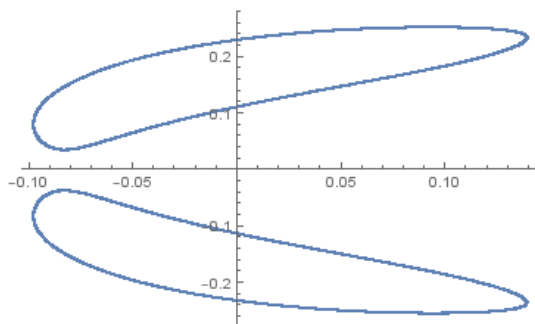


Fig. 2. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the fixed value of total energy $E_{\text{tot}} = 0.06125$. The trajectory is still regular — the system is chaos free.

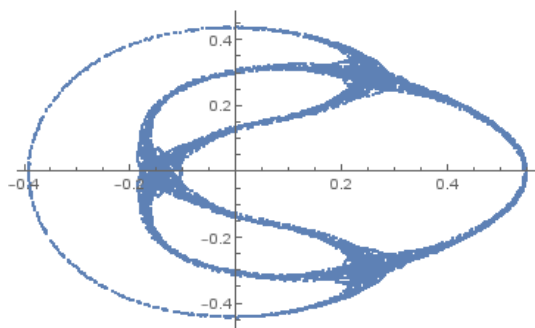


Fig. 3. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.10125$. The trajectory still remains regular.

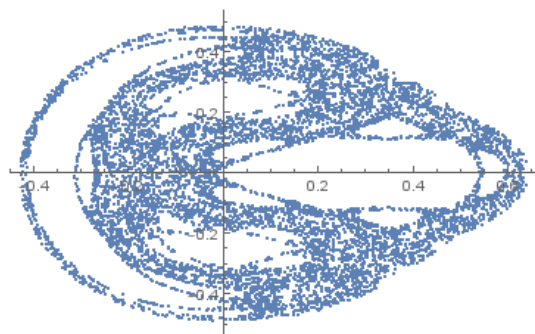


Fig. 4. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.125$. The system becomes mixed: chaotic and ordered simultaneously.

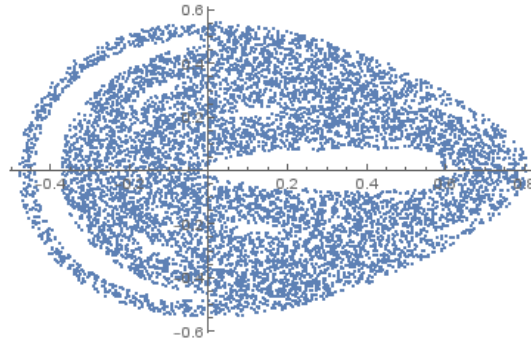


Fig. 5. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.15125$. The system becomes chaotic.

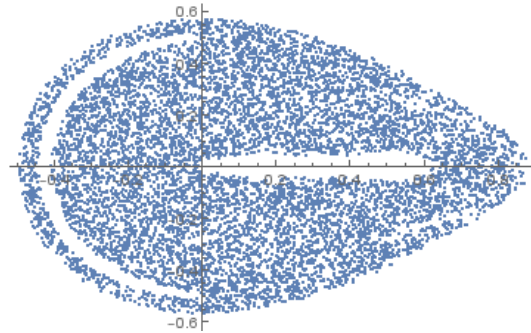


Fig. 6. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.16245$. The chaos increases.

3. Canonically deformed Galilei space-time

In this section, we very shortly recall the basic facts associated with the (twisted) canonically deformed Galilei Hopf algebra $\mathcal{U}_\theta(\mathcal{G})$ and with the corresponding quantum space-time [12]. Firstly, it should be noted that in accordance with the Drinfeld twist procedure [20], the algebraic sector of Hopf structure $\mathcal{U}_\theta(\mathcal{G})$ remains undeformed

$$[K_{ij}, K_{kl}] = i(\delta_{il} K_{jk} - \delta_{jl} K_{ik} + \delta_{jk} K_{il} - \delta_{ik} K_{jl}), \quad (10)$$

$$[K_{ij}, V_k] = i(\delta_{jk} V_i - \delta_{ik} V_j), \quad [K_{ij}, \Pi_\rho] = i(\eta_{j\rho} \Pi_i - \eta_{i\rho} \Pi_j), \quad (11)$$

$$[V_i, V_j] = [V_i, \Pi_j] = 0, \quad [V_i, \Pi_0] = -i\Pi_i, \quad [\Pi_\rho, \Pi_\sigma] = 0, \quad (12)$$

where K_{ij} , Π_0 , Π_i and V_i can be identified with rotation, time translation, momentum and boost operators respectively. Besides, the coproducts and

antipodes of such algebra take the form:

$$\Delta_{\theta}(\Pi_{\rho}) = \Delta_0(\Pi_{\rho}), \quad \Delta_{\theta}(V_i) = \Delta_0(V_i), \quad (13)$$

$$\begin{aligned} \Delta_{\theta}(K_{ij}) = \Delta_0(K_{ij}) - \theta^{kl} [(\delta_{ki}\Pi_j - \delta_{kj}\Pi_i) \otimes \Pi_l \\ + \Pi_k \otimes (\delta_{li}\Pi_j - \delta_{lj}\Pi_i)], \end{aligned} \quad (14)$$

$$S(\Pi_{\rho}) = -\Pi_{\rho}, \quad S(K_{ij}) = -K_{ij}, \quad S(V_i) = -V_i, \quad (15)$$

while the corresponding quantum space-time can be defined as the representation space, so-called Hopf modules (see, *e.g.*, [10, 11]), for the canonically deformed Hopf structure $\mathcal{U}_{\theta}(\mathcal{G})$; it looks as follows:

$$[t, \hat{x}_i] = 0, \quad [\hat{x}_i, \hat{x}_j] = i\theta_{ij} \quad (16)$$

and for the deformation parameter θ approaching zero it becomes commutative.

4. Classical Henon–Heiles system on canonically deformed space-time

Let us now turn to the Henon–Heiles model defined on quantum space-time (16). In the first step of our construction, we extend the canonically deformed space to the whole algebra of momentum and position operators as follows (see, *e.g.*, [21–24])⁴:

$$\{\hat{x}_1, \hat{x}_2\} = 2\theta, \quad \{\hat{p}_i, \hat{p}_j\} = 0, \quad \{\hat{x}_i, \hat{p}_j\} = \delta_{ij}. \quad (17)$$

One can check that relations (17) satisfy the Jacobi identity and for the deformation parameter θ approaching zero become classical.

Next, by analogy to the commutative case, we define the corresponding Hamiltonian function by⁵

$$H(\hat{p}, \hat{x}) = \frac{1}{2} \sum_{i=1}^2 (\hat{p}_i^2 + \hat{x}_i^2) + \hat{x}_1^2 \hat{x}_2 - \frac{1}{3} \hat{x}_2^3, \quad (18)$$

with the noncommutative operators $(\hat{x}_i, \hat{p}_i)$ represented by the classical ones (x_i, p_i) as [24–26]

$$\hat{x}_1 = x_1 - \theta p_2, \quad (19)$$

$$\hat{x}_2 = x_2 + \theta p_1, \quad (20)$$

$$\hat{p}_1 = p_1, \quad \hat{p}_2 = p_2. \quad (21)$$

⁴ The correspondence relations are $\{\cdot, \cdot\} = \frac{1}{i} [\cdot, \cdot]$.

⁵ Such a construction of deformed Hamiltonian function (by replacing the commutative variables (x_i, p_i) by noncommutative ones $(\hat{x}_i, \hat{p}_i)$) is well-known in the literature — see, *e.g.*, [21–23].

Consequently, we have

$$H(p, x) = \frac{1}{2M(\theta)} (p_1^2 + p_2^2) + \frac{1}{2}M(\theta)\Omega^2(\theta) (x_1^2 + x_2^2) - S(\theta)L \\ + (x_1 - \theta p_2)^2 (x_2 + \theta p_1) - \frac{1}{3} (x_2 + \theta p_1)^3, \quad (22)$$

where

$$L = x_1 p_2 - x_2 p_1, \quad (23)$$

$$1/M(\theta) = 1 + \theta^2, \quad (24)$$

$$\Omega(\theta) = \sqrt{1 + \theta^2} \quad (25)$$

and

$$S(\theta) = \theta. \quad (26)$$

Further, using formula (22), one gets the following canonical Hamiltonian equations of motion

$$\dot{x}_1 = \frac{1}{M(\theta)} p_1 + S(\theta) x_2 + \left[(x_1 - \theta p_2)^2 - (x_2 + \theta p_1)^2 \right] \theta, \quad (27)$$

$$\dot{x}_2 = \frac{1}{M(\theta)} p_2 - S(\theta) x_1 - 2 (x_2 + \theta p_1) (x_1 - \theta p_2) \theta, \quad (28)$$

$$\dot{p}_1 = -M(\theta)\Omega^2(\theta)x_1 + S(\theta)p_2 - 2 (x_2 + \theta p_1) (x_1 - \theta p_2), \quad (29)$$

$$\dot{p}_2 = -M(\theta)\Omega^2(\theta)x_2 + S(\theta)p_1 - (x_1 - \theta p_2)^2 + (x_2 + \theta p_1)^2, \quad (30)$$

which for deformation parameter running to zero become classical.

Similarly to the undeformed case, we find numerically the Poincaré maps in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$. However, this time, apart from parameter E_{tot} , we take under consideration the parameter of deformation θ . Consequently, we derive the Poincaré sections of phase space parameterized by pair (E_{tot}, θ) for $\theta = 0.5, 1, 2$ and six values of total energy E_{tot} . In such a way, we detect chaos in the model only for $\theta = 0.5$ and for $E_{\text{tot}} = 0.160178$, $E_{\text{tot}} = 0.1607445$ and $E_{\text{tot}} = 0.16245$, respectively (see for chaotic scenario figures 7–12). In the case of $\theta = 1$ as well as $\theta = 2$, the system remains ordered⁶.

⁶ As in the undeformed case, the calculations are performed for single trajectory with initial condition $x_1(0) = (2E_{\text{tot}})^{\frac{1}{2}}$ and $x_2(0) = p_1(0) = p_2(0) = 0$.

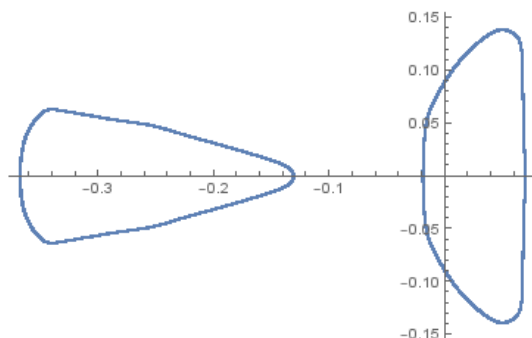


Fig. 7. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.15125$. The trajectory is completely regular — there is no chaos in the system.

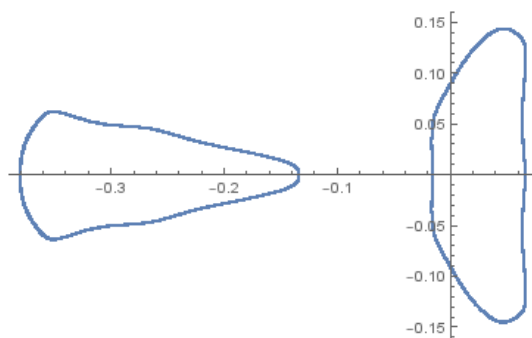


Fig. 8. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the fixed value of total energy $E_{\text{tot}} = 0.1568$. The trajectory is still regular — the system is chaos free.

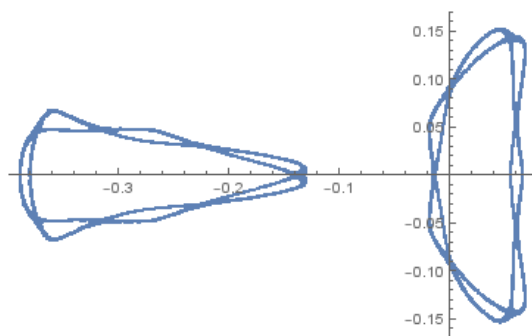


Fig. 9. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.1596125$. The trajectory still remains regular.

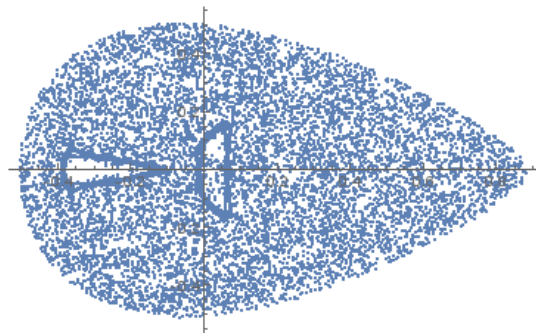


Fig. 10. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.160178$. The system suddenly becomes chaotic.

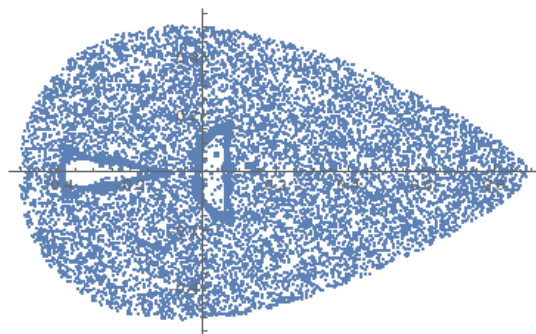


Fig. 11. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.1607445$. The chaos increases.

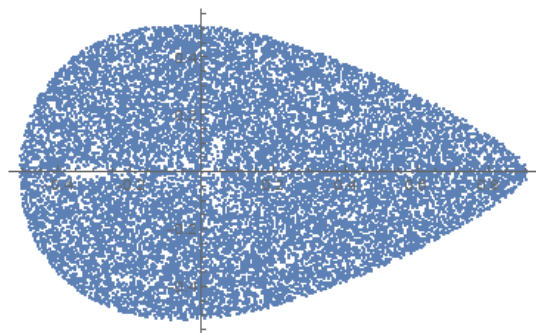


Fig. 12. The Poincaré map in two-dimensional phase space (x_2, p_2) for section $x_1 = 0$ and for the total energy $E_{\text{tot}} = 0.16245$. The system becomes totally chaotic.

5. Final remarks

In this article, we provide the canonically deformed Henon–Heiles system, *i.e.*, we define the proper Hamiltonian function as well as we derive the corresponding equations of motion. We also demonstrate (with the use of the Poincaré section method) that for deformation parameter $\theta = 0.5$ and for particular values of control parameter E_{tot} , the analyzed model becomes chaotic.

As a next step of presented here investigations, one can consider the canonical deformation of so-called generalized Henon–Heiles systems given by the following Hamiltonian function:

$$H(p, x) = \frac{1}{2} (p_1^2 + p_2^2) + \delta x_1^2 + (\delta + \Omega)x_2^2 + \alpha x_1^2 x_2 + \alpha \beta x_2^3 \quad (31)$$

with arbitrary coefficients α , β , δ and Ω , respectively. It should be noted that the properties of commutative models described by function (31) are quite interesting. For example, it is well-known (see, *e.g.*, [27–30] and references therein) that such systems remain integrable only in the Sawada–Kotera case: with $\beta = 1/3$ and $\Omega = 0$, in the KdV case: with $\beta = 2$ and arbitrary Ω as well as in the Kaup–Kupershmidt case: with $\beta = 16/3$ and $\Omega = 15\delta$. Besides, there has been provided in articles [31] and [32]⁷ the different types of integrable perturbations of mentioned above (integrable) models such as, for example, q^{-2} perturbations, the Ramani series of polynomial deformations and the rational perturbations. Consequently, the impact of the canonical deformation (3) on the above dynamical structures (in fact) seems to be very interesting. For this reason, the works in this direction already started and are in progress.

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REFERENCES

- [1] E.N. Lorenz, *J. Atmos. Sci.* **20**, 130 (1963).
- [2] M. Henon, C. Heiles, *Astrophys. J.* **69**, 73 (1964).
- [3] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Dover Publications, 1982.
- [4] G. Duffing, *Erzwungene Schwingungen bei Veränderlicher Eigenfrequenz*, F. Vieweg u. Sohn, Braunschweig 1918.

⁷ See also references therein.

- [5] D.A. Wells, *Theory and Problems of Lagrangian Dynamics*, New York: McGraw-Hill, 1967, pp. 13–14, 24, and 320–321.
- [6] V.I. Arnold, *Problem in Mathematical Methods of Classical Mechanics*, 2nd ed., New York: Springer-Verlag, 1989, p. 109.
- [7] W.P. Schleich, *Quantum Optics in Phase Space*, Wiley-VCH, Berlin 2001.
- [8] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics*, Wiley, New York 1989.
- [9] M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer-Verlag, Berlin 1989.
- [10] R. Oeckl, *J. Math. Phys.* **40**, 3588 (1999).
- [11] M. Chaichian, P.P. Kulish, K. Nashijima, A. Tureanu, *Phys. Lett. B* **604**, 98 (2004) [arXiv:hep-th/0408069].
- [12] M. Daszkiewicz, *Mod. Phys. Lett. A* **23**, 505 (2008) [arXiv:0801.1206 [hep-th]].
- [13] S. Zakrzewski, *Commun. Math. Phys.* **185**, 285 (1997) [arXiv:q-alg/9602001].
- [14] Y. Brihaye, E. Kowalczyk, P. Maslanka, arXiv:math/0006167 [math.QA].
- [15] J. Lukierski, A. Nowicki, H. Ruegg, V.N. Tolstoy, *Phys. Lett. B* **264**, 331 (1991).
- [16] S. Giller *et al.*, *Phys. Lett. B* **286**, 57 (1992).
- [17] J. Lukierski, M. Woronowicz, *Phys. Lett. B* **633**, 116 (2006) [arXiv:hep-th/0508083].
- [18] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, *Commun. Math. Phys.* **150**, 495 (1992).
- [19] P. Aschieri, L. Castellani, A.M. Scarfone, *Eur. Phys. J. C* **7**, 159 (1999) [arXiv:q-alg/9709032].
- [20] V.G. Drinfeld, *Soviet Math. Dokl.* **32**, 254 (1985); *Algebra i Analiz* **1**, 114 (1989) (in Russian only).
- [21] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, *Phys. Rev. Lett.* **86**, 2716 (2001) [arXiv:hep-th/0010175].
- [22] M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, *Eur. Phys. J. C* **36**, 251 (2004) [arXiv:hep-th/0212259].
- [23] S. Gangopadhyay, A. Saha, A. Halder, *Phys. Lett. A* **379**, 2956 (2015) [arXiv:1412.3581 [hep-th]].
- [24] J.M. Romero, J.A. Santiago, J.D. Vergara, *Phys. Lett. A* **310**, 9 (2003) [arXiv:hep-th/0211165].
- [25] Y. Miao, X. Wang, S. Yu, *Ann. Phys.* **326**, 2091 (2011) [arXiv:0911.5227 [math-ph]].
- [26] A. Kijanka, P. Kosinski, *Phys. Rev. D* **70**, 127702 (2004) [arXiv:hep-th/0407246].
- [27] T. Bountis, H. Segur, F. Vivaldi, *Phys. Rev. A* **25**, 1257 (1982).

- [28] Y.F. Chang, M. Tabor, J. Weiss, *J. Math. Phys.* **23**, 531 (1982).
- [29] S. Wojciechowski, *Phys. Lett. A* **100**, 277 (1984).
- [30] A.P. Fordy, *Physica D* **52**, 204 (1990).
- [31] A. Ballesteros, A. Blasco, *Ann. Phys.* **325**, 2787 (2010)
[arXiv:1011.3005 [math-ph]].
- [32] A. Ballesteros, A. Blasco, F.J. Herranz, *J. Phys: Conf. Ser.* **597**, 012013
(2015) [arXiv:1503.09187 [nlin.SI]].